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On Hermite-Hadamard Type Inequalities Via Fractional Integral Operators

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Abstract. In this paper, we give new definitions related to fractional integral operators for two variables functions using the class of integral operators. We are interested to give the Hermite–Hadamard inequality for a rectangle in plane via convex functions on co-ordinates involving fractional integral operators.

1. Introduction

The most well-known inequality related to the integral mean of a convex function is the Hermite Hadamard inequality. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then the following double inequality is known in the literature as the Hermite-Hadamard's inequality for convex functions [8]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The inequalities (1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Many generalizations and extensions of the Hermite-Hadamard inequality exist in the literature; (see [3],[4] and [30]) and references therein.

Let us consider a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on Δ if for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$, it satisfies the following inequality:

$$f(tx + (1-t)\,z, ty + (1-t)\,w) \leq t\,f(x, y) + (1-t)\,f(z, w).$$

A modification for convex function on Δ was defined by Dragomir [29], as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$.

A formal definition for co-ordinated convex function may be stated as follows:

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Definition 1.1. *A function $f : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x,u),(y,v) \in \Delta$ and $t,s \in [0,1]$, if it satisfies the following inequality:*

$$f(tx+(1-t)\,y, su+(1-s)\,v) \tag{2}$$

$$\leq \; ts\, f(x,u)+t(1-s)f(x,v)+s(1-t)f(y,u)+(1-t)(1-s)f(y,v).$$

Note that every convex function $f : \Delta \to \mathbb{R}$ is co-ordinated convex but the converse is not generally true (see, [29]).

In [29], Dragomir proved the following inequality which is Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1.2. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\; & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\; & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq\; & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx+\frac{1}{b-a}\int_a^b f(x,d)dx \right.\\
& \left. +\frac{1}{d-c}\int_c^d f(a,y)dy+\frac{1}{d-c}\int_c^d f(b,y)dy\right] \\
\leq\; & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

For recent developments about Hermite-Hadamard's inequality for some convex functions on the co-ordinates, please refer to ([2],[5],[6],[11]-[13] and [15]-[23]). Also for several inequalities for convex functions on the co-ordinates see the references ([9],[10],[14],[24] and [25]).

In [28], Raina defined the following results connected with the general class of fractional integral operators:

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x)=\mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (\rho,\lambda>0;|x|<\mathcal{R}), \tag{3}$$

where the coefficients $\sigma(k)$ ($k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$) is a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. With the help of (3), Raina and Agarwal et al. defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(x)=\int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right]\varphi(t)dt, \;\; x>a, \tag{4}$$

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-x)^{\rho}\right]\varphi(t)dt, \quad x<b, \tag{5}$$

where $\lambda,\rho>0,\omega\in\mathbb{R}$, and $\varphi(t)$ is such that the integrals on the right side exists.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(x)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right] < \infty. \tag{6}$$

In fact, for $\varphi\in L(a,b)$, we have

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(x)\right\|_1 \le \mathfrak{M}(b-a)^{\lambda}\left\|\varphi\right\|_1 \tag{7}$$

and

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(x)\right\|_1 \le \mathfrak{M}(b-a)^{\lambda}\left\|\varphi\right\|_1, \tag{8}$$

where

$$\left\|\varphi\right\|_p := \left(\int_a^b \left|\varphi(t)\right|^p dt\right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order α defined by (see, [1, 27, 31])

$$\left(I^{\alpha}_{a^+}\varphi\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt \quad (x>a;\alpha>0) \tag{9}$$

and

$$\left(I^{\alpha}_{b^-}\varphi\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt \quad (x<b;\alpha>0) \tag{10}$$

follow easily by setting

$$\lambda=\alpha,\ \sigma(0)=1,\ \text{and } w=0 \tag{11}$$

in (4) and (5), and the boundedness of (9) and (10) on $L(a,b)$ is also inherited from (7) and (8), (see, [26]).

