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Generalized Integral Transforms with the Translation Operator on Function Space

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Abstract. Main goal of this paper is to establish various basic formulas for the generalized integral transform involving the generalized convolution product. In order to establish these formulas, we use the translation operator which was introduced in [9]. It was not easy to establish basic formulas for the generalized integral transforms because the generalized Brownian motion process used in this paper has the nonzero mean function. In this paper, we can easily establish various basic formulas for the generalized integral transform involving the generalized convolution product via the translation operator.

1. Introduction

In the mathematical field of functional analysis, the isomorphism properties are very important subjects. Also, these properties have been studying in various paper. That is to say, let T be the transform on an abstract space. The following basic formulas are meaningful subjects to many mathematicians;

$$T^{-1}T(F) = F = TT^{-1}(F)$$

as well as basic formulas

$$T(F * G) = T(F)T(G) \ \text{ and } \ T(F) * T(G) = T(FG)$$

where T^{-1} is the inverse transform and $*$ is the convolution product with respect to the transform T.

The function space $C_{a,b}[0,T]$ induced by a generalized Brownian motion was introduced by J. Yeh in [21] and was used extensively in [4–6, 8, 9, 11, 19]. The theory of the generalized integral transform $\mathcal{F}_{\gamma,\beta}$ on function space was studied and developed in various papers [5, 6, 8, 9, 11]. In particular, the authors gave a necessary and sufficient condition that a functional F in $L^2(C_{a,b}[0,T])$ has an integral transform $\mathcal{F}_{\gamma,\beta}(F)$ also belonging to $L^2(C_{a,b}[0,T])$, see [6]. Previous researches have attempted to establish various basic formulas with respect to the generalized integral transform, the generalized convolution product and the

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inverse transform. But, there are some difficulties to establish these basic formulas because the generalized Brownian motion used in this paper has the nonzero mean function $a(t)$. In [8], the authors established the inverse integral transform and a basic formula as follows;

$$\mathcal{F}_{\gamma,\beta}^{-1} = \mathcal{F}_{-i\gamma,1} \circ \mathcal{F}_{i\gamma,1} \circ \mathcal{F}_{-\frac{\gamma}{\beta},\frac{1}{\beta}}$$

and

$$\mathcal{F}_{\gamma,\beta}(F * G)_{\gamma}(y) = \mathcal{F}_{\gamma,\beta}(F * 1)_{\gamma}(y)\mathcal{F}_{\gamma,\beta}(1 * G)_{\gamma}(y)$$

for functionals on function space $C_{a,b}[0,T]$, where $\circ$ is the composition of transforms. But, these formulas are very complicated, namely, the inverse integral transform is obtained the composition of three generalized integral transforms and the basic formula is obtained by the concept of generalized convolution product. Recently, the authors improved the inverse integral transform by using the translation operator T_c as follows;

$$\mathcal{F}_{\gamma,\beta}^{-1} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ T_c$$

where $T_c(F)(x) = F(x + ca)$ and $c = -\frac{\gamma}{\beta}(1 + i)$, see [9]. However, various basic formulas with respect to the generalized integral transform and the generalized convolution product are not improved.

In this paper, we introduce the class $\mathcal{A}$ which is a dense set in $L^2(C_{a,b}[0,T])$. We then establish various improved basic formulas for the generalized integral transform(GIT) and the generalized convolution product(GCP). Finally, we give some extended formulas for the GIT. However, when $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, the general function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$ and so most of the results in [7, 10, 12, 14–16, 20] follow immediately from the results in this paper.

The Wiener process used in [1–3, 7, 10, 12, 14–17] is stationary in time and is free of drift while the stochastic process used in this paper as well as in [4–6, 8, 9, 11, 19], is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [18].

2. Preliminaries

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$. The generalized Brownian motion process Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 in [22], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. We then complete this function space to obtain $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

A subset E of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\rho E \in \mathcal{W}(C_{a,b}[0,T])$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere [13].

Let

$$L_{a,b}^2[0,T] = \left\{ v : \int_0^T |v^2(s)| db(s) < \infty \text{ and } \int_0^T |v^2(s)| d|a|(s) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$. Then $(L_{a,b}^2[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space with the norm $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ and $(u,v)_{a,b} = \int_0^T u(t)\overline{v(t)} d[b(t) + |a|(t)]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)\overline{v(t)}d[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], \|\cdot\|_{a,b}) \subset (L^2_{0,b}[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

For each $v \in L^2_{a,b}[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund(PWZ) stochastic integral. Note that the properties of the PWZ integral studied several times in many papers. For more details see, [5, 6, 8, 11].

In this paper, let $K_{a,b}[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t = 0$ and whose real and imaginary parts are elements of $C_{a,b}[0,T]$; namely,

$$K_{a,b}[0,T] = \{x : [0,T] \to \mathbb{C} \mid x(0) = 0, \text{ Re}(x) \in C_{a,b}[0,T] \text{ and } \text{Im}(x) \in C_{a,b}[0,T]\}.$$

Thus clearly $C_{a,b}[0,T]$ is a subspace of $K_{a,b}[0,T]$.

Throughout this paper we will assume that each functional $F : C_{a,b}[0,T] \to \mathbb{C}$ we consider is scale-invariant measurable and that

$$\int_{C_{a,b}[0,T]} |F(\rho x)| d\mu(x) < \infty$$

for each $\rho > 0$.

