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Some Generalized Gronwall-Bellman-Bihari Type Integral Inequalities with Application to Fractional Stochastic Differential Equation

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Abstract. In this paper some new Gronwall-Bellman-Bihari type integral inequalities with singular as well as non-singular kernels have been discussed, generalizing some already existing results. As an application of the derived results, the behaviour of solution of the fractional stochastic differential equation has been discussed.

1. Introduction

Integral inequalities play a vital role in the discussion of the quantitative as well as qualitative behaviour of solutions of differential equations. Among others Gronwall-Bellman and Bihari have gained a significant attention to analyze the behaviour of solutions of the certain differential equations, fractional differential equations, fractional stochastic differential equations. Due to the richness such inequalities have gained alot of attention of researchers, mathematicians and scientists. These inequalities are either generalized, modified, and extended in various directions using different techniques [1–6, 8]. Our aim, in this paper, is to discuss some qualitative properties of the solution of the fractional stochastic differential equation. For this purpose we are needed for an inequality involving singular as well as non-singular kernel(both) such an idea is floated by Qiong Wu [7]. Inspired by the idea of Qiong Wu, we have tried to develop Gronwall-Bellman-Bihari type inequalities involving singular as well as non singular kernels. This paper is organized in such a way that, after this Introduction in Section 2 we formulate the main results and some related consequences. And in Section 3 we give the application of the derived result to discuss the behaviour of solution of a certain fractional stochastic differential equation(SDE).

2. Main Results

Lemma 2.1. *[4] Assume that $a \geq 0$, $p \geq q \geq 0$, with $p \neq 0$. Then, for an $\lambda > 0$, we have*

$$a^{\frac{q}{p}} \leq \frac{qa}{p}\lambda^{\frac{q-p}{p}} + \frac{p-q}{p}\lambda^{\frac{q}{p}}.$$

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Theorem 2.2. *Let $a(t)$, $b(t)$, $g(t)$ and $u(t)$ be non-negative functions on $I = [0, T)$, $T \leq +\infty$, $\alpha \in (0, 1)$. Moreover, if $a(t)$, $u(t)$ are locally summable on I. If $b(t)$ and $g(t)$ are non-decreasing continuous functions on I bounded by a constant $M > 0$; let $p \geq q > 0$ and $\lambda > 0$ such that:*

$$u^p(t) \leq a(t) + b(t)\int_0^t u^q(s)ds + g(t)\int_0^t (t-s)^{\alpha-1}u^q(s)ds, \tag{1}$$

then we have the following explicitly bound for u

$$u(t) \leq \left\{\widehat{a}(t) + \sum_{k=1}^{\infty}\sum_{i=0}^{k}\frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}}\right]^k [b(t)]^{k-i}[\Gamma(\alpha)g(t)]^i}{p^k\Gamma(i\alpha+k-i)}\int_0^t (t-s)^{i\alpha-i-1+k}\widehat{a}(s)ds\right\}^{\frac{1}{p}}, \tag{2}$$

provided that

$$\widehat{a}(t) := a(t) + \frac{(p-q)\lambda^{\frac{q}{p}}\left[\alpha t b(t) + t^{\alpha}g(t)\right]}{\alpha p}$$

Proof. On letting the right hand side of (1) by $v(t)$, we have

$$u^p(t) \leq v(t) \tag{3}$$

Application of Lemma 2.1 yields:

$$\begin{aligned}
v(t) &\leq a(t) + b(t)\int_0^t v^{\frac{q}{p}}(s)ds + g(t)\int_0^t (t-s)^{\alpha-1}v^{\frac{q}{p}}(s)ds \\
&\leq a(t) + b(t)\int_0^t \frac{q\lambda^{\frac{q-p}{p}}v(s) + (p-q)\lambda^{\frac{q}{p}}}{p}ds \\
&\quad + g(t)\int_0^t \frac{q\lambda^{\frac{q-p}{p}}v(s) + (p-q)\lambda^{\frac{q}{p}}}{p}(t-s)^{\alpha-1}ds. \\
&= a(t) + \frac{(p-q)\lambda^{\frac{q}{p}}}{p\alpha}(\alpha t b(t) + t^{\alpha}g(t)) + \frac{q\lambda^{\frac{q-p}{p}}b(t)}{p}\int_0^t v(s)ds \\
&\quad + \frac{q\lambda^{\frac{q-p}{p}}g(t)}{p}\int_0^t (t-s)^{\alpha-1}v(s)ds \\
&= \widehat{a}(t) + \frac{q\lambda^{\frac{q-p}{p}}b(t)}{p}\int_0^t v(s)ds + \frac{q\lambda^{\frac{q-p}{p}}g(t)}{p}\int_0^t (t-s)^{\alpha-1}v(s)ds
\end{aligned}$$