In [7], Yaldiz and Sarikaya proved the following inequality which is Hermite-Hadamard inequality for fractional integral operators:

Theorem 1.3. *Let $\varphi:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $a<b$, then the following inequalities for fractional integral operators hold:*

$$\begin{aligned} &\varphi\left(\frac{a+b}{2}\right) \\ \le\ & \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(b)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(a)\right] \\ \le\ & \frac{\varphi(a)+\varphi(b)}{2} \end{aligned} \tag{12}$$

with $\lambda>0$.

Now, we establish new definitions related to fractional integral operators for two variables functions:

Definition 1.4. *Let $f \in L_1([a,b]\times[c,d])$. The fractional integral operators for two variables functions with $p = (p_1, p_2)$, $\lambda = (\lambda_1, \lambda_2)$, $p, \lambda \in [0,\infty)^2$; $w = (w_1, w_2) \in \mathbb{R}^2$; $\sigma = (\sigma_1, \sigma_2)$; and $a, c \geq 0$ defined by*

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(x,y) := \int_a^x \int_c^y (x-t)^{\lambda_1-1}(y-s)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1 (x-t)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2 (y-s)^{\rho_2}\right] f(t,s)dsdt, (x>a,\ y>c);$$

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(x,y) := \int_a^x \int_y^d (x-t)^{\lambda_1-1}(s-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1 (x-t)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2 (s-y)^{\rho_2}\right] f(t,s)dsdt, (x>a,\ y<d);$$

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(x,y) := \int_x^b \int_c^y (t-x)^{\lambda_1-1}(y-s)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1 (t-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2 (y-s)^{\rho_2}\right] f(t,s)dsdt, (x<b,\ y>c)$$

and

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(x,y) := \int_x^b \int_y^d (t-x)^{\lambda_1-1}(s-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1 (t-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2 (s-y)^{\rho_2}\right] f(t,s)dsdt, (x<b,\ y<d).$$

Similar the above definition, we introduce the following integrals:

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f\left(x,\frac{c+d}{2}\right) = \int_a^x (x-t)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1 (x-t)^{\rho_1}\right] f\left(t,\frac{c+d}{2}\right)dt, \quad x>a;$$

$$\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f\left(x,\frac{c+d}{2}\right) = \int_x^b (t-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1 (t-x)^{\rho_1}\right] f\left(t,\frac{c+d}{2}\right)dt, \quad x<b;$$

$$\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f\left(\frac{a+b}{2},y\right) = \int_c^y (y-s)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2 (y-s)^{\rho_2}\right] f\left(\frac{a+b}{2},t\right)dt, \quad y>c$$

and

$$\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f\left(\frac{a+b}{2},y\right) = \int_y^d (s-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2 (s-y)^{\rho_2}\right] f\left(\frac{a+b}{2},t\right)dt, \quad y<d.$$

In this paper, we are interested to give the Hermite–Hadamard inequality for a rectangle in plane via convex functions on co-ordinates involving fractional integral operators. We also study some properties of mappings associated with the Hermite–Hadamard inequality for convex functions on co-ordinates.

2. Hermite Hadamard Type Inequalities for Fractional Integral Operators

In this section, we will give Hermite-Hadamard type inequalities for fractional integral operators by using co-ordinated convex functions. During the this work we use the following symbols for $m = 0, 1$

$$\mathcal{A}_m(t) := \mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+m}\left[\omega_1 (b-a)^{\rho_1} t^{\rho_1}\right], \qquad \mathcal{B}_m(s) := \mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+m}\left[\omega_2 (d-c)^{\rho_2} s^{\rho_2}\right].$$

Theorem 2.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a co-ordinated convex on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b$, $0 \le c < d$ and $f \in L_1(\Delta)$. Then the following inequalities hold:*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \tag{13}$$

$$\le \frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}$$

$$\times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]$$

$$\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$

where $p=(p_1,p_2)$, $\lambda=(\lambda_1,\lambda_2)$, $p,\lambda \in [0,\infty)^2$; $w=(w_1,w_2)\in\mathbb{R}^2$; $\sigma=(\sigma_1,\sigma_2)$.