We are now ready to state the definition of the GIT $\mathcal{F}_{\gamma,\beta}$ and the GCP $(F * G)_\gamma$ used in [6, 8, 9].

Definition 2.1. *Let F and G be functionals defined on $K_{a,b}[0,T]$. For each pair of nonzero complex numbers γ and β, the GIT $\mathcal{F}_{\gamma,\beta}F$ of F, and the GCP $(F * G)_\gamma$ of F and G are defined by*

$$\mathcal{F}_{\gamma,\beta}F(y) \equiv \mathcal{F}_{\gamma,\beta}(F)(y) = \int_{C_{a,b}[0,T]} F(\gamma x + \beta y) d\mu(x), \tag{1}$$

for $y \in K_{a,b}[0,T]$ and

$$(F * G)_\gamma(y) = \int_{C_{a,b}[0,T]} F\Big(\frac{y + \gamma x}{\sqrt{2}}\Big) G\Big(\frac{y - \gamma x}{\sqrt{2}}\Big) d\mu(x), \tag{2}$$

for $y \in K_{a,b}[0,T]$ if they exist.

Remark 2.2. *(i) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\mathcal{F}_{\gamma,\beta}$ is the integral transform used by Kim and Skoug [15]. In particular, $\mathcal{F}_{1,i}$ is the Fourier-Wiener transform introduced by Cameron in [1] and used by Cameron and Martin in [2]. Also $\mathcal{F}_{\sqrt{2},i}$ is the modified Fourier-Wiener transform used by Cameron and Martin in [3].*

(ii) When $\gamma = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\gamma,\beta}$ is the generalized Fourier-Wiener function space transform introduced in [5].

Throughout this paper, in order to ensure that various integrals exist, we will assume that $\beta = c + id$ is a nonzero complex number satisfying the inequality

$$Re(1 - \beta^2) = 1 + d^2 - c^2 > 0 \tag{3}$$

with $\gamma^2+\beta^2=1$. Note that $\beta=c+id$ satisfies equality (3) if and only if $(c,d)\in\mathbb{R}^2$ lies in the open region determined by the hyperbola $c^2-d^2=1$ containing the d-axis. Next, let

$$\gamma=\sqrt{1-\beta^2},\quad -\pi/4<arg(\gamma)<\pi/4$$

and note that $\gamma^2+\beta^2=1$ and $Re(\gamma^2)=Re(1-\beta^2)>0$.

We shall analyze the condition introduced in [6, 15] to obtain some basic formulas for the GITs and the GCPs. In one-parameter Wiener space $C_0[0,T]$ (i.e., where $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$ for this study), the existence of the integral transform depends on the size of β (see [15]). However, on function space, the existence of the integral transform depends on the pairs of (γ,β), for more detailed see [6, Theorem 7].

Now we introduce a class $\mathcal{A}$ of functionals which is used in this paper. In order to do this, let

$$\mathcal{G}=\{(\gamma,\beta)\in\mathbb{C}\times\mathbb{C}\ :\ \mathcal{F}_{\gamma,\beta}(F)\in L^2(C_{a,b}[0,T]),\ F\in L^2(C_{a,b}[0,T])\}.$$

Also, for fixed positive integer m, let $\mathbb{E}_0^{(m)}$ be the class of functionals of the form

$$F(x)=f(\langle\alpha_1,x\rangle,\cdots,\langle\alpha_m,x\rangle)=f(\langle\vec{\alpha},x\rangle)$$

where $\{\alpha_1,\cdots,\alpha_m\}$ is an orthonormal set in $L^2_{a,b}[0,T]$ and f is an entire function on $\mathbb{C}^m$ and

$$|f(\vec{u})|\le L_F\exp\Big\{K_F\sum_{j=1}^m|u_j|\Big\}$$

for some positive real numbers L_F and K_F. One can easily check that for all nonzero complex numbers γ and β (and hence $(\gamma,\beta)\in\mathcal{G}$), the integral transform $\mathcal{F}_{\gamma,\beta}(F)$ exists and is an element of $\mathbb{E}_0^{(m)}$, for more details see [8]. Also, $\mathbb{E}_0^{(m)}\subset L^2(C_{a,b}[0,T])$ for all $m=1,2,\cdots$. From now on, we list some results and definitions from [5, 6]. Let $\{\alpha_1,\alpha_2,\cdots\}$ be any complete orthonormal set of functions in the separable Hilbert space $(L^2_{a,b}[0,T],\|\cdot\|_{a,b})$, and for each $j=1,2,\cdots$, let

$$A_j\equiv\int_0^T\alpha_j(t)da(t)\quad\text{and}\quad B_j\equiv\int_0^T\alpha_j^2(t)db(t).$$

We note that for each $j=1,2,\cdots$,

$$0<B_j=\int_0^T\alpha_j^2(t)db(t)\le\int_0^T\alpha_j^2(t)d[b(t)+|a|(t)]=\|\alpha_j\|^2_{a,b}=1,$$

while A_j may be positive, negative or zero. If $a(t)\equiv 0$ on $[0,T]$, then $A_j=0$ and $B_j=1$ for all $j=1,2,\cdots$. For each $m=0,1,2,\cdots$ and for each $j=1,2,\cdots$, let $H_m^j(u)$ denote the generalized Hermite polynomial

$$H_m^j(u)\equiv(-1)^m(m!)^{-\frac12}(B_j)^{\frac m2}\exp\Big\{\frac{(u-A_j)^2}{2B_j}\Big\}\frac{d^m}{du^m}\Big(\exp\Big\{-\frac{(u-A_j)^2}{2B_j}\Big\}\Big).\tag{4}$$