For locally summable function Φ

$$\mathfrak{B}\Phi(t) := \frac{q\lambda^{\frac{q-p}{p}}b(t)}{p}\int_0^t \Phi(s)ds + \frac{q\lambda^{\frac{q-p}{p}}g(t)}{p}\int_0^t (t-s)^{\alpha-1}\Phi(s)ds.$$

Equivalently,

$$v(t) \leq \widehat{a}(t) + \mathfrak{B}v(t).$$

Iterating the inequality for some n, one has

$$v(t) \leq \sum_{k=0}^{n-1}\mathfrak{B}^k\widehat{a}(t) + \mathfrak{B}^n v(t) \tag{4}$$

One should prove that:

$$\mathfrak{B}^n v(t) \leq \sum_{i=0}^{n} \frac{\binom{n}{i} b^{n-i}(t)\left[\Gamma(\alpha)g(t)\right]^i \left[q\lambda^{\frac{q-p}{p}}\right]^n}{p^n\Gamma(i\alpha+n-i)} \int_0^t (t-s)^{i\alpha-i-1+n} v(s)ds. \tag{5}$$

The proof follows the induction criteria on n. For $n=1$ the result trivially holds. Suppose it holds for some $n=k$. Furthermore, if $b(t)$, $g(t)$ are non-negative and nondecreasing therefore for $n=k+1$, one has

$$\begin{aligned}
\mathfrak{B}^{k+1}v(t) &= \mathfrak{B}(\mathfrak{B}^k v(t)) \\
&\leq \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i}\left[q\Gamma(\alpha)\lambda^{\frac{q-p}{p}} g(t)\right]^i}{p^k\Gamma(i\alpha+k-i)} \\
&\quad \times\left\{\frac{q\lambda^{\frac{q-p}{p}} b(t)}{p}\int_0^t\int_0^s (s-\tau)^{i\alpha-(i+1-k)} v(\tau)d\tau ds + \right. \\
&\quad \left. \frac{q\lambda^{\frac{q-p}{p}} g(t)}{p}\int_0^t\int_0^s (t-s)^{\alpha-1}(s-\tau)^{i\alpha-(i+1-k)} v(\tau)d\tau ds\right\} \\
&= \mathfrak{C}(t)+\mathfrak{G}(t),
\end{aligned} \tag{6}$$

$$\mathfrak{C}(t) := \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i+1}\left[q\Gamma(\alpha)\lambda^{\frac{q-p}{p}} g(t)\right]^i}{p^{k+1}\Gamma(i\alpha+k-i)}\int_0^t\int_0^s (s-\tau)^{i\alpha-(i+1-k)} v(\tau)d\tau ds$$

$$\mathfrak{G}(t) := \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i}\left[q\lambda^{\frac{q-p}{p}} g(t)\right]^{i+1}\left[\Gamma(\alpha)\right]^i}{p^{k+1}\Gamma(i\alpha+k-i)}\int_0^t\int_0^s (s-\tau)^{i\alpha-(i+1-k)}(t-s)^{\alpha-1} v(\tau)d\tau ds$$

By solving $\mathfrak{C}(t)$ and $\mathfrak{G}(t)$ we acquire inequality (5).

Change of integration order yields the following:

$$\begin{aligned}
\mathfrak{C}(t) &= \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i+1}\left[q\Gamma(\alpha)\lambda^{\frac{q-p}{p}} g(t)\right]^i}{p^{k+1}\Gamma(i\alpha+k-i)}\int_0^t\int_0^s (s-\tau)^{i\alpha-(i+1-k)} v(\tau)d\tau ds \\
&= \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i+1}\left[q\Gamma(\alpha)\lambda^{\frac{q-p}{p}} g(t)\right]^i}{p^{k+1}\Gamma(i\alpha+k-i)}\int_0^t\int_\tau^t (s-\tau)^{i\alpha-(i+1-k)} v(\tau)ds d\tau \\
&= \frac{\binom{k}{0}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k+1}}{p^{k+1}\Gamma(k+1)}\int_0^t (t-\tau)^k v(\tau)d\tau + \sum_{i=1}^{k} \frac{\binom{k}{i} b^{k-i+1}(t)\left[q\lambda^{\frac{q-p}{p}}\right]^{k+1}\left[\Gamma(\alpha)g(t)\right]^i}{p^{k+1}\Gamma(i\alpha+k-i+1)} \\
&\quad \times\int_0^t (t-\tau)^{i\alpha-i+k} v(\tau)d\tau
\end{aligned} \tag{7}$$