Proof. According to (2) with $x=ta+(1-t)b$, $y=(1-t)a+tb$, $u=sc+(1-s)d$, $v=(1-s)c+sd$ and $t_1=s_1=\frac{1}{2}$, we find that

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{4}[f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd) \tag{14}$$

$$+f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)].$$

Multiplying both sides of (14) by $t^{\lambda_1-1}s^{\lambda_2-1}\mathcal{A}_0(t)\mathcal{B}_0(s)$, then integrating with respect to (t,s) on $[0,1]\times[0,1]$, we obtain

$$\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\le \frac{1}{4}\left\{\int_0^1\int_0^1 t^{\lambda_1-1}s^{\lambda_2-1}\mathcal{A}_0(t)\mathcal{B}_0(s)[f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)]dsdt\right.$$

$$\left.+\int_0^1\int_0^1 t^{\lambda_1-1}s^{\lambda_2-1}\mathcal{A}_0(t)\mathcal{B}_0(s)\left[f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)\right]dsdt\right\}.$$

Using the change of variable in the last integrals, we have

$$4\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$

$$\le \frac{1}{(b-a)^{\lambda_1}(d-c)^{\lambda_2}}$$

$$\times\left\{\int_a^b\int_c^d (b-x)^{\lambda_1-1}(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}[\omega_2(d-y)^{\rho_2}]f(x,y)dydx\right.$$

$$+\int_a^b\int_c^d (b-x)^{\lambda_1-1}(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}[\omega_2(y-c)^{\rho_2}]f(x,y)dydx$$

$$+\int_a^b\int_c^d (x-a)^{\lambda_1-1}(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}[\omega_2(d-c)^{\rho_2}]f(x,y)dydx$$

$$\left.+\int_a^b\int_c^d (x-a)^{\lambda_1-1}(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}[\omega_2(d-c)^{\rho_2}]f(x,y)dydx\right\}$$

which gives the left hand side of inequality in (13). Now we prove the right hand side of inequality in (13). For this purpose we first note that if f is a co-ordinated convex on Δ, then we can write by using (2) with $x=a$, $y=b$, $u=c$ and $v=d$

$$f(ta+(1-t)b,sc+(1-s)d) \leq tsf(a,c)+s(1-t)f(b,c)+t(1-s)f(a,d)+(1-t)(1-s)f(b,d),$$

$$f(ta+(1-t)b,(1-s)c+sd) \leq t(1-s)f(a,c)+(1-t)(1-s)f(b,c)+tsf(a,d)+(1-t)sf(b,d),$$

$$f((1-t)a+tb,sc+(1-s)d) \leq (1-t)sf(a,c)+stf(b,c)+(1-t)(1-s)f(a,d)+t(1-s)f(b,d)$$

and

$$f((1-t)a+tb,(1-s)c+sd) \leq (1-t)(1-s)f(a,c)+t(1-s)f(b,c)+(1-t)sf(a,d)+tsf(b,d).$$

By adding these inequalities, we get

$$\begin{aligned}&f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)\\&+f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)\\\leq\ &f(a,c)+f(b,c)+f(a,d)+f(b,d).\end{aligned} \tag{15}$$

Multiplying both sides of (15) by $t^{\lambda_1}s^{\lambda_2}\mathcal{A}_0(t)\mathcal{B}_0(s)$, then integrating with respect to (t,s) on $[0,1]\times[0,1]$ we obtain

$$\begin{aligned}&\int_0^1\int_0^1 t^{\lambda_1}s^{\lambda_2}\mathcal{A}_0(t)\mathcal{B}_0(s)\left[f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)\right.\\&\left.+f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)\right]dsdt\\\leq\ &\int_0^1\int_0^1 t^{\lambda_1}s^{\lambda_2}\mathcal{A}_0(t)\mathcal{B}_0(s)[f(a,c)+f(b,c)+f(a,d)+f(b,d)]dsdt.\end{aligned}$$

Then by using the change of variable we have

$$\begin{aligned}&\frac{1}{(b-a)^{\lambda_1}(d-c)^{\lambda_2}}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]\\\leq\ &[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\,\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right].\end{aligned}$$