Then the set, for each $j=1,2,\cdots$,

$$\Big\{(2\pi B_j)^{-\frac14}H_m^j(u)\exp\Big\{-\frac{(u-A_j)^2}{4B_j}\Big\}:m=0,1,\cdots\Big\}$$

is a complete orthonormal set in $L_2(\mathbb{R})$. Now we define

$$\phi_{(m,k)}(x)\equiv H_m^k(\langle\alpha_k,x\rangle),\qquad m=0,1,2,\cdots,k=1,2,\cdots,$$

and

$$\Phi_{(m_1,\cdots,m_k)}(x) \equiv \phi_{(m_1,1)}(x)\phi_{(m_2,2)}(x)\cdots\phi_{(m_k,k)}(x) = \prod_{j=1}^{k} H^j_{m_j}(\langle\alpha_j, x\rangle). \tag{5}$$

The functionals in (5) are called the generalized Fourier-Hermite functionals. The authors showed that the generalized Fourier-Hermite functionals forms a complete orthonormal set in $L^2(C_{a,b}[0,T])$. That is to say, let F be any functionals on $C_{a,b}[0,T]$ with

$$\int_{C_{a,b}[0,T]} |F(x)|^2 d\mu(x) < \infty,$$

and for $N = 1, 2, \cdots$, let

$$F_N(x) = \sum_{m_1,\cdots,m_N=0}^{N} A^F_{(m_1,\cdots,m_N)}\Phi_{(m_1,\cdots,m_N)}(x) \tag{6}$$

where $A^F_{(m_1,\cdots,m_N)}$ is the generalized Fourier-Hermite coefficient,

$$A^F_{(m_1,\cdots,m_N)} \equiv \int_{C_{a,b}[0,T]} F(x)\Phi_{(m_1,\cdots,m_N)}(x)d\mu(x). \tag{7}$$

Then

$$\int_{C_{a,b}[0,T]} |F_N(x) - F(x)|^2 d\mu(x) \to 0$$

as $N \to \infty$ and

$$F(x) = l.i.m._{N\to\infty}F_N(x) = l.i.m._{N\to\infty} \sum_{m_1,\cdots,m_N=0}^{N} A^F_{(m_1,\cdots,m_N)}\Phi_{(m_1,\cdots,m_N)}(x) \tag{8}$$

is called the generalized Fourier-Hermite series expansion of F. Next, the generalized Fourier-Hermite functional $\Phi_{(m_1,\cdots,m_N)}$ satisfies all definitions of the class $\mathbb{E}_0^{(N)}$ for each $N = 1, 2, \cdots$ and likewise that the functional $F_N(x)$ also belong to $\cup_{N=1}^{\infty}\mathbb{E}_0^{(N)}$. Let $\mathcal{A} = \cup_{N=1}^{\infty}\mathbb{E}_0^{(N)}$. Then the class $\mathcal{A}$ is dense in $L^2(C_{a,b}[0,T])$ since the fact that $\mathcal{M} \equiv \{\Phi_{(m_1,\cdots,m_N)}\}_{N=1}^{\infty}$ is an orthonormal set in $L^2(C_{a,b}[0,T])$ [6]. Hence by using general theories in vector space, we could extend all results and formulas of the space $\mathcal{A}$ to the $L^2(C_{a,b}[0,T])$.

We close this section by stating some results and formulas which are used in this paper.

The following lemma was established in [11].

Lemma 2.3. *Let the function $a = a(t)$ be a function which satisfies the following condition*

$$\int_0^T |a'(t)|^2 d|a|(t) < \infty.$$

Then the mean function a can be written like as $a(t) = \int_0^t z(s)db(s)$ where $z(s) = \frac{a'(s)}{b'(s)} \in L^2_{a,b}[0,T]$. Let F be a μ-integrable functional defined on $K_{a,b}[0,T]$. Then for nonzero complex number c, $F(x + ca)$ is μ-integrable and

$$\int_{C_{a,b}[0,T]} F(x + ca)d\mu(x) = \exp\Big\{-\frac{c^2 + 2c}{2}\Big(\frac{a'}{b'}, a'\Big)\Big\}\int_{C_{a,b}[0,T]} F(x)\exp\{c\langle\frac{a'}{b'}, x\rangle\}d\mu(x)$$

where $(z, a') = \int_0^T z(s)da(s)$ for some $z \in L^2_{a,b}[0,T]$.

The formula (9) below is called the Fubini theorem with respect to the function space integrals, see [11].

Lemma 2.4. *Let F be μ-integrable defined on $K_{a,b}[0,T]$. Then for all complex numbers γ and β,*

$$\int_{C^2_{a,b}[0,T]} F(\gamma x + \beta y) d(\mu \times \mu)(x, y) = \int_{C_{a,b}[0,T]} F(\sqrt{\gamma^2 + \beta^2} w + ca) d\mu(w) \tag{9}$$

where $c = \gamma + \beta - \sqrt{\gamma^2 + \beta^2}$.