Similarly

$$\begin{aligned}
\mathfrak{G}(t) &= \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i}\left[q\lambda^{\frac{q-p}{p}} g(t)\right]^{i+1}[\Gamma(\alpha)]^{i}}{p^{k+1}\Gamma(i\alpha+k-i)} \int_0^t \int_0^s (s-\tau)^{i\alpha-i-1+k}(t-s)^{\alpha-1} v(\tau) d\tau ds \\
&= \sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k-i}\left[q\lambda^{\frac{q-p}{p}} g(t)\right]^{i+1}[\Gamma(\alpha)]^{i}}{p^{k+1}\Gamma(i\alpha+k-i)} \int_0^t \int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{i\alpha-i-1+k} v(\tau) ds d\tau \\
&= \frac{\binom{k}{k}\left[q\lambda^{\frac{q-p}{p}} g(t)\right]^{k+1}[\Gamma(\alpha)]^{k+1}}{p^{k+1}\Gamma(k\alpha+\alpha)} \int_0^t (t-\tau)^{k\alpha+\alpha-1} v(\tau) d\tau \\
&\quad + \sum_{i=1}^{k} \frac{\binom{k}{i-1} b^{k-i+1}(t)[\Gamma(\alpha) g(t)]^{i}\left[q\lambda^{\frac{q-p}{p}}\right]^{k+1}}{p^{k+1}\Gamma(i\alpha+k-i+1)} \int_0^t (t-\tau)^{i\alpha-i+k} v(\tau) d\tau. \qquad (8)
\end{aligned}$$

Combining (7) and (8), then (6) has the form

$$\begin{aligned}
\mathfrak{B}^{k+1} v(t) &\leq \frac{\binom{k}{0}\left[q\lambda^{\frac{q-p}{p}} b(t)\right]^{k+1}}{p^{k+1}\Gamma(k+1)} \int_0^t (t-\tau)^{k} v(\tau) d\tau \\
&\quad + \sum_{i=0}^{k} \frac{\left[\binom{k}{i-1}+\binom{k}{i}\right] b^{k-i+1}(t)[\Gamma(\alpha) g(t)]^{i}\left[q\lambda^{\frac{q-p}{p}}\right]^{k+1}}{p^{k+1}\Gamma(i\alpha+k-i+1)} \int_0^t (t-\tau)^{i\alpha-i+k} v(\tau) d\tau \\
&\quad + \frac{\binom{k}{k}\left[q\lambda^{\frac{q-p}{p}} g(t)\right]^{k+1}[\Gamma(\alpha)]^{k+1}}{p^{k+1}\Gamma(k\alpha+\alpha)} \int_0^t (t-\tau)^{k\alpha+\alpha-1} v(\tau) d\tau \\
&= \sum_{i=0}^{k+1} \frac{\binom{k+1}{i}[\Gamma(\alpha) g(t)]^{i} b^{k+1-i}(t)\left[q\lambda^{\frac{q-p}{p}}\right]^{k+1}}{p^{k+1}\Gamma(i\alpha+k-i+1)} \int_0^t (t-\tau)^{i\alpha-i+k} v(\tau) d\tau.
\end{aligned}$$

This proves the validity of (5) for $n = k+1$.

We claim that $\mathfrak{B}^n v(t) \to 0$ as $n \to \infty$. For

$$\mathfrak{H}_n(t) := \sum_{i=0}^{n} \frac{\binom{n}{i} b^{n-i}(t)\,[\Gamma(\alpha) g(t)]^{i}\left[q\lambda^{\frac{q-p}{p}}\right]^{n}}{p^{n}\Gamma(i\alpha+n-i)} \int_0^t (t-s)^{i\alpha-i-1+n} v(s) ds. \qquad (9)$$

Consider, $x_i = i\alpha + n - i$ then this sequence is decreasing on $[0, n]$ for $i \in [0, n]$. It may be easily seen that $\max x_i = n$; $\min x_i = n\alpha$ and hence $\Gamma(n\alpha) = \Gamma(\min x_i) < \Gamma(x_i)$; $\Gamma(n) = \Gamma(\max x_i) > \Gamma(x_i)$ for $i \in [0, n]$. In this case equation (9) can be written as:

$$\mathfrak{H}_n(t) \leq \sum_{i=0}^{n} \frac{\binom{n}{i} b^{n-i}(t) g^{i}(t)\left[q\lambda^{\frac{q-p}{p}}\Gamma(\alpha)\right]^{n}}{p^{n}\Gamma(n\alpha)} \int_0^t (t-s)^{i\alpha-i-1+n} v(s) ds. \qquad (10)$$

Similarly, as above, by considering $y_i = i\alpha - i - 1 + n$ we can see that $\min y_i = n\alpha - 1$ and $\max y_i = n - 1$ for sufficient large n. Furthermore, for an arbitrary T, we have

$$(t-s)^{y_i} \leq \begin{cases} t^{\min y_i} (= t^{n\alpha-1}), & t \in [0,1]; \\ t^{\max y_i} (= t^{n-1}), & t \in [1,T). \end{cases} \qquad (11)$$

