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Triangular Hesitant Fuzzy Preference Relations and Their Applications in Multi-Criteria Group Decision-Making

Yan Yang[a], Junhua Hu[b], Yongmei Liu[b], Xiaohong Chen[b]

[a] *School of Business, Nanjing Audit University, Nanjing 211815, China*
[b] *School of Business, Central South University, Changsha 410083, China*

Abstract. In this paper, we present a novel multi-criteria decision-making (MCDM) methodology for assessing several alternatives under the triangular hesitant fuzzy environment. A scientific evaluation and prioritization approach is proposed by solving the MCDM problems with triangular hesitant fuzzy preference relations (THFPRs). Firstly, the concepts of THFPRs are defined, and a series of aggregation operators is introduced and their corresponding properties are discussed. Then, we define the consistency of the THFPRs and propose two methods to measure consistency. Furthermore, we construct an MCDM model using THFPR (MCDM-THFPR) to help decision makers assess and prioritise alternatives in the decision making process. Lastly, the validity and feasibility of the proposed MCDM-THFPR method for the MCDM are verified by a comparison with two previous approaches, along with certain discussions.

1. Introduction

In the real world, a situation in which a person can efficiently offer a judgment of every alternative may be considered an exception rather than the rule. Many studies on multi-criteria decision-making (MCDM) have been conducted [1–10]. Decision makers (DMs) are typically more confident by simply comparing than explaining or scoring them in the MCDM process. Consequently, preference relations (PRs), as introduced by Blin [11], have recently captured a considerable attentions of many experts given their capability to yield accurate results. The PRs are generally divided into two types, namely, the complementary preference relation (CPR) [12] and the reciprocal multiplicative preference relation (RPR) [13]. The PRs have also been developed to many extensions, such as probabilistic hesitant fuzzy preference relation [14–16], asymmetric hesitant fuzzy preference relation[17], intuitionistic fuzzy preference relation [18], interval-valued intuitionistic fuzzy preference relation [19], 2-tuple fuzzy linguistic preference relations [20], single-valued trapezoidal neutrosophic preference relations [21], triangular fuzzy preference relation (TFPR) [22, 23], multiplicative triangular hesitant fuzzy preference relations [24]; and neutrosophic fuzzy preference relations [25].

Over the years, PRs have been widely applied in decision-making problems. Chiclana et al. [26, 27] studied integrating RPRs into multipurpose decision-making problems. Urena et al. [28, 29] made

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Email address: hujunhua@csu.edu.cn (Junhua Hu)

progress towards estimating missing preferences in decision-making and proposed a group decision-making approach using incomplete reciprocal intuitionistic fuzzy preference relations.

However, experts express difficulties in real decision-making problems in terms of their preference degrees for one alternative over others using an exact number. Hesitant fuzzy set (HFS) is a useful tool for solving this issue, and has been developed extensively [30–32]. Xia and Xu [33] extended the fuzzy preference relations (FPRs) to hesitant fuzzy preference relations (HFPRs), in which several possible preference values can be considered a hesitant fuzzy number (HFN). Recently, various types of HFPRs, such as hesitant fuzzy linguistic preference relations [34], extended hesitant fuzzy linguistic preference relations [35]; and hesitant-intuitionistic fuzzy preference relations [36], are proposed. Aggregation operators, including hesitant-intuitionistic fuzzy weighted averaging (hesitant-IFWA) operators [36] have also been proposed. Xu and Xia [37] provided distance and similarity measures for HFPRs, and Zhu and Xu [34] suggested consistency measures for hesitant fuzzy linguistic preference relations. Many group decision-making methods have also been developed using HFPRs [38, 39].

The PR consistency, which is its important property, is required to ensure that an approach produces consistent results. Multiplicative and additive consistencies of FPRs [27, 40], are strict types of consistency. Experts have proposed many definitions of consistency for FPRs and their extensions; for example, Wang and Xu [35] proposed the consistency measures of PRs in extended hesitant fuzzy linguistic environments. Herrera-Viedma et al. [40] presented a new characterization of the consistency property based on the additive transitivity property of the FPRs. Zhu [41] developed methods for measuring consistency in HFPRs; Zhang et al. [42] discussed consistency in probabilistic linguistic preference relation.