3. Some formulas for the GIT and the GCP

As mentioned in Section 1 above, establishing basic formulas for GITs and GCPs has proven to be difficult for functionals on function space. Hereafter, the operator T_c(which is called the translation operator) is defined as specified below, to solve these difficulties. We then establish basic formulas with respect to the GITs and the GCPs. Define an operator T_c from $\mathcal{A}$ into $\mathcal{A}$ by

$$T_c(F)(x) = F(x + ca) \tag{10}$$

for $x \in C_{a,b}[0,T]$ and complex number c. We have the following properties for T_c as follows;

(i) The operator T_c is well-defined for all complex number c from Lemma 2.3. Also, for all nonzero complex numbers $(\gamma, \beta) \in \mathcal{G}$, and $F \in \mathcal{A}$, $\mathcal{F}_{\gamma,\beta}(T_c(F))$ and $T_c(\mathcal{F}_{\gamma,\beta}(F))$ are well-defined.

(ii) The operator T_c is a bounded linear operator on $\mathcal{A}$ and it has an inverse operator T_{-c} for all complex number c.

(iii) When $a(t) \equiv 0$ on $[0,T]$, the operator T_c is the identity operator for all complex number c.

In order to establish the first main result in this paper, we need the following lemma which plays a key role finding various basic formulas for the GITs and the GCPs.

Lemma 3.1. *Let F be an element of $\mathcal{A}$. Let $c_1 = -\gamma(1 - \sqrt{2})$ and let $c_2 = -\gamma$. Then for all $(\gamma, \beta) \in \mathcal{G}$,*

$$\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(y/\sqrt{2}) = \int_{C_{a,b}[0,T]} F\Big(\frac{\beta y}{\sqrt{2}} + \gamma x - \gamma(1 - \sqrt{2})a\Big) d\mu(x) \tag{11}$$

and

$$\mathcal{F}_{\gamma,\beta}(T_{c_2}(F))(y/\sqrt{2}) = \int_{C_{a,b}[0,T]} F\Big(\frac{\beta y}{\sqrt{2}} + \gamma x - \gamma a\Big) d\mu(x) \tag{12}$$

for $y \in C_{a,b}[0,T]$. Furthermore, $\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))$ and $\mathcal{F}_{\gamma,\beta}(T_{c_2}(F))$ are elements of $\mathcal{A}$ because $(\gamma, \beta) \in \mathcal{G}$.

Proof. Using equations (1) and (10), we can easily obtain equations (11) and (12) as desired. Also, for all complex number c and $(\gamma, \beta) \in \mathcal{G}$, we see that $\mathcal{F}_{\gamma,\beta}(F)$ and $T_c(F)$ are elements of $\mathcal{A}$ for all $F \in \mathcal{A}$ and hence we have the desired results. □

Let F and G be elements of $\mathcal{A}$ and hence we let $F \in E_0^{(n)}$ and $G \in E_0^{(m)}$. If $n = m$, then $(F * G)_\gamma$ always exists and is an element of $E_0^{(m)}$. And so it is an element of $\mathcal{A}$. Otherwise, if $n \neq m$, then by using some properties of the space $E_0^{(N)}$, we can consider that $F, G \in E_0^{(m_0)}$ for some m_0. For example, let

$$F(x) = f(\langle \alpha_1, x \rangle, \langle \alpha_2, x \rangle) \in E_0^{(2)}$$

and let

$$G(x) = g(\langle \alpha_1, x \rangle, \langle \alpha_3, x \rangle, \langle \alpha_4, x \rangle) \in E_0^{(3)}.$$

Then we can express F and G as elements of $E_0^{(4)}$ by choosing $r(u_1, u_2, u_3, u_4) = f(u_1, u_2)$ and $s(u_1, u_2, u_3, u_4) = g(u_1, u_3, u_4)$. Hence the GCP $(F * G)_\gamma$ always exists and it is an element of $\mathcal{A}$.

The following theorem is one of the main results in this paper. This theorem tells us that the integral transform of the convolution product is the product of their transforms.

Theorem 3.2. *Let F and G be elements of $\mathcal{A}$. Then for all $(\gamma,\beta)\in\mathcal{G}$,*

$$\mathcal{F}_{\gamma,\beta}(F*G)_\gamma(y)=\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(y/\sqrt{2})\mathcal{F}_{\gamma,\beta}(T_{c_2}(G))(y/\sqrt{2}) \tag{13}$$

for $y\in C_{a,b}[0,T]$, where c_1 and c_2 are as in Lemma 3.1 above. Furthermore, $\mathcal{F}_{\gamma,\beta}(F*G)_\gamma$ is an element of $\mathcal{A}$.