In the light of (11), inequality (10) has the form

$$\mathfrak{H}_n(t) \leq \frac{\left[q\lambda^{\frac{q-p}{p}}\Gamma(\alpha)\right]^n (b(t)+g(t))^n}{p^n\Gamma(n\alpha)} \max\{t^{n\alpha-1}, t^{n-1}\} \int_0^t v(s)ds. \tag{12}$$

But, $b(t)$ and $g(t)$ are both bounded by a constant M; $v(s)$ is locally summable over $[0,T)$ and $\Gamma(n\alpha)$ is growing faster than $[\Gamma(\alpha)]^n$ for sufficient large n, so

$$\mathfrak{H}_n(t) \leq \left[\frac{2Mq\lambda^{\frac{q-p}{p}}}{p}\right]^n \frac{[\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \max\{t^{n\alpha-1}, t^{n-1}\} \int_0^t v(s)ds \to 0 \quad (\text{as } n \to \infty),$$

that is, $\mathfrak{B}^n v(t) \leq \mathfrak{H}_n(t) \to 0$ as $n \to \infty$ and hence (4) is rewritten as:

$$v(t) \leq \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i} b^{k-i}(t)\,[\Gamma(\alpha)g(t)]^i \left[q\lambda^{\frac{q-p}{p}}\right]^k}{p^k\Gamma(i\alpha+k-i)} \int_0^t (t-s)^{i\alpha-i-1+k}\widehat{a}(s)ds. \tag{13}$$

$$\mathfrak{L}(t;\beta) := \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i} b^{k-i}(t)\,[\Gamma(\alpha)g(t)]^i \left[q\lambda^{\frac{q-p}{p}}\right]^k}{p^k\Gamma(i\alpha+k-i)} \int_0^\beta (\beta-s)^{i\alpha-i-1+k}ds.$$

We claim that $\mathfrak{L}(t;\beta)$ is convergent for $t \in [0,T)$.

$$\begin{aligned}
\mathfrak{L}(t;\beta) &= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}}\right]^k b^{k-i}(t)\,[\Gamma(\alpha)g(t)]^i \beta^{i\alpha+k-i}}{p^k(i\alpha+k-i)(i\alpha+k-i-1).....(i\alpha+1)\Gamma(i\alpha+1)}. \\
&= \sum_{i=0}^{\infty} \frac{\left[q\Gamma(\alpha)g(t)\lambda^{\frac{q-p}{p}}\beta^\alpha\right]^i}{\Gamma(i\alpha+1)p^i} \sum_{r=0}^{\infty} \frac{(i+r)(i+r-1)...(i+1)\left[q\beta\lambda^{\frac{q-p}{p}}b(t)\right]^r}{r!p^r(i\alpha+r)(i\alpha+r-1).....(i\alpha+1)}. \\
&\leq \sum_{i=0}^{\infty} \frac{\left[q\Gamma(\alpha)g(t)\lambda^{\frac{q-p}{p}}\beta^\alpha\right]^i}{\Gamma(i\alpha+1)p^i} \sum_{r=0}^{\infty} \frac{\left[q\beta\lambda^{\frac{q-p}{p}}b(t)\right]^r}{\alpha^r r!p^r} \\
&= E_\alpha\left(\frac{q\Gamma(\alpha)g(t)\lambda^{\frac{q-p}{p}}\beta^\alpha}{p}\right)\exp\left(\frac{q\beta\lambda^{\frac{q-p}{p}}b(t)}{p\alpha}\right) \\
&\leq E_\alpha\left(\frac{qM\Gamma(\alpha)\lambda^{\frac{q-p}{p}}\beta^\alpha}{p}\right)\exp\left(\frac{qM\beta\lambda^{\frac{q-p}{p}}}{p\alpha}\right) =: \widehat{\mathfrak{L}}(M;\beta)
\end{aligned}$$

$$\begin{aligned}
\mathfrak{L}(\widehat{a};t;\beta) &:= \widehat{a}(t) + \sum_{k=1}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i} b^{k-i}(t)\,[\Gamma(\alpha)g(t)]^i \left[q\lambda^{\frac{q-p}{p}}\right]^k}{p^k\Gamma(i\alpha+k-i+1)} \int_0^\beta \frac{d}{d\beta}(\beta-s)^{i\alpha-i+k}\widehat{a}(s)ds. \\
&\leq \widehat{a}(t) + \sum_{k=1}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i} M^k\,[\Gamma(\alpha)]^i \left[q\lambda^{\frac{q-p}{p}}\right]^k}{p^k\Gamma(i\alpha+k-i+1)} \int_0^\beta \frac{d}{d\beta}(\beta-s)^{i\alpha-i+k}\widehat{a}(s)ds. \\
&= \widehat{a}(t) + \int_0^\beta \widehat{a}(s)\frac{d}{d\beta}\mathfrak{L}(\widehat{a};M;\beta)ds
\end{aligned} \tag{14}$$

