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Stancu Type Generalization of Szász-Durrmeyer Operators Involving Brenke-Type Polynomials

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Abstract. In the present paper, we introduce a Stancu type generalization of Szász- Durrmeyer operators including Brenke type polynomials. We give convergence properties of these operators via Korovkin's theorem and the order of convergence by using a classical approach. As an example, we consider a Stancu type generalization of the Durrmeyer type integral operators including Hermite polynomials of variance v. Then, we obtain the rates of convergence by using the second modulus of continuity. Also, for these operators including Hermite polynomials of variance v, we present a Voronovskaja type theorem and r-th order generalization of these positive linear operators.

1. Introduction

Durrmeyer [8] introduced a variant of the Bernstein polynomials for integrable functions on $[0,1]$. Many authors considered Durrmeyer type operators and further generalizations presenting better approximation result. Some are in [1], [4], [5], [10], [11].

For $f \in C[0,\infty)$, a Durrmeyer variant of Szász-Mirakyan operators was proposed by Mazhar et al. [15] by following

$$(S_n^* f)(x) = n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\, dt. \tag{1}$$

In [13], Jakimovski et al. introduced linear positive operators including Appell polynomials. In [7], Ciupa defined the following Durrmeyer type integral modification of the operators introduced by Jakimovski et al.

$$(P_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{2}$$

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under the assumption $\frac{a_k}{g(1)} \geq 0,\ k = 0, 1, 2, ...$ where $\lambda \geq 0,\ g(1) \neq 0$. Appell polynomials $p_k(x)$ have the generating function representation in the form of

$$g(t)e^{xt} = \sum_{k=0}^{\infty} p_k(x)t^k,$$

where $g(t)$ is an analytic function in the disc $|t| < R\ \ (R > 1)$

$$g(t) = \sum_{r=0}^{\infty} a_r t^r\ \ ,\ \ a_0 \neq 0.$$

By the motivation of these works, many authors introduced linear positive operators via generating functions and their further extentions for example, we refer the readers to [12], [21], [22], [23], [24]. In [25], Varma et al. introduced Durrmeyer variant of generalized Szász operators including Brenke type polynomials studied in [22]

$$L_n^*(f;x) = \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\,dt \tag{3}$$

under the restrictions

$$\begin{array}{ll} (i) & A(1) \neq 0,\ \frac{a_{k-n}b_n}{A(1)} \geq 0,\ \ 0 \leq n \leq k,\ \ k = 0, 1, 2, ..., \\ (ii) & B : [0, \infty) \to (0, \infty), \\ (iii) & (5) \text{ and the power series } (6) \text{ and } (7) \text{ are converge for } |t| < R,\ \ (R > 1) \end{array} \tag{4}$$

where Brenke type polynomials $p_k(x)$ are generated by [6]

$$A(t)B(xt) = \sum_{k=0}^{\infty} p_k(x)t^k, \tag{5}$$

where the functions $A(t)$ and $B(t)$ are analytic in the disk $|t| < R,\ \ (R > 1)$

$$A(t) = \sum_{n=0}^{\infty} a_n t^n\ \ ,\ \ a_0 \neq 0, \tag{6}$$

$$B(t) = \sum_{n=0}^{\infty} b_n t^n\ \ ,\ \ b_n \neq 0\ \ \ (n \geq 0) \tag{7}$$

from which, we get

$$p_k(x) = \sum_{n=0}^{k} a_{k-n} b_n x^n,\ \ \ k = 0, 1, 2, ...$$

Recently, Mishra et al. [16] introduced Stancu type generalization of the Baskakov-Szász operators and studied approximation properties of these operators. Inspired by such operators, we consider Stancu type generalization of the Durrmeyer type integral operators (3) including Brenke type polynomials as follows

$$L_n^{(\alpha,\beta,\lambda)}(f;x) = \frac{1}{A(1)B(nx)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f\left(\frac{nt+\alpha}{n+\beta}\right) dt \tag{8}$$

under the restrictions (4), where $x \geq 0,\ \lambda \geq 0$, $n \in \mathbb{N}$ and α, β parameters satisfy the condition $0 \leq \alpha \leq \beta$. For $\alpha = \beta = 0$, the operator (8) turns to the operator (3).

We organize the paper as follows.

In section 2, we first give some lemmas to obtain convergence properties of the operators (8) by means of Korovkin's theorem and then we prove the main theorem. We compute the rate of convergence of the operators $L_n^{(\alpha,\beta,\lambda)}(f)$ to f by means of the modulus of continuity. In section 3, as a special case of the operators (8), we define a Stancu type generalization of Szász-Durrmeyer operators including Hermite polynomials of variance v which are Brenke type polynomials. Then, we obtain the rate of convergence of these operators with the help of the second modulus of continuity. Furthermore, we give a Voronovskaja type theorem and *r-th* order generalization of these operators including Hermite polynomials of variance v.