Proof. Since $(F*G)_\gamma$ is an element of $\mathcal{A}$ and $(\gamma,\beta)\in\mathcal{G}$, the right-hand side of equation (13) exists and is an element of $\mathcal{A}$. Now using equations (1) and (2), it follows that for $y\in C_{a,b}[0,T]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}(F*G)_\gamma(y)\\
&=\int_{C_{a,b}[0,T]}(F*G)_\gamma(\gamma x+\beta y)d\mu(x)\\
&=\int_{C_{a,b}[0,T]}\int_{C_{a,b}[0,T]}F\Big(\frac{\gamma x+\beta y+\gamma z}{\sqrt{2}}\Big)G\Big(\frac{\gamma x+\beta y-\gamma z}{\sqrt{2}}\Big)d\mu(z)d\mu(x)\\
&=\int_{C_{a,b}[0,T]}\int_{C_{a,b}[0,T]}F\Big(\frac{\beta y}{\sqrt{2}}+\gamma\Big(\frac{x+z}{\sqrt{2}}\Big)\Big)G\Big(\frac{\beta y}{\sqrt{2}}+\gamma\Big(\frac{x-z}{\sqrt{2}}\Big)\Big)d\mu(z)d\mu(x).
\end{aligned}$$

But $w_1=\frac{x+z}{\sqrt{2}}+(1-\sqrt{2})a$ and $w_2=\frac{x-z}{\sqrt{2}}+a$ are independent generalized Brownian motion processes and hence we have

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}(F*G)_\gamma(y)\\
&=\int_{C_{a,b}[0,T]}\int_{C_{a,b}[0,T]}F\Big(\frac{\beta y}{\sqrt{2}}+\gamma w_1-\gamma(1-\sqrt{2})a\Big)G\Big(\frac{\beta y}{\sqrt{2}}+\gamma w_2-\gamma a\Big)d\mu(w_1)d\mu(w_2)\\
&=\int_{C_{a,b}[0,T]}F\Big(\frac{\beta y}{\sqrt{2}}+\gamma w_1-\gamma(1-\sqrt{2})a\Big)d\mu(w_1)\int_{C_{a,b}[0,T]}G\Big(\frac{\beta y}{\sqrt{2}}+\gamma w_2-\gamma a\Big)d\mu(w_2)
\end{aligned} \tag{14}$$

for $y\in C_{a,b}[0,T]$. Using equations (11) and (12) in Lemma 3.1, the last expression in equation (14) yields equation (13) as desired. □

The Fourier-Wiener function space transform introduced in [5] is a special case of our integral transform. We can apply our main results to the Fourier-Wiener function space transform as follows.

Corollary 3.3. *Let $\mathcal{F}_{\sqrt{2},i}$ be the generalized Fourier-Wiener function space transform introduced in [5]. Let F and G be elements of $\mathcal{A}$. Assume that $(\sqrt{2},i)\in\mathcal{G}$. Then*

$$\mathcal{F}_{\sqrt{2},i}(F*G)_{\sqrt{2}}(y)=\mathcal{F}_{\sqrt{2},i}(T_{c_3}(F))(y/\sqrt{2})\mathcal{F}_{\sqrt{2},i}(T_{c_4}(G))(y/\sqrt{2}) \tag{15}$$

for $y\in C_{a,b}[0,T]$, where $c_3=2-\sqrt{2}$ and $c_4=-\sqrt{2}$. Furthermore, the following equations immediately follows from equations (13) *and* (15) *by letting $G(y)=F(y)$ on $C_{a,b}[0,T]$. Let $F\in\mathcal{A}$. Then for all $(\gamma,\beta)\in\mathcal{G}$,*

$$\mathcal{F}_{\gamma,\beta}(F*F)_\gamma(y)=\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(y/\sqrt{2})\mathcal{F}_{\gamma,\beta}(T_{c_2}(F))(y/\sqrt{2})$$

for $y\in C_{a,b}[0,T]$, where c_1 and c_2 are as in Lemma 3.1 above. In particular, if $(\sqrt{2},i)\in\mathcal{G}$, then

$$\mathcal{F}_{\sqrt{2},i}(F*F)_{\sqrt{2}}(y)=\mathcal{F}_{\sqrt{2},i}(T_{c_3}(F))(y/\sqrt{2})\mathcal{F}_{\sqrt{2},i}(T_{c_4}(F))(y/\sqrt{2})$$

for $y\in C_{a,b}[0,T]$.

We can obtain the another formula without the concept of translation operator T_c by using the following Lemma 3.4. The proof of Lemma 3.4 was established in [9].

Lemma 3.4. *Let $F \in \mathcal{A}$. Then for all $(\gamma,\beta) \in \mathcal{G}$ and for all complex number c,*

$$\mathcal{F}_{\gamma,\beta}(T_c(F))(y) = \exp\Big\{-\frac{c^2+2c\gamma}{2\gamma^2}(\frac{a'}{b'},a') - \frac{c\beta}{\gamma^2}\langle\frac{a'}{b'},y\rangle\Big\}\mathcal{F}_{\gamma,\beta}(F^*_{c/\gamma^2})(y) \tag{16}$$

for $y \in C_{a,b}[0,T]$, where $F^*_c(x) = F(x)\exp\{c\langle\frac{a'}{b'},x\rangle\}$.

In Theorem 3.2, we established a basic formula by using the translation operator T_c. In our next Theorem 3.5, we obtain another basic formula without the concept of translation operator T_c with exponential weighed.