As the Mittag-Leffler function $E_\alpha(\beta^\alpha)$ is an entire function in β^α and the exponential function, $\exp(\beta)$, is uniformly continuous in β, $\widehat{a}(t)$ is summable function over $t > 0$ therefore $\mathfrak{L}(\widehat{a}; t; \beta) < \infty$. A combination of (3) and (13) yields the desired result (2). □

Corollary 2.3. *Under the conditions of Theorem 2.2, furthermore, if $a(t)$ is non-decreasing on $[0, T)$, then*

$$u(t) \leq \sqrt[p]{\widehat{a}(t) E_\alpha\left(\frac{q\Gamma(\alpha)g(t)\lambda^{\frac{q-p}{p}} t^\alpha}{p}\right) \exp\left(\frac{qt\lambda^{\frac{q-p}{p}} b(t)}{p\alpha}\right)} \tag{15}$$

Proof. Suppose $a(t)$ is non-decreasing on $[0, T)$ and hence so is $\widehat{a}(t)$ on $[0, T)$ therefore from inequality (2)

$$\begin{aligned} u^p(t) &\leq \widehat{a}(t) + \sum_{k=1}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}}\right]^k [b(t)]^{k-i}[\Gamma(\alpha)g(t)]^i}{p^k\Gamma(i\alpha + k - i)} \int_0^t (t-s)^{i\alpha - i - 1 + k}\widehat{a}(s)ds \\ &\leq \widehat{a}(t) \sum_{k=0}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}}\right]^k [b(t)]^{k-i}[\Gamma(\alpha)g(t)]^i}{p^k\Gamma(i\alpha + k - i)} \int_0^t (t-s)^{i\alpha - i - 1 + k}ds \\ &= \widehat{a}(t) \sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}}\right]^k [tb(t)]^{k-i}[t^\alpha\Gamma(\alpha)g(t)]^i}{p^k\Gamma(i\alpha + k - i + 1)} \\ &\leq \widehat{a}(t) E_\alpha\left(\frac{q\Gamma(\alpha)g(t)\lambda^{\frac{q-p}{p}} t^\alpha}{p}\right) \exp\left(\frac{qt\lambda^{\frac{q-p}{p}} b(t)}{p\alpha}\right). \end{aligned}$$

□

Remark 2.4. *For $p = q = 1$ and $b(t) \equiv 0$, Theorem 2.2 reduces to [3, Theorem 1]. For $p = q = 1$ and $g(t) \equiv 0$, Corollary 2.3 reduces to [3, Theorem A]. For $p = q = 1$ and $b(t) \equiv 0$, $g(t) \equiv c$, a constant, Theorem 2.2 reduces to [3, Corollary 1]. For $p = q = 1$ and $b(t) \equiv 0$, Corollary 2.3 reduces to [3, Corollary 2]*

Theorem 2.5. *Under the conditions of Theorem 1, furthermore, assume that $p \geq 1$ is a constant, $L \in C(R_+^2, R_+)$ with $0 \leq L(s, u) - L(s, v) \leq A(u - v)$ for $u \geq v \geq 0$, where $A > 0$ is a Lipschitz constant such that:*

$$u^p(t) \leq a(t) + b(t)\int_0^t u^q(s)ds + g(t)\int_0^t (t-s)^{\alpha-1}L(s, u^q(s))ds, \tag{16}$$

then we have the following explicit estimate of u

$$u(t) \leq \left\{\widetilde{a}(t) + \sum_{k=1}^{\infty}\sum_{i=0}^{k} \frac{\binom{k}{i}\left[q\lambda^{\frac{q-p}{p}}\right]^k [b(t)]^{k-i}[A\Gamma(\alpha)g(t)]^i}{p^k\Gamma(i\alpha + k - i)} \int_0^t (t-s)^{i\alpha - i - 1 + k}\widetilde{a}(s)ds\right\}^{\frac{1}{p}}, \tag{17}$$

$$\widetilde{a}(t) := a(t) + \frac{(p-q)\lambda^{\frac{q}{p}} tb(t)}{p} + g(t)\int_0^t (t-s)^{\alpha-1}L\left(s, \frac{(p-q)\lambda^{\frac{q}{p}}}{p}\right)ds$$