In this way the proof is completed. □

Theorem 2.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a co-ordinated convex on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b$, $0 \le c < d$ and $f \in L_1(\Delta)$. Then the following inequalities hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \le\ & \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1} f\left(b, \frac{c+d}{2}\right) + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1} f\left(a, \frac{c+d}{2}\right)\right] \\ & + \frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2} f\left(\frac{a+b}{2}, d\right) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2} f\left(\frac{a+b}{2}, c\right)\right] \\ \le\ & \frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]} \\ & \times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega} f(b,d) + \mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega} f(b,c) + \mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega} f(a,d) + \mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega} f(a,c)\right] \\ \le\ & \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}\varphi(a,c) + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}\varphi(a,d) + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1} f(b,d) + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1} f(b,c)\right] \\ & + \frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2} f(a,c) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2} f(a,d) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2} f(b,d) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2} f(b,c)\right] \\ \le\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} \end{aligned} \tag{16}$$

where $p = (p_1, p_2)$, $\lambda = (\lambda_1, \lambda_2)$, $p, \lambda \in [0,\infty)^2$; $w = (w_1, w_2) \in \mathbb{R}^2$; $\sigma = (\sigma_1, \sigma_2)$.

Proof. Since $f : \Delta \to \mathbb{R}$ is co-ordinated convex on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b$, $0 \le c < d$, it follows that the mapping $g_x : [c,d] \to \mathbb{R}$, $g_x(y) = f(x,y)$ is convex on $[c,d]$ for all $x \in [a,b]$. Then by using inequalities (12), we can write for $x \in [a,b]$

$$\begin{aligned} & g_x\left(\frac{c+d}{2}\right) \le \frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2} g_x(d) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2} g_x(c)\right] \\ \le\ & \frac{g_x(c) + g_x(d)}{2}. \end{aligned}$$

That is for $x \in [a,b]$,

$$\begin{aligned} & f\left(x, \frac{c+d}{2}\right) \\ \le\ & \frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f(x,y)dy \right. \\ & \left. + \int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right] f(x,y)dy\right] \\ \le\ & \frac{f(x,c) + f(x,d)}{2}. \end{aligned} \tag{17}$$

Then multiplying both sides of (17) by $\frac{(b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}$ and $\frac{(x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}$ respectively and then integrating with respect to x over $[a,b]$, we get

$$\begin{aligned}
&\frac{1}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\int_a^b (b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right] f\left(x,\frac{c+d}{2}\right)dx \qquad (18)\\
\leq\ & \frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\\
&\left[\int_a^b\int_c^d (b-x)^{\lambda_1-1}(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right]f(x,y)dydx\right.\\
&\left.+\int_a^b\int_c^d (b-x)^{\lambda_1-1}(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right]f(x,y)dydx\right]\\
\leq\ & \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\int_a^b (b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]f(x,c)dx\right.\\
&\left.+\int_a^b (b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]f(x,c)dx\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\int_a^b (x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right] f\left(x,\frac{c+d}{2}\right)dx \qquad (19)\\
\leq\ & \frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\\
&\left[\int_a^b\int_c^d (x-a)^{\lambda_1-1}(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right]f(x,y)dydx\right.\\
&\left.+\int_a^b\int_c^d (x-a)^{\lambda_1-1}(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right]f(x,y)dydx\right]\\
\leq\ & \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\int_a^b (x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]f(x,c)dx\right.\\
&\left.+\int_a^b (x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]f(x,d)dx\right].
\end{aligned}$$

In a similar way appling for the mapping $g_y : [a,b] \to \mathbb{R}$, $g_y(x) = f(x,y)$, we have

$$\frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f\left(\frac{a+b}{2},y\right)dy \tag{20}$$
$$\leq \frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}$$
$$\times\left[\int_a^b\int_c^d (b-x)^{\lambda_1-1}(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f(x,y)dydx\right.$$
$$\left.+\int_a^b\int_c^d (x-a)^{\lambda_1-1}(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f(x,y)dydx\right]$$
$$\leq \frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f(a,y)dy\right.$$
$$\left.+\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f(b,y)dy\right]$$

and

$$\frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right] f\left(\frac{a+b}{2},y\right)dy \tag{21}$$
$$\leq \frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}$$
$$\left[\int_a^b\int_c^d (b-x)^{\lambda_1-1}(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right] f(x,y)dydx\right.$$
$$\left.+\int_a^b\int_c^d (x-a)^{\lambda_1-1}(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right] f(x,y)dydx\right]$$
$$\leq \frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right] f(a,y)dy\right.$$
$$\left.+\int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right] f(b,y)dy\right].$$