2. Approximation properties of the operators $L_n^{(\alpha,\beta,\lambda)}$

First of all, we give some lemmas in order to prove the main theorem and then, we compute the rate of convergence of the operators $L_n^{(\alpha,\beta,\lambda)}(f)$ to f by means of a classical approach.

Lemma 2.1. *For the operators $L_n^{(\alpha,\beta,\lambda)}$, we have*

$$L_n^{(\alpha,\beta,\lambda)}(1;x) = 1, \tag{9}$$

$$L_n^{(\alpha,\beta,\lambda)}(t;x) = \frac{\lambda+\alpha+1}{n+\beta} + \frac{A'(1)}{A(1)(n+\beta)} + \frac{nx}{n+\beta}\frac{B'(nx)}{B(nx)}, \tag{10}$$

$$\begin{aligned} L_n^{(\alpha,\beta,\lambda)}\left(t^2;x\right) &= \frac{n^2x^2}{(n+\beta)^2}\frac{B''(nx)}{B(nx)} + \frac{2nx}{(n+\beta)^2}\frac{B'(nx)}{B(nx)}\left\{\frac{A'(1)}{A(1)} + \lambda+\alpha+2\right\} \\ &\quad + \frac{1}{(n+\beta)^2}\left\{\frac{A''(1)}{A(1)} + 2(\lambda+\alpha+2)\frac{A'(1)}{A(1)} \right. \\ &\quad \left. +\alpha^2 + 2(\lambda+1)\alpha + (\lambda+1)(\lambda+2)\right\}. \end{aligned} \tag{11}$$

Proof. Using the generating function of Brenke type polynomials given by (5), we obtain

$$\sum_{k=0}^{\infty} p_k(nx) = A(1)B(nx), \tag{12}$$

$$\sum_{k=0}^{\infty} kp_k(nx) = A'(1)B(nx) + nxA(1)B'(nx), \tag{13}$$

$$\begin{aligned} \sum_{k=0}^{\infty} k^2 p_k(nx) &= n^2x^2A(1)B''(nx) + nxB'(nx)\{2A'(1)+A(1)\} \\ &\quad + B(nx)\{A''(1)+A'(1)\}. \end{aligned} \tag{14}$$

For $f(t) = 1$, the operator (8) is as follows

$$L_n^{(\alpha,\beta,\lambda)}(1;x) = \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}dt$$

from the definition of the gamma function, it becomes

$$\begin{aligned} L_n^{(\alpha,\beta,\lambda)}(1;x) &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}dt \\ &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx) \\ &= 1. \end{aligned}$$

For $f(t)=t$, the operator (8) reduces to

$$\begin{aligned} L_n^{(\alpha,\beta,\lambda)}(t;x) &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &= \frac{1}{(n+\beta)A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+2}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k+1}dt \\ &+ \frac{\alpha}{(n+\beta)A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}dt. \end{aligned}$$

By using the gamma function and the equalities (12) and (13), we have the equality (10). Similarly, for $f(t)=t^2$, by means of the equalities (12), (13) and (14), one can easily obtain the equality (11). □

Lemma 2.2. *For each $x\in[0,\infty)$, it follows from the results in Lemma 2.1 ,*

$$\begin{aligned} &\Omega_n^{(\alpha,\beta,\lambda)}(x) := L_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right) \\ &= \left\{\frac{n^2}{(n+\beta)^2}\frac{B''(nx)}{B(nx)} - \frac{2n}{n+\beta}\frac{B'(nx)}{B(nx)} + 1\right\}x^2 \\ &+ \left\{\frac{2n}{(n+\beta)^2}\frac{B'(nx)}{B(nx)}\left\{\frac{A'(1)}{A(1)}+\lambda+\alpha+2\right\} - \frac{2}{n+\beta}\left\{\frac{A'(1)}{A(1)}+\lambda+\alpha+1\right\}\right\}x \\ &+ \frac{1}{(n+\beta)^2}\left\{\frac{A''(1)}{A(1)} + 2(\lambda+\alpha+2)\frac{A'(1)}{A(1)} + \alpha^2 + 2(\lambda+1)\alpha + (\lambda+1)(\lambda+2)\right\}. \end{aligned}$$

Theorem 2.3. *Assume that*

$$E := \left\{f:\ x\in[0,\infty),\frac{f(x)}{1+x^2}\ \textit{is convergent as}\ x\to\infty\right\}$$

and

$$\lim_{y\to\infty}\frac{B'(y)}{B(y)} = 1 \quad \textit{and} \quad \lim_{y\to\infty}\frac{B''(y)}{B(y)} = 1. \tag{15}$$

Let $f\in C[0,\infty)\cap E$. Then

$$\lim_{n\to\infty}L_n^{(\alpha,\beta,\lambda)}(f;x) = f(x),$$

and the operators $L_n^{(\alpha,\beta,\lambda)}$ converge uniformly in each compact subset of $[0,\infty)$.