Proof. On letting the right hand side of (16) by $w(t)$, we have

$$u^p(t) \leq w(t) \tag{18}$$

Application of Lemma 2.1, inequality (18) yield:

$$\begin{aligned} w(t) &\le a(t) + b(t)\int_0^t \frac{q\lambda^{\frac{q-p}{p}}w(s) + (p-q)\lambda^{\frac{q}{p}}}{p}ds \\ &\quad + g(t)\int_0^t (t-s)^{\alpha-1} L\left(s, \frac{q\lambda^{\frac{q-p}{p}}w(s) + (p-q)\lambda^{\frac{q}{p}}}{p}\right)ds \\ &= a(t) + b(t)\int_0^t \frac{q\lambda^{\frac{q-p}{p}}w(s) + (p-q)\lambda^{\frac{q}{p}}}{p}ds \\ &\quad + g(t)\int_0^t (t-s)^{\alpha-1} \left\{L\left(s, \frac{q\lambda^{\frac{q-p}{p}}w(s) + (p-q)\lambda^{\frac{q}{p}}}{p}\right)\right. \\ &\quad \left. + L\left(s, \frac{(p-q)\lambda^{\frac{q}{p}}}{p}\right) - L\left(s, \frac{(p-q)\lambda^{\frac{q}{p}}}{p}\right)\right\}ds \\ &\le a(t) + b(t)\int_0^t \frac{q\lambda^{\frac{q-p}{p}}w(s) + (p-q)\lambda^{\frac{q}{p}}}{p}ds \\ &\quad + g(t)\int_0^t (t-s)^{\alpha-1} \left\{\frac{Aq\lambda^{\frac{q-p}{p}}w(s)}{p} + L\left(s, \frac{(p-q)\lambda^{\frac{q}{p}}}{p}\right)\right\}ds \\ &= \widetilde{a}(t) + \frac{q\lambda^{\frac{q-p}{p}}b(t)}{p}\int_0^t w(s)ds + \frac{Aq\lambda^{\frac{q-p}{p}}g(t)}{p}\int_0^t (t-s)^{\alpha-1}w(s)ds \end{aligned} \tag{19}$$

Application of Theorem 2.2 to (19), yields the desired result (17). □

Corollary 2.6. *Under the conditions of Theorem 2.5, furthermore, if $a(t)$ is non-decreasing on $[0, T)$, then*

$$u(t) \le \sqrt[p]{\widetilde{a}(t)E_\alpha\left(\frac{qA\Gamma(\alpha)g(t)\lambda^{\frac{q-p}{p}}t^\alpha}{p}\right)\exp\left(\frac{qt\lambda^{\frac{q-p}{p}}b(t)}{p\alpha}\right)} \tag{20}$$

Remark 2.7. *For $q = 1$; $b(t) \equiv 0$ and $g(t) \to \frac{h(t)}{\Gamma(\alpha)}$, Theorem 2.5 reduces to [2, Theorem 2] and Corollary 2.6 reduces to [2, Corollary 3].*

3. Application

In this section we shall try to discuss the following Stochastic differential equation:

$$d(f(x(t))) = b(t, f(x(t)))dt + \sigma_1(t, f(x(t)))dt^\alpha + \sigma_2(t, f(x(t)))dB_t \tag{21}$$

where $0 < \alpha < 1$ and B_t is standrad Brownian motion.

Theorem 3.1. *Let (Ω, F, P) be a complete probability space with a filtration $[F_t]_{t\geqslant 0}$. Let $B(t) := (B_1(t), ..., B_m(t))^T$ be an $m-$dimensional Browanian motion defined on space $\mathbb{R}^n$. Let $0 \leqslant t < T < \infty$ and x_0 a random variable such that $E|x_0|^2 < \infty$; let $b(.,.), \sigma_1(.,.) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that the following linear Growth and Lipschitz conditions, respectively, are satisfied*

$$|b(t, f(x))|^2 \vee |\sigma_1(t, f(x))|^2 \vee |\sigma_2(t, f(x))|^2 \le K^2\left(1 + |f(x)|^2\right) \tag{22}$$

$$|b(t, f(x)) - b(t, f(y))| \vee |\sigma_1(t, f(x)) - \sigma_1(t, f(y))| \vee |\sigma_2(t, f(x)) - \sigma_2(t, f(y))| \leq L|f(x) - f(y)| \quad (23)$$

for some constants $K, L > 0$. Then equation (21) *has a t-continuous solution and*

$$E\left[\int_0^T |f(x)|^2 dt\right] < \infty.$$

Proof. The integral form of the stochastic differential equation (21) is

$$f(x(t)) = f(x_0) + \int_0^t b(s, f(x(s)))ds + \alpha \int_0^t (t-s)^{\alpha-1}\sigma_1(s, f(x(s)))ds + \int_0^t \sigma_2(s, f(x(s)))dB_s. \quad (24)$$