Adding the inequalities (18)-(21), we obtain

$$
\begin{aligned}
&\frac{1}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1\,(b-a)^{\rho_1}\right]}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f\left(b,\frac{c+d}{2}\right)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f\left(a,\frac{c+d}{2}\right)\right] \qquad (22)\\
&+\frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2\,(d-c)^{\rho_2}\right]}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f\left(\frac{a+b}{2},d\right)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f\left(\frac{a+b}{2},c\right)\right]\\
\leq\;&\frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1\,(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2\,(d-c)^{\rho_2}\right]}\\
&\times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d+;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]\\
\leq\;&\frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1\,(b-a)^{\rho_1}\right]}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f(b,c)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f(b,d)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f(a,c)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f(a,d)\right]\\
&+\frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_{2+1}}\left[\omega_2\,(d-c)^{\rho_2}\right]}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f(a,d)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f(b,d)+\mathcal{J}^{\sigma_2}_{\rho_1,\lambda_2,d-;\omega_2}f(a,c)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f(b,c)\right]
\end{aligned}
$$

where $p=(p_1,p_2)$, $\lambda=(\lambda_1,\lambda_2)$, $p,\lambda\in[0,\infty)^2$; $w=(w_1,w_2)\in\mathbb{R}^2$; $\sigma=(\sigma_1,\sigma_2)$. Thus, we proved the second and the third inequalities in (16).

Now, using the left side of inequality in (12), we also have

$$
\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\;&\frac{1}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1\,(b-a)^{\rho_1}\right]}\left[\int_a^b(b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1\,(b-x)^{\rho_1}\right]f\left(x,\frac{c+d}{2}\right)dx\right.\\
&\left.+\int_a^b(x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1\,(x-a)^{\rho_1}\right]f\left(x,\frac{c+d}{2}\right)dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\leq\;&\frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2\,(d-c)^{\rho_2}\right]}\left[\int_c^d(d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2\,(d-y)^{\rho_2}\right]f\left(\frac{a+b}{2},y\right)dy\right.\\
&\left.+\int_c^d(y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2\,(y-c)^{\rho_2}\right]f\left(\frac{a+b}{2},y\right)dy\right].
\end{aligned}
$$

By adding these inequalities, we get

$$\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq \;& \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f\left(b,\frac{c+d}{2}\right)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f\left(a,\frac{c+d}{2}\right)\right] \\
& +\frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f\left(\frac{a+b}{2},d\right)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f\left(\frac{a+b}{2},c\right)\right]
\end{aligned}$$

which gives the first inequality in (16).

Finally, using the right hand side of inequality in (12), we can state

$$\begin{aligned}
& \frac{1}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\int_a^b (b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]f(x,c)dx\right. \\
& \left.+\int_a^b (x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]f(x,c)dx\right] \\
\leq \;& \frac{f(a,c)+f(b,c)}{2},
\end{aligned} \tag{23}$$

$$\begin{aligned}
& \frac{1}{2(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\int_a^b (b-x)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(b-x)^{\rho_1}\right]f(x,d)dx\right. \\
& \left.+\int_a^b (x-a)^{\lambda_1-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1}\left[\omega_1(x-a)^{\rho_1}\right]f(x,d)dx\right] \\
\leq \;& \frac{f(a,d)+f(b,d)}{2},
\end{aligned} \tag{24}$$

$$\begin{aligned}
& \frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right]f(a,y)dy\right. \\
& \left.+\int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right]f(a,y)dy\right] \\
\leq \;& \frac{f(a,c)+f(a,d)}{2}
\end{aligned} \tag{25}$$

and

$$\frac{1}{2(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\left[\int_c^d (d-y)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(d-y)^{\rho_2}\right]f(b,y)dy\right. \tag{26}$$

$$\left.+\int_c^d (y-c)^{\lambda_2-1}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2}\left[\omega_2(y-c)^{\rho_2}\right]f(a,y)dy\right]$$

$$\leq \frac{f(b,c)+f(b,d)}{2}$$

which give by addition (23)-(26), the last inequality in (16). □

3. Fractional Integral Operators for Co-ordinated Convex Functions

Firstly, we give the following lemma for our results.