Proof. From the results obtained in Lemma 2.1 and the assumptions (15),

$$\lim_{n\to\infty} L_n^{(\alpha,\beta,\lambda)}\left(t^i;x\right) = x^i,\ i=0,1,2,$$

holds where this convergence holds uniformly in each compact subset of $[0,\infty)$. Then, applying the universal Korovkin type Theorem 4.1.4 (vi) given in [3] gives the desired result. □

Let $f \in \widetilde{C}[0,\infty)$ and $\delta > 0$. The modulus of continuity of f denoted by $\omega(f;\delta)$ is defined by

$$\omega(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} \left|f(x) - f(y)\right|,$$

where $\widetilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$. Then, for any $\delta > 0$ and each $x \in [0,\infty)$, we have the following inequality

$$\left|f(x) - f(y)\right| \le \omega(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{16}$$

Now, we can calculate the order of approximation by means of the modulus of continuity as follows.

Theorem 2.4. *For $f \in \widetilde{C}[0,\infty) \cap E$, we have*

$$\left|L_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right| \le 2\omega\left(f;\sqrt{\Omega_n^{(\alpha,\beta,\lambda)}(x)}\right),$$

where

$$\begin{aligned} \Omega_n^{(\alpha,\beta,\lambda)}(x) &:= L_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right) \qquad (17)\\ &= \left\{\frac{n^2}{(n+\beta)^2}\frac{B''(nx)}{B(nx)} - \frac{2n}{n+\beta}\frac{B'(nx)}{B(nx)} + 1\right\}x^2 \\ &+ \left\{\frac{2n}{(n+\beta)^2}\frac{B'(nx)}{B(nx)}\left\{\frac{A'(1)}{A(1)} + \lambda + \alpha + 2\right\} - \frac{2}{n+\beta}\left\{\frac{A'(1)}{A(1)} + \lambda + \alpha + 1\right\}\right\}x \\ &+ \frac{1}{(n+\beta)^2}\left\{\frac{A''(1)}{A(1)} + 2(\lambda+\alpha+2)\frac{A'(1)}{A(1)} + \alpha^2 + 2(\lambda+1)\alpha + (\lambda+1)(\lambda+2)\right\}. \end{aligned}$$

Proof. Using linearity of the operators $L_n^{(\alpha,\beta,\lambda)}$, (9) and (16), we get

$$\begin{aligned}
&\left|L_n^{(\alpha,\beta,\lambda)}(f;x)-f(x)\right| \\
&\leq \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left|f\left(\frac{nt+\alpha}{n+\beta}\right)-f(x)\right|dt \\
&\leq \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left(\frac{\left|\frac{nt+\alpha}{n+\beta}-x\right|}{\delta}+1\right)\omega(f;\delta)\,dt \\
&\leq \left\{1+\frac{1}{A(1)B(nx)\delta}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\right. \\
&\left.\times\int_0^{\infty}e^{-nt}t^{\lambda+k}\left|\frac{nt+\alpha}{n+\beta}-x\right|dt\right\}\omega(f;\delta). \qquad (18)
\end{aligned}$$

By applying the Cauchy-Schwarz inequality for integration, one may write

$$\int_0^{\infty}e^{-nt}t^{\lambda+k}\left|\frac{nt+\alpha}{n+\beta}-x\right|dt\leq\left(\frac{\Gamma(\lambda+k+1)}{n^{\lambda+k+1}}\right)^{1/2}\left(\int_0^{\infty}e^{-nt}t^{\lambda+k}\left(\frac{nt+\alpha}{n+\beta}-x\right)^2dt\right)^{1/2},$$

and, it concludes that

$$\begin{aligned}
&\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left|\frac{nt+\alpha}{n+\beta}-x\right|dt \\
&\leq\sum_{k=0}^{\infty}p_k(nx)\left(\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\right)^{1/2}\left(\int_0^{\infty}e^{-nt}t^{\lambda+k}\left(\frac{nt+\alpha}{n+\beta}-x\right)^2dt\right)^{1/2}. \qquad (19)
\end{aligned}$$