By the method of Picard-Lindelof approximation, define $x^{(0)}(t) = x_0$ and $x^{(k)}(t) = x^{(k)}(t, w)$ inductively as follows:

$$f(x^{(k+1)}(t)) = f(x_0) + \int_0^t b(s, f(x^{(k)}(s)))ds + \alpha \int_0^t (t-s)^{\alpha-1}\sigma_1(s, f(x^{(k)}(s)))ds + \int_0^t \sigma_2(s, f(x^{(k)}(s)))dB_s. \quad (25)$$

Applications of the inequality, $|x+y+z|^2 \leq 3|x|^2 + 3|y|^2 + 3|z|^2$, Cauchy Schwartz inequality and Itô's Isometry yield the following:

$$E|f(x^{(k+1)}(t)) - f(x^{(k)}(t))|^2 \leq 3TE\int_0^t \left[b(s, f(x^{(k)}(s))) - b(s, f(x^{(k-1)}(s)))\right]^2 ds$$
$$+3\alpha t^\alpha E\int_0^t (t-s)^{\alpha-1}[(\sigma_1(s, f(x^{(k)}))) - \sigma_1(s, f(x^{(k-1)}(s)))]^2 ds$$
$$+3E\int_0^t [(\sigma_2(s, f(x^{(k)}))) - \sigma_2(s, f(x^{(k-1)}(s)))]^2 ds. \quad (26)$$

Repeating applications of (23) on each integral of the right hand side of (26) yield the following:

$$E|f(x^{(k+1)}(t)) - f(x^{(k)}(t))|^2 \leq 3L^2(1+T)\int_0^t E|f(x^{(k)}(t)) - f(x^{(k-1)}(t))|^2 ds + 3L^2\alpha T^\alpha \times \int_0^t (t-s)^{\alpha-1}E|f(x^{(k)}(t)) - f(x^{(k-1)}(t))|^2 ds. \quad (27)$$

For summable function $\Psi(t)$ define an operator $\mathfrak{G}$ defined by:

$$\mathfrak{G}\Psi(t) := 3L^2(1+T)\int_0^t \Psi(s)ds + 3L^2\alpha T^\alpha \int_0^t (t-s)^{\alpha-1}\Psi(s)ds. \quad (28)$$

From (27) and (28) repeating iteration yields:

$$E|f(x^{(k+1)}(t)) - f(x^{(k)}(t))|^2 \leq \mathfrak{G}(E|f(x^{(k)}(t)) - f(x^{(k-1)}(t))|^2)$$
$$\leq ... \leq \mathfrak{G}^{k-1}(E|f(x^{(2)}(t)) - f(x^{(1)}(t))|^2) \leq \mathfrak{G}^k(E|f(x^{(1)}(t)) - f(x^{(0)}(t))|^2). \quad (29)$$

As, $E|f(x^{(1)}(t)) - f(x^{(0)}(t))|^2$ is locally summable therefore in the light of (5), (12) and (29)

$$\begin{aligned} E|f(x^{(k+1)}(t)) - f(x^{(k)}(t))|^2 &\leq \mathfrak{G}^k(E|f(x^{(1)}(t)) - f(x^{(0)}(t))|^2) \\ &= \frac{[\Gamma(\alpha)]^k \max\{t^{k\alpha-1}, t^{k-1}\}[3L^2(1+T+\alpha T^\alpha)]^k}{\Gamma(k\alpha)} \\ &\quad \times \int_0^t E|f(x^{(1)}(t)) - f(x^{(0)}(t))|^2 ds. \end{aligned} \tag{30}$$

Again, from (25) applications of the inequalities (22), $|x+y+z|^2 \leq 3|x|^2 + 3|y|^2 + 3|z|^2$, Cauchy Schwartz inequality, Itô's Isometry yield the following:

$$E|f(x^{(1)}(t)) - f(x^{(0)}(t))|^2 \leq 3K^2(1 + E|f(x_{(0)})|^2)[(1+T)t + (Tt)^\alpha]. \tag{31}$$

A combination of (30) and (31) yield the following:

$$\begin{aligned} &\sup_{0\leq t\leq T} E|f(x^{(k+1)}(t)) - f(x^{(k)}(t))|^2 \leq \frac{M_0[\Gamma(\alpha)]^k}{\Gamma(k\alpha)} \\ &\times \max\{T^{k\alpha-1}, T^{k-1}\}[3L^2(1+T+\alpha T^\alpha)]^k, \end{aligned} \tag{32}$$

provided that

$$M_0 := 3K^2(1 + E|f(x_0)|^2)\left(\frac{T^2+T^3}{2} + \frac{T^{2\alpha+1}}{\alpha+1}\right),$$