Lemma 3.1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\leq a<b$, $0\leq c<d$. If $\frac{\partial^2 f}{\partial t\partial s}\in L_1(\Delta)$, then the following inequalities hold:*

$$\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}$$

$$+\frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}$$

$$\times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]-A$$

$$=\frac{(b-a)(d-c)}{4\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(b-a)^{\rho_2}\right]}$$

$$\times\left\{\int_0^1\int_0^1 t^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt\right.$$

$$-\int_0^1\int_0^1(1-t)^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt$$

$$-\int_0^1\int_0^1 t^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(1-s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt$$

$$\left.+\int_0^1\int_0^1(1-t)^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(1-s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt\right\},$$

where $p = (p_1, p_2)$, $\lambda = (\lambda_1, \lambda_2)$, $p, \lambda \in [0, \infty)^2$; $w = (w_1, w_2) \in \mathbb{R}^2$; $\sigma = (\sigma_1, \sigma_2)$ *and*

$$\begin{aligned} A &= \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]} \Big[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1} f(b,c) + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1} f(b,d) \\ &\quad + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1} f(a,c) + \mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1} f(a,d)\Big] + \frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (d-c)^{\rho_2}\right]} \\ &\quad \times \Big[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2} f(b,d) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2} f(a,d) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2} f(a,c) + \mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2} f(b,c)\Big]. \end{aligned}$$

Proof. Integrating by parts, we obtain

$$\begin{aligned} I_1 &= \int_0^1 \int_0^1 t^{\lambda_1} s^{\lambda_2} \mathcal{A}_1(t)\, \mathcal{B}_1(s) \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) ds dt \qquad (27) \\ &= \int_0^1 s^{\lambda_2}\, \mathcal{B}_1(s) \left\{ t^{\lambda_1}\mathcal{A}_1(t) \frac{1}{a-b}\frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)\Big|_0^1 \right. \\ &\quad \left. -\frac{\lambda_1}{a-b}\int_0^1 t^{\lambda_1-1}\mathcal{A}_0(t)\frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)dt \right\} ds \\ &= \int_0^1 s^{\lambda_2}\, \mathcal{B}_1(s) \left\{ -\frac{1}{b-a}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right] \frac{\partial f}{\partial s}(a, sc+(1-s)d) \right. \\ &\quad \left. +\frac{\lambda_1}{b-a}\int_0^1 t^{\lambda_1-1}\mathcal{A}_0(t)\frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)dt \right\} ds \\ &= -\frac{1}{b-a}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right] \int_0^1 s^{\lambda_2}\, \mathcal{B}_1(s)\frac{\partial f}{\partial s}(a, sc+(1-s)d)ds \\ &\quad +\frac{\lambda_1}{b-a}\int_0^1\int_0^1 t^{\lambda_1-1}s^{\lambda_2}\, \mathcal{B}_1(s)\frac{\partial f}{\partial s}(ta+(1-t)b, sc+(1-s)d)dsdt. \\ &= \frac{1}{(b-a)(d-c)}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (d-c)^{\rho_2}\right] f(a,c) \\ &\quad -\frac{1}{(b-a)(d-c)}\int_0^1 t^{\lambda_1-1}\mathcal{A}_0(t)\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (d-c)^{\rho_2}\right] f(ta+(1-t)b, c)dt \\ &\quad -\frac{1}{(b-a)(d-c)}\int_0^1 s^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]\mathcal{B}_0(s) f(a, sc+(1-s)d)dt \\ &\quad +\frac{1}{(b-a)(d-c)}\int_0^1\int_0^1 t^{\lambda_1-1}s^{\lambda_2-1}\mathcal{A}_0(t)\mathcal{B}_0(s) f(ta+(1-t)b, sc+(1-s)d)dsdt. \end{aligned}$$