By Cauchy-Schwarz inequality for sum on the right hand side of (19), it follows

$$\begin{aligned}
&\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left|\frac{nt+\alpha}{n+\beta}-x\right|dt \\
&\leq\sqrt{A(1)B(nx)}\left(A(1)B(nx)L_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right)\right)^{1/2} \\
&=A(1)B(nx)\left(L_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right)\right)^{1/2} \\
&=A(1)B(nx)\left(\Omega_n^{(\alpha,\beta,\lambda)}(x)\right)^{1/2}, \qquad (20)
\end{aligned}$$

where $\Omega_n^{(\alpha,\beta,\lambda)}(x)$ is as in the equality (17). When we consider (20) in (18), we obtain

$$\left|L_n^{(\alpha,\beta,\lambda)}(f;x)-f(x)\right|\leq\left\{1+\frac{1}{\delta}\sqrt{\Omega_n^{(\alpha,\beta,\lambda)}(x)}\right\}\omega(f;\delta).$$

Choosing $\delta = \sqrt{\Omega_n^{(\alpha,\beta,\lambda)}(x)}$ gives us

$$\left|L_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right| \leq 2\omega\left(f; \sqrt{\Omega_n^{(\alpha,\beta,\lambda)}(x)}\right)$$

We note that $\Omega_n^{(\alpha,\beta,\lambda)}(x)$ goes to zero when $n \to \infty$ under the assumptions (15). □

3. Special Cases of the Operators $L_n^{(\alpha,\beta,\lambda)}$ and Further Properties

The Hermite polynomials $H_k^v(x)$ of variance v [20] are generated by

$$e^{\frac{-vt^2}{2}+xt} = \sum_{k=0}^{\infty} \frac{H_k^v(x)}{k!} t^k \tag{21}$$

and their explicit representation is given by

$$H_k^v(x) = \sum_{r=0}^{\left[\frac{k}{2}\right]} \left(-\frac{v}{2}\right)^r \frac{k!}{r!(k-2r)!} x^{k-2r}, \tag{22}$$

where [.] denotes the integer part. It is clear that Hermite polynomials $H_k^v(x)$ of variance v are Brenke-type polynomials for

$$A(t) = e^{\frac{-vt^2}{2}} \text{ and } B(t) = e^t.$$

So, the operators (8) return to

$$H_n^{(\alpha,\beta,\lambda)}(f;x) = e^{-nx+\frac{v}{2}} \sum_{k=0}^{\infty} \frac{H_k^{(v)}(nx)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f\left(\frac{nt+\alpha}{n+\beta}\right) dt \tag{23}$$

which gives a Stancu type generalization of the Durrmeyer type integral operators (3) including Hermite polynomials of variance v where $x \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}$. For $v \leq 0$, the restrictions (4) and assumptions (15) for the operators $H_n^{(\alpha,\beta,\lambda)}$ are verified.

Now, we give the moments and recurrence relation for central moments of the operators (23).

Lemma 3.1. *For the operators $H_n^{(\alpha,\beta,\lambda)}$, we have*

$$H_n^{(\alpha,\beta,\lambda)}(1;x) = 1, \tag{24}$$

$$H_n^{(\alpha,\beta,\lambda)}(t;x) = \frac{\lambda+\alpha+1-v+nx}{n+\beta},$$

$$\begin{aligned} H_n^{(\alpha,\beta,\lambda)}\left(t^2;x\right) = \frac{1}{(n+\beta)^2} \Big\{ & n^2x^2 + n(-2v+2\lambda+2\alpha+4)x \\ & +v^2 - v(2\lambda+2\alpha+5) + \alpha^2 + (\lambda+1)(2\alpha+\lambda+2) \Big\}, \end{aligned} \tag{25}$$

$$\begin{aligned} H_n^{(\alpha,\beta,\lambda)}\left(t^3;x\right) = \frac{1}{(n+\beta)^3} \Big\{ & (nx-v)^3 + (3\lambda+3\alpha+9)(nx-v)^2 \\ & + (nx-v)\left(3\alpha^2 + 3(2\lambda+4)\alpha + 3\lambda^2 + 15\lambda + 18 - 3v\right) \\ & +\alpha^3 + 3(\lambda+1)\alpha^2 + 3\left(\lambda^2+3\lambda+2-v\right)\alpha + (\lambda+3)\left(\lambda^2+3\lambda+2-3v\right) \Big\}, \end{aligned}$$

$$H_n^{(\alpha,\beta,\lambda)}\left(t^4;x\right)=\frac{1}{(n+\beta)^4}\Big\{(nx-v)^4+4(\lambda+\alpha+4)(nx-v)^3$$
$$+6\left(\lambda^2+7\lambda+\alpha^2+6\alpha+2\lambda\alpha+12-v\right)(nx-v)^2$$
$$+\Big[(\lambda+1)\left(6\alpha^2+4\alpha+3\lambda+9\right)+(\lambda+2)\left(6\alpha^2+12\alpha+4\lambda+15\right)$$
$$+(\lambda+1)(\lambda+2)(4\lambda+12\alpha+15)+4\alpha^3+6\alpha^2$$
$$-12v(\alpha+\lambda+4)+4\lambda+4\alpha+11+2(\lambda+2)(\lambda+4\alpha+4)\Big](nx-v)$$
$$+3v^2-6v\left(\lambda^2+7\lambda+\alpha^2+6\alpha+2\lambda\alpha+12\right)$$
$$+\alpha^4+(\lambda+1)\left(4\alpha^3+6\alpha^2(\lambda+2)\right)\Big\}.$$