Theorem 3.5. *Let $F \in \mathcal{A}$. Then for all $(\gamma,\beta) \in \mathcal{G}$,*

$$\mathcal{F}_{\gamma,\beta}(F*G)_\gamma(y) = \exp\Big\{\frac{\beta}{\gamma}(\sqrt{2}-1)\langle\frac{a'}{b'},y\rangle\Big\}\mathcal{F}_{\gamma,\beta}(F^*_{(\sqrt{2}-1)/\gamma})(y/\sqrt{2})\mathcal{F}_{\gamma,\beta}(G^*_{-1/\gamma})(y/\sqrt{2}) \tag{17}$$

for $y \in C_{a,b}[0,T]$, where $F^*_{(\sqrt{2}-1)/\gamma}$ and $G^*_{-1/\gamma}$ are as in Lemma 3.4 above.

Proof. From Lemma 3.4, we can calculate as follows;

$$\begin{aligned}&\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(y/\sqrt{2})\\ &= \exp\Big\{-\frac{1}{2}(\frac{a'}{b'},a') + \frac{\beta}{\gamma}(1-\sqrt{2})\langle\frac{a'}{b'},\frac{y}{\sqrt{2}}\rangle\Big\}\mathcal{F}_{\gamma,\beta}(F^*_{(\sqrt{2}-1)/\gamma})(y/\sqrt{2})\end{aligned}$$

with c by replacing $-\gamma(1-\sqrt{2})$, and

$$\mathcal{F}_{\gamma,\beta}(T_{c_2}(G))(y/\sqrt{2}) = \exp\Big\{\frac{1}{2}(\frac{a'}{b'},a') + \frac{\beta}{\gamma}\langle\frac{a'}{b'},\frac{y}{\sqrt{2}}\rangle\Big\}\mathcal{F}_{\gamma,\beta}(G^*_{-1/\gamma})(y/\sqrt{2})$$

with c by replacing $-\gamma$. Hence by applying Theorem 3.2, we obtain equation (17) as desired. □

4. Some formulas for the GITs and the GCPs via the inverse GIT

In this section, we use the inverse integral transform to establish various formulas involving the GCPs. The following Theorem 4.1 was established in [9].

Theorem 4.1. *Let F be an element of $\mathcal{A}$ and let $c = -\frac{\gamma}{\beta}(1+i)$ for nonzero complex numbers γ and β. Then for all $(\gamma,\beta) \in \mathcal{G}$ with $(i\frac{\gamma}{\beta},\frac{1}{\beta}) \in \mathcal{G}$,*

$$\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_c(\mathcal{F}_{\gamma,\beta}F))(y) = F(y) = \mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}(T_cF))(y) \tag{18}$$

for $y \in C_{a,b}[0,T]$. This tells us that the inverse integral transform $\mathcal{F}^{-1}_{\gamma,\beta}$ of generalized integral transform $\mathcal{F}_{\gamma,\beta}$ is given by

$$\mathcal{F}^{-1}_{\gamma,\beta} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}} \circ T_c.$$

Now we are ready to draw conclusions about the convolution product $(F*G)_\gamma$.

Theorem 4.2. *Let c_1 and c_2 be as in Theorem 3.2, and let c be as in Theorem 4.1. Let F and G be elements of $\mathcal{A}$. Then for all (γ,β) as in Theorem 4.1,*

$$(F*G)_\gamma(y) = \mathcal{F}^{-1}_{\gamma,\beta}(\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(\cdot/\sqrt{2})\mathcal{F}_{\gamma,\beta}(T_{c_2}(G))(\cdot/\sqrt{2}))(y) \tag{19}$$

for $y \in C_{a,b}[0,T]$.

Proof. Using equations (13) and (18), we can easily obtain equation (19) as desired. □

The following corollary follows from Theorem 4.2 by letting $G(y) = F(y)$ or by letting $G(y)$ is identically one on $C_{a,b}[0,T]$.

Corollary 4.3. *Let c_1, c_2 and c be as in Theorem 4.2. Let $F \in \mathcal{A}$. Then for all (γ,β) as in Theorem 4.1,*

$$(F*F)_\gamma(y) = \mathcal{F}_{\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(\cdot/\sqrt{2})\mathcal{F}_{\gamma,\beta}(T_{c_2}(F)))(\cdot/\sqrt{2}))(y),$$

$$(F*1)_\gamma(y) = \mathcal{F}_{\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(\cdot/\sqrt{2}))(y) = T_{c_1}(F)(y/\sqrt{2})$$

and

$$(1*F)_\gamma(y) = \mathcal{F}_{\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}(T_{c_2}(F))(\cdot/\sqrt{2}))(y) = T_{c_2}(F)(y/\sqrt{2})$$

for $y \in C_{a,b}[0,T]$.

In our next theorem, we obtain a basic formula for the GCP with respect to the GITs.

Theorem 4.4. *Let c be as in Theorem 4.1. Let F and G be elements of $\mathcal{A}$. Then for all (γ,β) as in Theorem 4.1,*

$$(\mathcal{F}_{\gamma,\beta}F * \mathcal{F}_{\gamma,\beta}G)_\gamma(y) = \mathcal{F}_{\gamma,\beta}(T_{c_1}(F)(\cdot/\sqrt{2})T_{c_2}(G)(\cdot/\sqrt{2}))(y) \tag{20}$$

for $y \in C_{a,b}[0,T]$, where c_1 and c_2 are as in Lemma 3.1 above.