In this way by integration by parts, we get

$$
\begin{aligned}
I_2 &= \int_0^1\int_0^1 (1-t)^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt \qquad (28)\\
&= -\frac{1}{(b-a)(d-c)}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f(b,c)\\
&\quad+\frac{1}{(b-a)(d-c)}\int_0^1 (1-t)^{\lambda_1-1}\mathcal{A}_0(1-t)\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f(ta+(1-t)b,c)dt\\
&\quad+\frac{1}{(b-a)(d-c)}\int_0^1 s^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{B}_0(s)f(b,sc+(1-s)d)ds\\
&\quad-\frac{1}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)^{\lambda_1-1}s^{\lambda_2-1}\mathcal{A}_0(1-t)\mathcal{B}_0(s)f(ta+(1-t)b,sc+(1-s)d)dsdt,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_0^1\int_0^1 t^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(1-s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt \qquad (29)\\
&= -\frac{1}{(b-a)(d-c)}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f(a,d)\\
&\quad+\frac{1}{(b-a)(d-c)}\int_0^1 t^{\lambda_1-1}\mathcal{A}_0(t)\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f(ta+(1-t)b,d)dt\\
&\quad+\frac{1}{(b-a)(d-c)}\int_0^1 (1-s)^{\lambda_2-1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{B}_0(1-s)f(a,sc+(1-s)d)ds\\
&\quad-\frac{1}{(b-a)(d-c)}\int_0^1\int_0^1 t^{\lambda_1-1}(1-s)^{\lambda_2-1}\mathcal{A}_0(t)\mathcal{B}_0(1-s)f(ta+(1-t)b,sc+(1-s)d)dsdt,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1\int_0^1 (1-t)^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(1-s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt \qquad (30)\\
&= \frac{1}{(b-a)(d-c)}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f(b,d)\\
&\quad-\frac{1}{(b-a)(d-c)}\int_0^1 (1-t)^{\lambda_1}\mathcal{A}_0(1-t)\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]f(ta+(1-t)b,d)dt
\end{aligned}
$$

$$-\frac{1}{(b-a)(d-c)}\int_0^1 (1-s)\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]\mathcal{B}_0(1-s)f(b,sc+(1-s)d)ds$$

$$+\frac{1}{(b-a)(d-c)}\int_0^1\int_0^1 (1-t)^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_0(1-t)\mathcal{B}_0(1-s)f(ta+(1-t)b,sc+(1-s)d)dsdt.$$

Using the change of variables for $t,s\in[0,1]$, $x=ta+(1-t)b$, $y=sc+(1-s)d$ from (27)-(30), we get

$$\begin{aligned}
&I_1-I_2-I_3+I_4 \qquad (31)\\
=\;& \frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (d-c)^{\rho_2}\right]}{(b-a)(d-c)}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\
&-\frac{\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]}{(b-a)(d-c)^{\lambda_2+1}}\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f(a,d)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f(b,d)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f(a,c)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f(b,c)\right]\\
&-\frac{\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (b-a)^{\rho_2}\right]}{(b-a)^{\lambda_1+1}(d-c)}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f(b,c)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f(b,d)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f(a,c)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f(a,d)\right]\\
&+\frac{1}{(b-a)^{\lambda_1+1}(d-c)^{\lambda_2+1}}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right].
\end{aligned}$$

Then, multiplying both sides of (31) by $\frac{(b-a)(d-c)}{4\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(b-a)^{\rho_2}\right]}$, thus we obtain desired result. □