Proof. From the generating function (21) of the Hermite polynomials of variance v, we can easily obtain the above equalities. □

Lemma 3.2. *If we define the central moment of degree m,*

$$\mu_{n,m}^{(\lambda)}(x)=H_n^{(\alpha,\beta,\lambda)}\left((t-x)^m;x\right)=e^{-nx+\frac{v}{2}}\sum_{k=0}^{\infty}\frac{H_k^{(v)}(nx)}{k!}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt \tag{26}$$

then, we have

$$\begin{aligned}\mu_{n,0}^{(\lambda)}(x)&=1,\\ \mu_{n,1}^{(\lambda)}(x)&=\frac{\lambda+\alpha+1-v-\beta x}{n+\beta},\\ \mu_{n,2}^{(\lambda)}(x)&=\frac{\beta^2}{(n+\beta)^2}x^2+\frac{x}{(n+\beta)^2}\left(2n+2\beta(v-\lambda-\alpha-1)\right)\\ &\quad+\frac{1}{(n+\beta)^2}\left\{v^2-v(2\lambda+2\alpha+5)+\alpha^2+(\lambda+1)(2\alpha+\lambda+2)\right\},\end{aligned} \tag{27}$$

$$\begin{aligned}\mu_{n,4}^{(\lambda)}(x)&=\frac{\beta^4}{(n+\beta)^4}x^4+\frac{12n\beta^2-4\beta^3(\lambda+\alpha+1-v)}{(n+\beta)^4}x^3\\ &+\frac{x^2}{(n+\beta)^4}\Big\{12n^2+24n\beta(v-\alpha-\lambda-2)\\ &+6\beta^2\left(v^2-v(2\lambda+2\alpha+5)+\alpha^2+(\lambda+1)(2\alpha+\lambda+2)\right)\Big\}\\ &+\frac{x}{(n+\beta)^4}\Big\{-12nv(2\lambda+2\alpha+7-v)+n\lambda(24\alpha+12\lambda+60)\\ &+n\left(12\alpha^2+48\alpha+72\right)\\ &+4\beta v\left(v^2-12v\lambda-12v\alpha-39v+3\alpha^2+3\lambda^2+6\lambda\alpha+15\alpha+18\lambda+27\right)\\ &-4\alpha\beta\left(\alpha^2+3\lambda\alpha+3\alpha+3\lambda^2+9\lambda+6\right)-4\beta\left(\lambda^3+6\lambda^2+11\lambda+6\right)\Big\}\\ &+\frac{1}{(n+\beta)^4}\Big\{v^4-2v^3(2\lambda+2\alpha+11)\\ &-v\left(12\lambda\alpha^2+146\lambda+106\alpha+42\lambda^2+72\alpha\lambda+168+30\alpha^2+4\lambda^3+12\lambda^2\alpha+4\alpha^3\right)\\ &+3v^2\left(2\lambda^2+18\lambda+2\alpha^2+16\alpha+4\lambda\alpha+41\right)\\ &+\alpha^4+(\lambda+1)\left(4\alpha^3+6\alpha^2(\lambda+2)\right)\Big\}.\end{aligned} \tag{28}$$

Moreover, for $n > m$, we have the following recurrence relation

$$\begin{aligned} & n(m+1)\,\mu_{n,m}^{(\lambda)}(x) + m(m+1)\,\mu_{n,m-1}^{(\lambda)}(x) + (m+1)\,\mu_{n,m}^{\prime(\lambda)}(x) \\ & = n(n+\beta)\left(\mu_{n,m+1}^{(\lambda+1)}(x) - \mu_{n,m+1}^{(\lambda)}(x)\right). \end{aligned} \tag{29}$$

Proof. By using the linearity of the operators $H_n^{(\alpha,\beta,\lambda)}$ and Lemma 3.1, one can easily obtain the equalities (27) and (28).