HFPRs are effective approaches for decision making that assist DMs in describing their preferences whilst allowing for uncertainty. Most current works use FPRs to address the hesitant fuzzy environments with exact and crisp values. However, precisely quantifying the preference of an expert with a crisp number many be difficult for this expert under many conditions, although this expert can provide the lower and upper values and the most possible value when comparing two alternatives. Recent extensions of the HFPRs are insufficient for solving these problems. This study develops the THFPRs, based on the PRs and triangular hesitant fuzzy sets (THFSs) [43], to overcome the drawbacks of existing methods.

The remainder of this paper is organised as follows. Section 2 briefly reviews several basic concepts. Section 3 introduces the THFPRs and derives certain corresponding operators. Section 4 defines the consistent THFPR; and provides two methods for measuring the consistency of the THFPR. Section 5 proposes the MCDM-THFPR model. Section 6 presents a numerical example to validate the proposed approach which is compared with two previous methods. Section 7 elaborates the conclusions of this study.

2. Preliminaries

This section begins with the definitions of HFSs and HFNs, triangular fuzzy numbers (TFNs); and comparison rules of the TFNs in Definitions (1-3). Then, we provide the axiomatic definitions of triangular hesitant fuzzy numbers (THFNs) in Definition 4; on the basis of Definitions (1, 2). Furthermore, the comparison rules of the THFNs are provided in Definition 5 in accordance with Definitions (3, 4). We introduce the TFPR in Definition 6 on the basis of the FPR presented in Definition 5.

Definition 2.1 [44] Owing to a fixed set $\tilde{A}_z = \left(\tilde{a}_{zij}\right)$, an HFS on k in terms of a function θ when applied to $\tilde{P}$ is defined as follows:

$$A = \{\langle x, h_A(x)\rangle | \, x \in X\}$$

where $h_A(x)$ is a set of values in [0,1] that denote the possible degrees of $x \in X$ to the set A. For convenience, Xu and Xia [31] labelled $h = h_A(x)$ an HFN and H the set of all HFNs.

Definition 2.2 [23] A TFN a can be defined by a triplet $a = \left(a^L, a^M, a^U\right)$. The membership function $\mu_a(x)$ is defined as follows:

$$\mu_a(x) = \begin{cases} \left(x - a^L\right)/\left(a^M - a^L\right), a^L \leq x \leq a^M \\ \left(x - a^U\right)/\left(a^M - a^U\right), a^M \leq x \leq a^U \\ 0, otherwise \end{cases}$$

where $0 < a^L \leq a^M \leq a^U \leq 1$; a^L and a^U represent the lower and upper values of the support of a, respectively; and a^M is the modal value.

Definition 2.3 [45] Let $a = \left(a^L, a^M, a^U\right)$ be a TFN, $m(a)$ is the centre of the mean value and $\sigma(a)$ is the variance if they satisfy

$$m(a) = \left(a^L + a^M + a^U\right)/3$$

$$\sigma(a) = \sqrt{\left(a^{L2} + a^{M2} + a^{U2} - a^L a^M - a^M a^U - a^L a^U\right)/18}$$

Clearly, $a_1 > a_2$ if $m(a_1) > m(a_2)$ and $\sigma(a_1) < \sigma(a_2)$, in which people prefer the number with a larger mean value and smaller variance.

Definition 2.4 [46] Let X be a fixed set, a THFS E on X is in terms of a function $\tilde{h}_{g(x)}$ that returns several triangular fuzzy values, denoted by:

$$E = \left\{ \left\langle x, \tilde{h}_{g(x)} \right\rangle \middle| x \in X \right\}$$

where $\tilde{h}_{g(x)}$ is a set of several TFNs that denote the possible membership degrees of the element $x \in X$ to the set E. For convenience, $\tilde{h}_{g(x)}$ is called a THFN. Moreover,

$$\tilde{h}_{g(x)} = \left\{ \left(\gamma^L, \gamma^M, \gamma^U\right) \middle| \gamma \in \tilde{h}_{g(x)} \right\}$$

where γ is a TFN; γ^L and γ^U represent the lower and upper values, correspondingly; and γ^M denotes the modal value. The TFNs are in an ascending order in the THFN, following Definition 2.3.