Proof. Using equations (13) and (18) it follows that for $y \in C_{a,b}[0,T]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}F * \mathcal{F}_{\gamma,\beta}G)_\gamma(y) \\
&= \mathcal{F}_{\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}T_{c_1}(F))(y/\sqrt{2})\mathcal{F}_{\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}T_{c_2}(G))(y/\sqrt{2}) \qquad (21)\\
&= T_{c_1}(F)(y/\sqrt{2})T_{c_2}(G)(y/\sqrt{2}).
\end{aligned}$$

Now taking the integral transform $\mathcal{F}_{\gamma,\beta}$ of each side of equation (21), we can obtain equation (20) as desired. □

From Theorem 4.4, we can obtain the following corollary by replacing $\mathcal{F}_{\gamma,\beta}$ with $\mathcal{F}_{\gamma,\beta}^{-1}$.

Corollary 4.5. *Let c be as in Theorem 4.1. Let F and G be elements of $\mathcal{A}$. Then for all (γ,β) as in Theorem 4.1,*

$$(\mathcal{F}_{\gamma,\beta}^{-1}F * \mathcal{F}_{\gamma,\beta}^{-1}G)_\gamma(y) = \mathcal{F}_{\gamma,\beta}^{-1}(T_{c_1}(F)(\cdot/\sqrt{2})T_{c_2}(G)(\cdot/\sqrt{2}))(y)$$

for $y \in C_{a,b}[0,T]$, where c_1 and c_2 are as in Lemma 3.1 above.

Remark 4.6. *In previous papers, all formulas and results from Theorem 3.2 to Theorem 4.4 were not established. They only expressed all results and formulas by the concept of GCPs $(F*1)_\gamma$ and $(1*G)_\gamma$. But, we used the translation operator T_c to obtain more simple expressions without the concept of GCP. However, we have*

$$\mathcal{F}_{\gamma,\beta}(T_{c_1}(F))(y/\sqrt{2}) = \mathcal{F}_{\gamma,\beta}(F*1)_\gamma(y),$$

and

$$\mathcal{F}_{\gamma,\beta}(T_{c_2}(F))(y/\sqrt{2}) = \mathcal{F}_{\gamma,\beta}(1*F)_\gamma(y)$$

for $y \in C_{a,b}[0,T]$, where $c_1 = -\gamma(1-\sqrt{2})$ and $c_2 = -\gamma$ form Corollary 4.3. This means that all formulas and results in previous papers are also obtained easily as corollaries. Hence our formulas and results in this paper are more generalized.

Remark 4.7. *In one-parameter Wiener space $C_0[0,T]$ (i.e., where $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$ for this study), $T_c(F)(x) = F(x)$ for all complex number c and hence we have*

$$\mathcal{F}_{\gamma,\beta}(F*G)_\gamma(y) = \mathcal{F}_{\gamma,\beta}(F)(y/\sqrt{2})\mathcal{F}_{\gamma,\beta}(G)(y/\sqrt{2}),$$

and

$$(\mathcal{F}_{\gamma,\beta}F * \mathcal{F}_{\gamma,\beta}G)_\gamma(y) = \mathcal{F}_{\gamma,\beta}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y).$$

5. More formulas for the GITs

In this section, we give some formulas for the GITs by use of the translation operator T_c. To simplify the expressions, we use following notations. Let $\{(\gamma_n, \beta_n)\}_{n=1}^{\infty}$ be a sequence in $\mathcal{G}$. For each $n = 1, 2, \cdots$, let

$$\tilde{\gamma}_n = \sqrt{\sum_{k=1}^{n} \gamma_k^2 \prod_{i=1}^{k} \beta_{i-1}^2}$$

and

$$\tilde{\beta}_n = \prod_{k=1}^{n} \beta_k$$

where $\beta_0 = 1$. Note that $\tilde{\gamma}_1 = \gamma_1$ and $\tilde{\beta}_1 = \beta_1$. Furthermore, $(\tilde{\gamma}_n, \tilde{\beta}_n) \in \mathcal{G}$ for all $n = 1, 2, \cdots$. For $n \geq 2$, define a function $H_n : \mathbb{C}^n \to \mathbb{C}$ by

$$H_n(z_1, \cdots, z_n) = \sum_{j=1}^{n} z_j - \Big(\sum_{j=1}^{n} z_j^2\Big)^{\frac{1}{2}}. \tag{22}$$

Note that H_n is a symmetric function for all $n = 2, 3, \cdots$.

In our next theorem, we establish the Fubini theorem for GITs.

Theorem 5.1. *Let F be as in Theorem 4.1. Assume that for (γ_1, β_1) and $(\gamma_2, \beta_2) \in \mathcal{G}$,*

$$(1 - \beta_1)\gamma_2 = (1 - \beta_2)\gamma_1.$$

Then

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y). \tag{23}$$

Furthermore, both of the expressions in (23) *are given by the expression*

$$\mathcal{F}_{\tilde{\gamma}_2,\tilde{\beta}_2}(T_{c_{m_2}}(F))(y) \tag{24}$$

for $y \in C_{a,b}[0, T]$, where $c_{m_2} = H_2(\gamma_1, \beta_1\gamma_2)$.