In order to obtain the recurrence relation, if we take derivative of the each side of the equality (26), we have

$$\begin{aligned} & \mu_{n,m}^{\prime(\lambda)}(x) \\ & = -ne^{-nx+\frac{v}{2}} \sum_{k=0}^{\infty} \frac{H_k^{(v)}(nx)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt \\ & + ne^{-nx+\frac{v}{2}} \sum_{k=0}^{\infty} \frac{H_k^{\prime(v)}(nx)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt \\ & - me^{-nx+\frac{v}{2}} \sum_{k=0}^{\infty} \frac{H_k^{(v)}(nx)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} \left(\frac{nt+\alpha}{n+\beta} - x\right)^{m-1} dt. \end{aligned} \tag{30}$$

From (21), it readily follows that

$$H_k^{\prime(v)}(nx) = kH_{k-1}^{(v)}(nx).$$

Using this in (30), we get

$$\begin{aligned} & \mu_{n,m}^{\prime(\lambda)}(x) \\ & = -n\mu_{n,m}^{(\lambda)}(x) - m\mu_{n,m-1}^{(\lambda)}(x) \\ & + ne^{-nx+\frac{v}{2}} \sum_{k=0}^{\infty} \frac{H_k^{(v)}(nx)}{k!} \frac{n^{\lambda+1+k+1}}{\Gamma(\lambda+1+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+1+k} \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt \end{aligned}$$

Applying integration by parts

$$e^{-nt} t^{\lambda+k+1} = u\,, \quad \left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt = dv,$$

gives (29). □

Now, let's calculate the rates of convergence of the operators $H_n^{(\alpha,\beta,\lambda)}$ to f by using the second modulus of continuity.

First of all, we give the following lemmas.

Lemma 3.3 (Gavrea and Raşa [9]). *Let $g \in C^2[0,a]$ and $(P_n)_{n\geq 0}$ be a sequence of linear positive operators with the property $P_n(1;x) = 1$. Then*

$$\left|P_n(g;x) - g(x)\right| \leq \left\|g'\right\| \sqrt{P_n\left((t-x)^2;x\right)} + \frac{1}{2}\left\|g''\right\| P_n\left((t-x)^2;x\right).$$

Lemma 3.4 (Zhuk [26]). *Let $f \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$. Let f_h be the second-order Steklov function attached to the function f. Then the following inequalities are satisfied*

(i) $\left\|f_h - f\right\| \leq \frac{3}{4}\omega_2(f;h)$
(ii) $\left\|f_h''\right\| \leq \frac{3}{2h^2}\omega_2(f;h)$.

Theorem 3.5. *For $f \in C[0,a]$, we have*

$$\left|H_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right| \leq \frac{2}{a}\left\|f\right\|h^2 + \frac{3}{4}\left(a + h^2 + 2\right)\omega_2(f;h),$$

where

$$h := h_n(x) = \sqrt[4]{H_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right)}$$

and the second modulus of continuity of $f \in C[a,b]$ is defined by

$$\omega_2(f;\delta) := \sup_{0<t\leq\delta}\left\|f(.+2t) - 2f(.+t) + f(.)\right\|$$

with the norm $\left\|f\right\| = \max_{x\in[a,b]}\left|f(x)\right|$.

Proof. Assume that f_h is the second-order Steklov function attached to the function f. From (24), we may write

$$\begin{aligned}&\left|H_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right| \qquad (31)\\ &\leq \left|H_n^{(\alpha,\beta,\lambda)}(f-f_h;x)\right| + \left|H_n^{(\alpha,\beta,\lambda)}(f_h;x) - f_h(x)\right| + \left|f_h(x) - f(x)\right|\\ &\leq 2\left\|f_h - f\right\| + \left|H_n^{(\alpha,\beta,\lambda)}(f_h;x) - f_h(x)\right|.\end{aligned}$$

In view of the fact that $f_h \in C^2[0,a]$, from Lemma 3.3, we get

$$\begin{aligned}&\left|H_n^{(\alpha,\beta,\lambda)}(f_h;x) - f_h(x)\right|\\ &\leq \left\|f_h'\right\|\sqrt{H_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right)} + \frac{1}{2}\left\|f_h''\right\|H_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right). \qquad (32)\end{aligned}$$

It follows from the Landau inequality and Lemma 3.4

$$\begin{aligned}\left\|f_h'\right\| &\leq \frac{2}{a}\left\|f_h\right\| + \frac{a}{2}\left\|f_h''\right\|\\ &\leq \frac{2}{a}\left\|f\right\| + \frac{3a}{4h^2}\omega_2(f;h).\end{aligned}$$

By using this inequality, the inequality (32) leads to by taking $h = \sqrt[4]{H_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right)}$

$$\begin{aligned}&\left|H_n^{(\alpha,\beta,\lambda)}(f_h;x) - f_h(x)\right|\\ &\leq \frac{2}{a}\left\|f\right\|h^2 + \frac{3a}{4}\omega_2(f;h) + \frac{3h^2}{4}\omega_2(f;h).\end{aligned}$$

From this inequality and Lemma 3.4, it follows

$$\left|H_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right| \leq \frac{2}{a}\left\|f\right\|h^2 + \frac{3}{4}\left(a + h^2 + 2\right)\omega_2(f;h),$$

which is desired. □

3.1. Voronovskaja-type theorem for the operators $H_n^{(\alpha,\beta,\lambda)}$

Now, we obtain a Voronovskaja-type asymptotic estimate of the operators (23).