Definition 2.5 [46] For a THFN $\tilde{h} = \left\{ \left(\gamma^L, \gamma^M, \gamma^U\right) \middle| \gamma \in \tilde{h} \right\}$, $s\left(\tilde{h}\right) = \frac{1}{\#\tilde{h}} \sum_{\gamma\in\tilde{h}} \left(\left(\lambda^L + \lambda^M + \lambda^U\right)/3\right)$ is called the score function of $\tilde{h}$, where $\#\tilde{h}$ is the number of the elements in $\tilde{h}$. For two HFNs $\tilde{h}_1$ and $\tilde{h}_2$, if $s\left(\tilde{h}_1\right) > s\left(\tilde{h}_2\right)$, then $\tilde{h}_1 > \tilde{h}_2$;if $s\left(\tilde{h}_1\right) = s\left(\tilde{h}_2\right)$, then $\tilde{h}_1 = \tilde{h}_2$.

Definition 2.6 [26, 47] An FPR P on a set of alternatives X is represented by a matrix $P \subset X \times X$ where $P = \left(p_{ij}\right)$ and $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $0 \leq p_{ij} \leq 1$, $\forall i, j \in \{1, 2, \ldots, n\}$. p_{ij} denotes the preference degree of alternative x_i over x_j.

Definition 2.7 [48] Let $A = \left(a_{ij}\right)_{n\times n}$ be an FPR, and $a_{ij} = \left(a_{ij}^L, a_{ij}^M, a_{ij}^U\right), i, j \in N$; Then, A is defined as a TFPR, if

$$a_{ii} = (0.5, 0.5, 0.5)$$

$$a_{ij}^L + a_{ji}^U = a_{ij}^M + a_{ji}^M = a_{ij}^U + a_{ji}^L = 1$$

and the complementary of a_{ij} is $a_{ij}^c = \left(1 - a_{ij}^U, 1 - a_{ij}^M, 1 - a_{ij}^L\right)$.

3. Triangular Hesitant Fuzzy Preference Relations (THFPRs)

In many practical situations, information is constantly uncertain and incomplete. This condition brings difficulty for experts to express their preference information using exact and crisp values. In this section, we present the THFPRs based on the TFPRs. Several aggregation operators and their properties are also provided.

Definition 3.1 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, and then the THFPR is defined as a matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times n}$ where $\tilde{a}_{ij}$ is a THFN with $\tilde{a}_{ij} = \left\{\gamma_{ij}^{h} \middle| h = 1,2,\ldots,\#\tilde{a}_{ij}\right\} = \left\{\left(\gamma_{ij}^{h^L}, \gamma_{ij}^{h^M}, \gamma_{ij}^{h^U}\right) \middle| h = 1,2,\ldots,\#\tilde{a}_{ij}\right\}$; $\tilde{a}_{ij}$ indicates several possible preference degrees of alternative x_i over alternative x_j; $\#\tilde{a}_{ij}$ is the number of TFNs in $\tilde{a}_{ij}$ and satisfies $\#\tilde{a}_{ij} = \#\tilde{a}_{ji}, 1 \le i \le j \le n$. Here, $\tilde{a}_{ij}^{h}$ is the hth least element in $\tilde{a}_{ij}$. $\tilde{a}_{ij}$ should satisfy $\gamma_{ij}^{h^L} + \gamma_{ji}^{\left(\#\tilde{a}_{ij}-h+1\right)^U} = 1$, $\gamma_{ij}^{h^U} + \gamma_{ji}^{\left(\#\tilde{a}_{ij}-h+1\right)^L} = 1$, $\gamma_{ij}^{h^M} + \gamma_{ji}^{\left(\#\tilde{a}_{ij}-h+1\right)^M} = 1$ when $1 \le i < j \le n$ and $\gamma_{ij}^{h^L} \le \gamma_{ij}^{h^M} \le \gamma_{ij}^{h^U} \in [0,1]$; moreover, $\gamma_{ii}^{h^L} = \gamma_{ii}^{h^M} = \gamma_{ii}^{h^U} = 0.5$ when $1 \le i \le n$