Theorem 3.2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\le a<b$, $0\le c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is co-ordinated convex function on Δ, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\
&+\frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (d-c)^{\rho_2}\right]}\\
&\left.\times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]-A\right|\\
\le\;& \frac{(b-a)(d-c)}{4\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1 (b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2 (b-a)^{\rho_2}\right]}\\
&\times\left\{\left(\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+3}\left[\omega_1 (b-a)^{\rho_1}\right]+\mathcal{F}^{\sigma_3}_{\rho_1,\lambda_1+3}\left[\omega_1 (b-a)^{\rho_1}\right]\right)\times\left(\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+3}\left[\omega_2 (d-c)^{\rho_2}\right]+\mathcal{F}^{\sigma_4}_{\rho_2,\lambda_2+3}\left[\omega_2 (d-c)^{\rho_2}\right]\right)\right.\\
&\left.\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right)\right\}
\end{aligned}$$

where $p=(p_1,p_2)$, $\lambda=(\lambda_1,\lambda_2)$, $p,\lambda\in[0,\infty)^2$; $w=(w_1,w_2)\in\mathbb{R}^2$; $\sigma=(\sigma_1,\sigma_2)$,

$$
\begin{aligned}
A &= \frac{1}{4(b-a)^{\lambda_1}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]}\left[\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f(b,c)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,a+;\omega_1}f(b,d)\right.\\
&\left.+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f(a,c)+\mathcal{J}^{\sigma_1}_{\rho_1,\lambda_1,b-;\omega_1}f(a,d)\right]+\frac{1}{4(d-c)^{\lambda_2}\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\\
&\times\left[\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f(b,d)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,c+;\omega_2}f(a,d)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f(a,c)+\mathcal{J}^{\sigma_2}_{\rho_2,\lambda_2,d-;\omega_2}f(b,c)\right]
\end{aligned}
$$

and

$$\sigma_3(k) := \frac{\sigma_1(k)\Gamma(\rho_1 k+\lambda_1+2)}{\Gamma(\rho_1 k+\lambda_1+1)}, \qquad \sigma_4(k) := \frac{\sigma_2(k)\Gamma(\rho_2 k+\lambda_2+2)}{\Gamma(\rho_2 k+\lambda_2+1)}.$$

Proof. From Lemma 3.1, we have

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\
&+\frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\\
&\left.\times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]-A\right|\\
\leq\ &\frac{(b-a)(d-c)}{4\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(b-a)^{\rho_2}\right]}\\
&\times\left\{\int_0^1\int_0^1 t^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(s)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dsdt\right.\\
&+\int_0^1\int_0^1(1-t)^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(s)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dsdt\\
&+\int_0^1\int_0^1 t^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dsdt\\
&\left.+\int_0^1\int_0^1(1-t)^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dsdt\right\}.
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is co-ordinated convex function on Δ, we can write

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\
&+\frac{1}{4(b-a)^{\lambda_1}(d-c)^{\lambda_2}\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(d-c)^{\rho_2}\right]}\\
&\left.\times\left[\mathcal{J}^{\sigma}_{\rho,\lambda,a+,c+;\omega}f(b,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,a+,d-;\omega}f(b,c)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,c+;\omega}f(a,d)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-,d-;\omega}f(a,c)\right]-A\right|
\end{aligned}
$$

$$\leq \frac{(b-a)(d-c)}{4\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[\omega_1(b-a)^{\rho_1}\right]\mathcal{F}^{\sigma_2}_{\rho_2,\lambda_2+1}\left[\omega_2(b-a)^{\rho_2}\right]}$$

$$\times\left\{\int_0^1\int_0^1\left[t^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(s)+(1-t)^{\lambda_1}s^{\lambda_2}\mathcal{A}_1(1-t)\mathcal{B}_1(s)+t^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(t)\,\mathcal{B}_1(1-s)\right.\right.$$

$$\left.+(1-t)^{\lambda_1}(1-s)^{\lambda_2}\mathcal{A}_1(1-t)\,\mathcal{B}_1(1-s)\right]x\left[ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\right.$$

$$\left.\left.+t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|dsdt\right\}.$$

In the above inequality calculating the integrals, we obtain desired result. □

Remark 3.3. *If we take $\lambda_1=\alpha$, $\lambda_2=\beta$, $\sigma_1(0)=1=\sigma_2(0)$, $w_1=0=w_2$ in Lemma 3.1, Theorem 2.1, Theorem 2.2, we have the inequalities which is proved by Sarikaya et al.in [21].*

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