Theorem 3.6. *Suppose that $f \in C^2[0,a]$. Then we have*

$$\lim_{n\to\infty} n\left[H_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right] = (\lambda+\alpha+1-v-\beta x)f'(x) + xf''(x).$$

Proof. *By the Taylor expansion, we may write*

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + (t-x)^2\eta(t;x),$$

where $\eta(t;x) \in C[0,a]$ and $\eta(t;x) \to 0$ as $t \to x$. Applying the operators (23) *to the above equality, we get*

$$H_n^{(\alpha,\beta,\lambda)}(f;x) - f(x) = f'(x)H_n^{(\alpha,\beta,\lambda)}(t-x;x) + \frac{f''(x)}{2!}H_n^{(\alpha,\beta,\lambda)}\left((t-x)^2;x\right) + H_n^{(\alpha,\beta,\lambda)}\left((t-x)^2\eta(t;x);x\right).$$

In view of Lemma 3.2, we have

$$\begin{aligned} &n\left[H_n^{(\alpha,\beta,\lambda)}(f;x) - f(x)\right] \\ &= nf'(x)\left[\frac{\lambda+\alpha+1-v-\beta x}{n+\beta}\right] \\ &+ n\left\{\frac{\beta^2}{(n+\beta)^2}x^2 + \frac{2}{(n+\beta)^2}(n+\beta(v-\lambda-\alpha-1))x\right. \\ &\left.+\frac{1}{(n+\beta)^2}\left[v^2 - v(2\lambda+2\alpha+5) + \alpha^2 + (\lambda+1)(2\alpha+\lambda+2)\right]\right\}\frac{f''(x)}{2!} \\ &+ nH_n^{(\alpha,\beta,\lambda)}\left((t-x)^2\eta(t;x);x\right). \end{aligned} \tag{33}$$

Now we shall show that

$$\lim_{n\to\infty} nH_n^{(\alpha,\beta,\lambda)}\left((t-x)^2\eta(t;x);x\right) = 0.$$

From the Cauchy-Schwarz inequality, we have

$$nH_n^{(\alpha,\beta,\lambda)}\left((t-x)^2\eta(t;x);x\right) \le \sqrt{n^2H_n^{(\alpha,\beta,\lambda)}\left((t-x)^4;x\right)}.\sqrt{H_n^{(\alpha,\beta,\lambda)}(\eta^2(t;x);x)}.$$

Direct calculations show that

$$\lim_{n\to\infty} n^2H_n^{(\alpha,\beta,\lambda)}\left((t-x)^4;x\right) = 12x^2$$

and, since $\eta(t;x) \in C[0,a]$ and $\eta(t;x) \to 0$ as $t \to x$, it follows

$$\lim_{n\to\infty} H_n^{(\alpha,\beta,\lambda)}\left(\eta^2(t;x);x\right) = \eta^2(x;x) = 0.$$

So, we obtain

$$\lim_{n\to\infty} nH_n^{(\alpha,\beta,\lambda)}\left((t-x)^2\eta(t;x);x\right) = 0 \tag{34}$$

and then, by taking limit as $n \to \infty$ in (33) and using (34), we end up the proof. □

3.2. An r-th order generalization of the operators $H_n^{(\alpha,\beta,\lambda)}$

In this subsection, we consider *r-th* order generalization of the positive linear operators $H_n^{(\alpha,\beta,\lambda)}$. This generalization was first given by Kirov and Popova [14]. In 2011, Agratini [2] studied a generalization of the *r-th* order of Stancu-type operators. Some results in this direction were found in [17], [18],[19].

Now, we define the following generalization of the positive linear operators $H_n^{(\alpha,\beta,\lambda)}$

$$H_{n,r}^{(\alpha,\beta,\lambda)}(f;x)=e^{-nx+\frac{v}{2}}\sum_{k=0}^{\infty}\frac{H_k^{(v)}(nx)}{k!}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}$$
$$\times\int_0^{\infty}e^{-nt}t^{\lambda+k}\sum_{j=0}^{r}f^{(j)}\left(\frac{nt+\alpha}{n+\beta}\right)\frac{\left(x-\frac{nt+\alpha}{n+\beta}\right)^j}{j!}dt \tag{35}$$

where $f\in C^r[0,A]$ $(0<A<1, r\in\mathbb{N}_0)$, $n\in\mathbb{N}$. Here $C^r[0,A]$ denotes the space of all functions of having continuous *r-th* order derivative $f^{(r)}$ on the segment $[0,A]$, $(0<A<1)$, where as usual, $f^{(0)}(x)=f(x)$.