Definition 3.2 Let $\tilde{A}_z = \left(\tilde{a}_{zij}\right)_{n\times n}(z = 1,2,\ldots,k)$be a collection of the THFPRs, where $\tilde{a}_{zij} = \left\{\left(\gamma_{zij}^{h^L}, \gamma_{zij}^{h^M}, \gamma_{zij}^{h^U}\right) \middle| h = 1,2,\ldots,\#\tilde{a}_{zij}\right\}$; furthermore, let $W = (w_1, w_2, \ldots, w_k)^T$ be the weight vector with $w_z \in [0,1]$ and $\sum\limits_{z=1}^{k} w_z = 1$. Then, the triangular hesitant fuzzy preference relation weighted averaging (THFPRWA) operator is defined as follows.

$$THFPRWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k\right) = \left(\left\{\overset{k}{\underset{z=1}{\oplus}}\left(w_z \tilde{a}_{zij}\right)\right\}\right)_{n\times n}$$

$$= \left(\left\{\cup_{\tilde{\gamma}_{zij}\in\tilde{a}_{zij}}\left(\sum_{z=1}^{k} w_z \tilde{\gamma}_{zij}^{L}, \sum_{z=1}^{k} w_z \tilde{\gamma}_{zij}^{M}, \sum_{z=1}^{k} w_z \tilde{\gamma}_{zij}^{U}\right)\right\}\right)_{n\times n}, i,j \in \{1,2,\ldots,n\}, z \in \{1,2,\ldots,k\}$$

Theorem 1 The TFPR $\tilde{P} = \left(\tilde{p}_{ij}\right)_{n\times n}$ that is aggregated from a collection of THFPRs $\tilde{A}_z = \left(\tilde{a}_{zij}\right)_{n\times n}$ where $z = 1,2,\ldots,k$ and $i \le j = 1,2,\ldots,n$, by the THFPRWA operator is still a THFPR, where $W = (w_1, w_2, \ldots, w_k)^T$ is the weight vector of $\tilde{A}_z$ with $w_z \in [0,1]$ and $\sum\limits_{z=1}^{k} w_z = 1$.

Proof.

(1) Let $\#\tilde{p}_{ij}$ and $\#\tilde{a}_{zij}$ be the number of TFNs in $\tilde{p}_{ij}$ and $\tilde{a}_{zij}$. In accordance with the THFPRWA operator, we obtain $\#\left(\tilde{p}_{ij}\right) = \prod\limits_{z=1}^{k}\left(\#\tilde{a}_{zij}\right) = \prod\limits_{z=1}^{k}\left(\#\tilde{a}_{zji}\right) = \#\left(\tilde{p}_{ji}\right)$.

(2) Let $\underset{(p)}{\gamma}{}_{ij}^{h} = \left(\underset{(p)}{\gamma}{}_{ij}^{h^L}, \underset{(p)}{\gamma}{}_{ij}^{h^M}, \underset{(p)}{\gamma}{}_{ij}^{h^U}\right)$ and $\underset{(a_z)}{\gamma}{}_{ij}^{t_z} = \left(\underset{(a_z)}{\gamma}{}_{ij}^{t_z^{l}}, \underset{(a_z)}{\gamma}{}_{ij}^{t_z^{M}}, \underset{(a_z)}{\gamma}{}_{ij}^{t_z^{U}}\right)$ be the hth TFNs of $\tilde{p}_{ij}$ and t_zth least TFN of $\tilde{a}_{zij}$, respectively. Then, we can obtain $\underset{(p)}{\gamma}{}_{ij}^{h^L} = w_1 \underset{(a_1)}{\gamma}{}_{ij}^{t_1^L} + w_2 \underset{(a_2)}{\gamma}{}_{ij}^{t_2^L} + \ldots + w_k \underset{(a_k)}{\gamma}{}_{ij}^{t_k^L} = w_1\left(1 - \underset{(a_1)}{\gamma}{}_{ji}^{\left(\#a_{1ij}-t_1+1\right)^U}\right) + w_2\left(1 - \underset{(a_2)}{\gamma}{}_{ji}^{\left(\#a_{2ij}-t_2+1\right)^U}\right) + \ldots + w_k\left(1 - \underset{(a_k)}{\gamma}{}_{ji}^{\left(\#a_{kij}-t_k+1\right)^U}\right) = 1 - \underset{(p)}{\gamma}{}_{ji}^{\left(\#p_{ij}-h+1\right)^U}$. Similarly, we attain $\underset{(p)}{\gamma}{}_{ij}^{(h)^M} = 1 - \underset{(p)}{\gamma}{}_{ji}^{\left(\#p_{ij}-h+1\right)^M}$ and $\underset{(p)}{\gamma}{}_{ij}^{(h)^U} = 1 - \underset{(p)}{\gamma}{}_{ji}^{\left(\#p_{ij}-h+1\right)^L}$.