Proof. Using equations (1), (9) and (22) it follows that for $y \in C_{a,b}[0, T]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y)\\
&= \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} F(\gamma_1 z + \beta_1\gamma_2 x + \beta_1\beta_2 y) d\mu(z) d\mu(x)\\
&= \int_{C_{a,b}[0,T]} F\Big(\sqrt{\gamma_1^2 + \beta_1^2\gamma_2^2}\, w + \beta_1\beta_2 y + H_2(\gamma_1, \beta_1\gamma_2)a\Big) d\mu(w).
\end{aligned}$$

On the other hand, equations (1), (9) and (22) again it follows that for $y \in C_{a,b}[0, T]$,

$$\begin{aligned}
&\mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y)\\
&= \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} F(\gamma_2 z + \beta_2\gamma_1 x + \beta_1\beta_2 y) d\mu(z) d\mu(x)\\
&= \int_{C_{a,b}[0,T]} F\Big(\sqrt{\gamma_2^2 + \beta_2^2\gamma_1^2}\, w + \beta_1\beta_2 y + H_2(\gamma_2, \beta_2\gamma_1)a\Big) d\mu(w).
\end{aligned}$$

Note that $\gamma_1^2 + \beta_1^2\gamma_2^2 = \gamma_2^2 + \beta_2^2\gamma_1^2$ and using equation (22), we have

$$H_2(\gamma_1, \beta_1\gamma_2) = H_2(\gamma_2, \beta_2\gamma_1),$$

which establish equation (23) as desired. Furthermore, equation (24) follows from equations (1) and (10) easily. □

In the last theorem in this paper, we give the n-dimensional version of Theorem 5.1. The proof of Theorem 5.2 immediately follows from Theorem 5.1 and the mathematical induction.

Theorem 5.2. *Let F be as in Theorem 5.1. Assume that for $(\gamma_n, \beta_n) \in \mathcal{G}$,*

$$\tilde{\gamma}_{n-1}(1-\beta_n) = (1-\tilde{\beta}_{n-1})\gamma_n.$$

Then

$$\mathcal{F}_{\gamma_n,\beta_n}(\cdots(\mathcal{F}_{\gamma_1,\beta_1}F))(y) = \mathcal{F}_{\gamma_1,\beta_1}(\cdots(\mathcal{F}_{\gamma_n,\beta_n}F))(y) \tag{25}$$

for $y \in C_{a,b}[0,T]$. Furthermore, both of the expressions in (25) are given by the expression

$$\mathcal{F}_{\tilde{\gamma}_n,\tilde{\beta}_n}(T_{c_{m_n}}(F))(y) \tag{26}$$

for $y \in C_{a,b}[0,T]$, where $c_{m_n} = H_n(\gamma_1, \beta_1\gamma_2, \tilde{\beta}_2\gamma_3, \cdots, \tilde{\beta}_{n-1}\gamma_n)$.

We close this paper by stating some observations with respect to the GIT as remarks. As possible, we adopt the definitions and notations of [6, 7, 10, 12, 14, 15] for the integral transforms.

Remark 5.3. *In [6, 7, 10, 12, 14, 15], the authors obtained the existence of the integral transform $\mathcal{F}_{\alpha,\beta}$ for several large classes of functionals F on Wiener space $C_0[0,T]$. In particular, they showed that*

$$\mathcal{F}_{\alpha,\beta}^{-1} = \mathcal{F}_{i\gamma/\beta,1/\beta},$$

$$\mathcal{F}_{\alpha,\beta}(F*G)_\alpha(y) = \mathcal{F}_{\alpha,\beta}(F)(y/\sqrt{2})\mathcal{F}_{\alpha,\beta}(G)(y/\sqrt{2}),$$

and

$$(\mathcal{F}_{\alpha,\beta}(F)*\mathcal{F}_{\alpha,\beta}(G))_\alpha(y) = \mathcal{F}_{\alpha,\beta}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y).$$

A major goal of the authors of [6, 8, 9, 11] was to generalize the concepts of the integral transform of the functionals of paths for the generalized Brownian motion process. However, as mentioned in Section 1, it is not easy to verify the existence of the inverse GIT and to establish some basic formulas because the generalized Brownian motion process has a drift term $a(t)$. Recently [9], the authors obtained that

$$\mathcal{F}_{\gamma,\beta}^{-1} = \mathcal{F}_{i\gamma/\beta,1/\beta} \circ T_c$$

for some complex number c where T_c is the translation operator. But some basic formulas were not established yet. However, by use of the translation operator T_c and the inverse integral transform, we could obtain those basic formulas, see equations (13) and (20) above.

Remark 5.4. *In [9], the authors asked some questions with respect to the GITs as follows;*
(A) Is there a pair (γ, β) so that

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma,\beta}(F)(y)$$

for $y \in C_{a,b}[0,T]$?
(B) Is there a pair (γ, β) and complex number c so that

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma,\beta}(T_c(F))(y)$$

for $y \in C_{a,b}[0,T]$?
(C) Are there pairs (γ_1, β_1) and (γ_2, β_2) so that

$$\mathcal{F}_{\gamma_2,\beta_2}(\mathcal{F}_{\gamma_1,\beta_1}F)(y) = \mathcal{F}_{\gamma_1,\beta_1}(\mathcal{F}_{\gamma_2,\beta_2}F)(y)$$

for $y \in C_{a,b}[0,T]$?

They said that the answer to questions (A) and (C) was negative. While the answer to question (B) was positive. But, we gave positive answers to questions (A) and (C) by using the translation operator T_c, see Section 5.

In views of these, the translation operator T_c is a key role to establish various basic formulas and it will be used to obtain various other relationships.

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