Note that taking $r=0$, we obtain the operators $H_n^{(\alpha,\beta,\lambda)}(f;x)$ defined by (23).

In order to give some approximation properties of the operators defined by (35) let us recall some definitions.

Definition 3.7. *Let f be a real valued continuous function defined on $[0,A]$. Then f is said to be Lipschitz continuous of order γ on $[0,A]$ if*

$$\left|f(x)-f(y)\right|\le M\left|x-y\right|^{\gamma}$$

for $x,y\in[0,A]$ with $M>0$ and $0<\gamma\le 1$. The set of Lipschitz continuous functions is denoted by $Lip_M(\gamma)$.

Theorem 3.8. *For any $f\in C^r[0,A]$ with the property $f^{(r)}\in Lip_M(\gamma)$,*

$$\left\|H_{n,r}^{(\alpha,\beta,\lambda)}(f;x)-f\right\|_{C[0,A]}\le\frac{M}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)\left\|H_n^{(\alpha,\beta,\lambda)}\left(|x-t|^{r+\gamma};x\right)\right\|_{C[0,A]} \tag{36}$$

where $B(\gamma,r)$ is Beta function defined by

$$B(\gamma,r)=\int_0^1 s^{\gamma-1}(1-s)^{r-1}dt,(\gamma,r>0). \tag{37}$$

Proof. From (35), we have

$$f(x)-H_{n,r}^{(\alpha,\beta,\lambda)}(f;x)$$
$$=e^{-nx+\frac{v}{2}}\sum_{k=0}^{\infty}\frac{H_k^{(v)}(nx)}{k!}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}$$
$$\times\int_0^{\infty}e^{-nt}t^{\lambda+k}\left[f(x)-\sum_{j=0}^{r}f^{(j)}\left(\frac{nt+\alpha}{n+\beta}\right)\frac{\left(x-\frac{nt+\alpha}{n+\beta}\right)^j}{j!}\right]dt. \tag{38}$$

Applying the Taylor's formula, we may write that

$$f(x)-\sum_{j=0}^{r}f^{(j)}\left(\frac{nt+\alpha}{n+\beta}\right)\frac{\left(x-\frac{nt+\alpha}{n+\beta}\right)^{j}}{j!} \tag{39}$$
$$=\frac{\left(x-\frac{nt+\alpha}{n+\beta}\right)^{r}}{(r-1)!}\int_{0}^{1}(1-s)^{r-1}\left[f^{(r)}\left(\frac{nt+\alpha}{n+\beta}+s\left(x-\frac{nt+\alpha}{n+\beta}\right)\right)-f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)\right]ds.$$

Because of $f^{(r)}\in Lip_M(\gamma)$, we obtain

$$\left|f^{(r)}\left(\frac{nt+\alpha}{n+\beta}+s\left(x-\frac{nt+\alpha}{n+\beta}\right)\right)-f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)\right|\leq Ms^{\gamma}\left|x-\frac{nt+\alpha}{n+\beta}\right|^{\gamma}. \tag{40}$$

On the other hand, from the the well-known expression of the Beta function, we get

$$\int_{0}^{1}(1-s)^{r-1}s^{\gamma}ds=B(1+\gamma,r)=\frac{\gamma}{\gamma+r}B(\gamma,r). \tag{41}$$

By considering (40) and (41) in (39), we have

$$\left|f(x)-\sum_{j=0}^{r}f^{(j)}\left(\frac{nt+\alpha}{n+\beta}\right)\frac{\left(x-\frac{nt+\alpha}{n+\beta}\right)^{j}}{j!}\right|\leq\frac{M}{(r-1)!}\frac{\gamma}{\gamma+r}B(\gamma,r)\left|x-\frac{nt+\alpha}{n+\beta}\right|^{r+\gamma}. \tag{42}$$

Taking (42) in (38) we arrive at (36). □

Now we take a function $g\in C[0,A]$ which is defined by $g(t)=|x-t|^{r+\gamma}$. Since $g(x)=0$, we can write

$$\left\|H_{n}^{(\alpha,\beta,\lambda)}\left(|x-t|^{r+\gamma};x\right)\right\|_{C[0,A]}=0.$$

Theorem 3.8 yields that for all $f\in C^{r}[0,A]$ such that $f^{(r)}\in Lip_M(\gamma)$, we obtain

$$\lim_{n\to\infty}\left\|H_{n,r}^{(\alpha,\beta,\lambda)}(f;x)-f\right\|_{C[0,A]}=0.$$

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