Thus, for any $m > n > 0$

$$\begin{aligned} \|f(x^{(m)}(t)) - f(x^{(n)}(t))\|^2_{L^2(\mathbb{P})} &\leq \sum_{k=n}^{m} \|f(x^{(k+1)}(t)) - f(x^{(k)}(t))\|^2_{L^2(\mathbb{P})} \\ &= \sum_{k=n}^{m} \int_0^T E|f(x^{(k+1)}(t)) - f(x^{(k)}(t))|^2 dt \\ &\leq M_1 \sum_{k=n}^{m} \frac{[\Gamma(\alpha)]^k \max\{T^{k\alpha-1}, T^{k-1}\}[3L^2(1+T+\alpha T^\alpha)]^k}{\Gamma(k\alpha)} \to 0 \end{aligned}$$

for sufficiently large m, n such that

$$M_1 := 3K^2(1 + E|f(x_0)|^2)\left(\frac{T^3+T^4}{2} + \frac{T^{3\alpha+1}}{\alpha+1}\right),$$

Doob's maximal inequality for martingale yields:

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbb{P}\left[\sup_{0\leq t\leq T} |f(x^{(k+1)}(t)) - f(x^{(k)}(t))| > \frac{1}{k^2}\right] \\ &\leq M_0 \sum_{k=1}^{\infty} \frac{k^4[\Gamma(\alpha)]^k \max\{T^{k\alpha-1}, T^{k-1}\}[3L^2(1+T+\alpha T^\alpha)]^k}{\Gamma(k\alpha)} < +\infty \end{aligned} \tag{33}$$

From (33) by Borel-Cantelli lemma

$$\mathbb{P}\left\{\sup_{0\leq t\leq T} |f(x^{(k+1)}(t)) - f(x^{(k)}(t))| > \frac{1}{k^2} \text{ for infinitely many } k\right\} = 0,$$

it follows that:

$$\mathbb{P}\left[\sup_{0\leq t\leq T} f\left(x^{(k+1)}(t)\right) - f\left(x^{(k)}(t)\right) > \epsilon\right] = 0 \text{ for each } \epsilon > 0,$$

so there exist a random variable $f(x(t))$ which is, almost surely uniformly continuous on $[0, T]$, such that

$$f(x^{(k)}(t)) = f(x^{(0)}(t)) + \sum_{n=0}^{k-1}(f(x^{(n+1)}(t)) - f(x^{(n)}(t)) \overset{k\to\infty}{\longrightarrow} f(x(t)).$$

By the continuity of $x^{(k)}(t)$ with respect to t, for any k, $f(x(t))$ is as well. Therefore,

$$f(x_0) + \int_0^t b(s, f(x^{(k)}(s)))ds + \alpha \int_0^t (t-s)^{\alpha-1}\sigma_1(s, f(x^{(k)}(s)))ds$$
$$+ \int_0^t \sigma_2(s, f(x^{(k)}(s)))dB_s \overset{k\to\infty}{\longrightarrow} f(x(t)),$$

for a stochastic process $x(t)$ satisfying (24). □

Theorem 3.2. *Under the conditions of Theorem 3.1, equation* (21) *has at most one solution.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of (21) with the initial conditions $x_i^{(0)}(t) = y_i, 1 \leq i \leq 2$. Applications of Cauchy-Schwartz inequality, the Itô Isometry, and Lipschitz condition yield:

$$E|f(x_1(t)) - f(x_2(t))|^2 \leq 3E|f(y_1) - f(y_2)| + 3L^2(1+T)\int_0^t E|f(x_1(s)) - f(x_2(s))|^2 ds$$
$$+3\alpha L^2 T^\alpha \int_0^t (t-s)^{\alpha-1} E|f(x_1(s)) - f(x_2(s))|^2 ds.$$

Application of (15) for $a(t) \to 3E|f(y_1) - f(y_2)|$; $b(t) \to 3L^2(1+T)$ and $g(t) \to 3\alpha L^2 T^\alpha$ yields

$$E|f(x_1(t)) - f(x_2(t))|^2 \leq 3E|f(y_1) - f(y_2)| \times E_\alpha(3\alpha L^2(tT)^\alpha\Gamma(\alpha))\exp\left(\frac{3L^2(1+T)t}{\alpha}\right). \tag{34}$$

Since, $x_1(t)$ and $x_2(t)$ are the solutions of (21) with the initial conditions $x_i^{(0)}(t) = y_i$, $1 \leq i \leq 2$, therefore $y_1 = y_2$ and hence (34) yields

$$E|f(x_1(t)) - f(x_2(t))|^2 = 0 \text{ for all } t > 0,$$

which proves the uniqueness. □

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