(3) Considering the above conditions, we can determine that $\underset{(p)}{\gamma}{}_{ij}^{h^L} = w_1 \underset{(a_1)}{\gamma}{}_{ij}^{t_1^L} + w_2 \underset{(a_2)}{\gamma}{}_{ij}^{t_2^L} + \ldots + w_k \underset{(a_k)}{\gamma}{}_{ij}^{t_k^L} \le w_1 \underset{(a_1)}{\gamma}{}_{ij}^{t_1^M} + w_2 \underset{(a_2)}{\gamma}{}_{ij}^{t_2^M} + \ldots + w_k \underset{(a_k)}{\gamma}{}_{ij}^{t_k^M} = \underset{(p)}{\gamma}{}_{ij}^{h^M} \le w_1 \underset{(a_1)}{\gamma}{}_{ij}^{t_1^U} + w_2 \underset{(a_2)}{\gamma}{}_{ij}^{t_2^U} + \ldots + w_k \underset{(a_k)}{\gamma}{}_{ij}^{t_k^U} = \underset{(p)}{\gamma}{}_{ij}^{h^U} \le w_1 \cdot 1 + w_2 \cdot 1 + \ldots + w_k \cdot 1 = 1$

Similarly, we obtain $\underset{(p)}{\gamma}{}_{ij}^{h^L} = w_1 \underset{(a_1)}{\gamma}{}_{ij}^{t_1^L} + w_2 \underset{(a_2)}{\gamma}{}_{ij}^{t_2^L} + \ldots + w_k \underset{(a_k)}{\gamma}{}_{ij}^{t_k^L} \ge w_1 \cdot 0 + w_2 \cdot 0 + \ldots + w_k \cdot 0 = 0$

Thus, we express $0 \le \underset{(p)}{\gamma}{}_{ij}^{h^L} \le \underset{(p)}{\gamma}{}_{ij}^{h^M} \le \underset{(p)}{\gamma}{}_{ij}^{h^U} \le 1$.

(4) When $1 \le i \le n$, we can obtain $\underset{(p)}{\gamma}{}_{ii}^{h^L} = w_1 \underset{(a_1)}{\gamma}{}_{ii}^{t_1^L} + w_2 \underset{(a_2)}{\gamma}{}_{ii}^{t_2^L} + \ldots + w_k \underset{(a_k)}{\gamma}{}_{ii}^{t_k^L} = w_1 \cdot 0.5 + w_2 \cdot 0.5 + \ldots + w_k \cdot 0.5 = 0.5$

Similarly, we can obtain $\underset{(p)}{\gamma}{}_{ii}^{h^M} = 0.5$ and $\underset{(p)}{\gamma}{}_{ii}^{h^U} = 0.5$, when $1 \le i \le n$.

Furthermore, the ordered weighted operator (OWA) which is a popular aggregation operator allows DMs to combine the PR values in accordance with a set of weights. We can extend the OWA operator to aggregate THFPRs.

Definition 3.3 Let $\tilde{A}_y = \left(\tilde{a}_{yij}\right)_{n\times n}(y = 1, 2, \ldots, k)$ be a collection of the THFPRs, where $\tilde{a}_{yij} = \left\{\left(\gamma_{yij}^{h^L}, \gamma_{yij}^{h^M}, \gamma_{yij}^{h^U}\right)\middle| h = 1, 2, \ldots, \#\tilde{a}_{yij}\right\}$, and then the THFPR ordered weighted averaging (THFPROWA) operator is defined as follows:

$THFPROWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k\right) = \left(\left\{\overset{k}{\underset{z=1}{\oplus}} \left(w_{zij}\tilde{a}_{zij}\right)\right\}\right)_{n\times n}$,

where $\left\{\overset{k}{\underset{z=1}{\oplus}} \left(w_{zij}\tilde{a}_{zij}\right)\right\} = \left\{\bigcup_{\tilde{\beta}_{zij}\in\tilde{a}_{zij}} \left(\sum\limits_{z=1}^{k} w_{zij}\tilde{\beta}_{zij}^{L}, \sum\limits_{z=1}^{k} w_{zij}\tilde{\beta}_{zij}^{M}, \sum\limits_{z=1}^{k} w_{zij}\tilde{\beta}_{zij}^{U}\right)\right\}$, $i, j \in \{1, 2, \cdots, n\}$, $z \in \{1, 2, \ldots, k\}$ and $\overset{k}{\underset{z=1}{\oplus}} \left(w_{zij}\tilde{a}_{zij}\right) = (0.5, 0.5, 0.5), 1 \le i = j \le n$. $\tilde{a}_{zij} = \left\{\left(\beta_{zij}^{h^L}, \beta_{zij}^{h^M}, \beta_{zij}^{h^U}\right)\middle| h = 1, 2, \ldots, \#\tilde{a}_{zij}\right\}$ is the zth largest element of the HTFN $\left\{\tilde{a}_{yij}\middle| y = 1, 2, \ldots, k\right\}$, and $W_{ij} = \left(w_{1ij}, w_{2ij}, \ldots, w_{kij}\right)^T$ is the weight vector that satisfies $w_{zij} \in [0, 1]$, $\sum\limits_{z=1}^{k} w_{zij} = 1$, $w_{zij} = Q(z/k) - Q((z-1)/k)$ and $z = 1, \ldots, k$, where $Q(\gamma) = \begin{cases} 0,\ \gamma < \alpha \\ (\gamma-\alpha)/(\beta-\alpha)\ ,\ \alpha \le \gamma \le \beta \\ 1,\ \gamma > \beta \end{cases}$

Clearly, $w_{zij} = w_{zji}$ when $\alpha = 0.25$ and $\beta = 0.75$. The weights can be obtained by the above mentioned equation. For example, if three THFPRs, that is, $\tilde{A}_1 = \left(\tilde{a}_{1ij}\right)_{4\times4}$, $\tilde{A}_2 = \left(\tilde{a}_{2ij}\right)_{4\times4}$, and $\tilde{A}_3 = \left(\tilde{a}_{3ij}\right)_{4\times4}$, are available, and $\widetilde{a}_{111} = (0.2, 0.5, 0.6)$, $\widetilde{a}_{211} = (0.4, 0.5, 0.6)$, and $\widetilde{a}_{311} = (0.3, 0.4, 0.7)$. We can calculate the score function of these three THFNs, and obtain the ranking result $\widetilde{a}_{211} > \widetilde{a}_{311} > \widetilde{a}_{111}$. Then $w_{211} = Q(1/3) - Q(0) = 1/6 - 0 = 1/6$, $w_{311} = Q(2/3) - Q(1/3) = 5/6 - 1/6 = 2/3$ and $w_{111} = Q(1) - Q(2/3) = 1 - 5/6 = 1/6$.

Theorem 2 If $\tilde{P} = \left(\tilde{p}_{ij}\right)_{n\times n} = \left(\left\{\left(\gamma_{ij}^{t^L}, \gamma_{ij}^{t^M}, \gamma_{ij}^{t^U}\right)\middle| t = 1, 2, \ldots, \#\tilde{p}_{ij}\right\}\right)_{n\times n}$ is a TFPR that is aggregated from a collection of THFPRs $\tilde{A}_y = \left(\tilde{a}_{yij}\right)_{n\times n}(y = 1, 2, \ldots, k)$, where $\tilde{a}_{yij} = \left\{\left(\gamma_{yij}^{h^L}, \gamma_{yij}^{h^M}, \gamma_{yij}^{h^U}\right)\middle| h = 1, 2, \ldots, \#\tilde{a}_{yij}\right\}$, $i \le j = 1, 2, \ldots, n$ by the THFPROWA operator, $\tilde{a}_{zij} = \left\{\left(\beta_{zij}^{h_z{}^L}, \beta_{zij}^{h_z{}^M}, \beta_{zij}^{h_z{}^U}\right)\middle| h_z = 1, 2, \ldots, \#\tilde{a}_{zij}\right\}$ is the zth largest element of the HTFN $\left\{\tilde{a}_{yij}\middle| y = 1, 2, \ldots, k\right\}$, then $\tilde{P}$ remains a THFPR.

Proof.

(1) Let $\#\tilde{p}_{ij}$ represents the number of TFNs in the THFN $\tilde{p}_{ij}$, $\#\tilde{a}_{yij}$ denotes the number of TFNs in the THFN $\tilde{a}_{yij}$ of the THFPR $\tilde{A}_y$ and $\#\tilde{a}_{zij}$ represents the number of TFNs in the THFN $\tilde{a}_{zij}$. In accordance with the THFPROWA operator, we can observe $\#\left(\tilde{p}_{ij}\right) = \prod\limits_{z=1}^{k}\left(\#\tilde{a}_{zij}\right) = \prod\limits_{y=1}^{k}\left(\#\tilde{a}_{yij}\right) = \prod\limits_{y=1}^{k}\left(\#\tilde{a}_{yji}\right) = \prod\limits_{z=1}^{k}\left(\#\tilde{a}_{zji}\right) = \#\left(\tilde{p}_{ji}\right)$

.(2) If $\gamma_{ij}^{t} = \left(\gamma_{ij}^{t^L}, \gamma_{ij}^{t^M}, \gamma_{ij}^{t^U}\right)$ is the hth least TFN of $\tilde{p}_{ij}$, then let $\beta_{ij}^{h_z} = \left(\beta_{ij}^{h_z{}^L}, \beta_{ij}^{h_z{}^M}, \beta_{ij}^{h_z{}^U}\right)$ be the h_zth least TFN of $\tilde{a}_{zij}$. We can obtain $\gamma_{ij}^{t^L} = w_{1ij}\beta_{1ij}^{h_{1q}{}^L} + w_{2ij}\beta_{2ij}^{h_{2q}{}^L} + \ldots + w_{kij}\beta_{kij}^{h_{kq}{}^L} = w_{1ij}\left(1 - \beta_{1ji}^{\left(\#a_{1ij}-h_1+1\right)^U}\right) + w_{2ij}\left(1 - \beta_{2ji}^{\left(\#a_{2ij}-h_2+1\right)^U}\right) + \ldots + w_{kij}\left(1 - \beta_{kji}^{\left(\#a_{kij}-h_k+1\right)^U}\right) = w_{1ij} + w_{2ij} + \ldots + w_{kij} - \left(w_{1ij}\beta_{1ji}^{\left(\#a_{1ij}-h_1+1\right)^U} + w_{2ij}\beta_{2ji}^{\left(\#a_{2ij}-h_2+1\right)^U} + \ldots + w_{kij}\beta_{kji}^{\left(\#a_{kij}-h_k+1\right)^U}\right) = 1 - \gamma_{ji}^{\left(\#p_{ij}-t+1\right)^U}$

Similarly, we obtain $\gamma_{ij}^{t^M} = 1 - \gamma_{ji}^{\left(\#p_{ij}-t+1\right)^M}$ and $\gamma_{ij}^{t^U} = 1 - \gamma_{ji}^{\left(\#p_{ij}-t+1\right)^L}$.

(3) Owing to the above mentioned conditions, we can easily achieve $\gamma_{ij}^{t^L} = w_{1ij}\beta_{1ij}^{h_{1q}{}^L} + w_{2ij}\beta_{2ij}^{h_{2q}{}^L} + \ldots + w_{kij}\beta_{kij}^{h_{kq}{}^L} \le w_{1ij}\beta_{1ij}^{h_1{}^M} + w_{2ij}\beta_{2ij}^{h_2{}^M} + \ldots + w_{kij}\beta_{kij}^{h_k{}^M} = \gamma_{ij}^{t^M} \le w_{1ij}\beta_{1ij}^{h_1{}^U} + w_{2ij}\beta_{2ij}^{h_2{}^U} + \ldots + w_{kij}\beta_{kij}^{h_k{}^U} = \gamma_{ij}^{t^U} \le w_1\cdot1 + w_2\cdot1 + \ldots + w_k\cdot1 = 1$

Similarly, we obtain $\gamma_{ij}^{t^L} = w_{1ij}\beta_{1ij}^{h_{1q}{}^L} + w_{2ij}\beta_{2ij}^{h_{2q}{}^L} + \ldots + w_{kij}\beta_{kij}^{h_{kq}{}^L} \ge w_1\cdot0 + w_2\cdot0 + \ldots + w_k\cdot0 = 0$

Thus, we obtain $0 \le \gamma_{ij}^{h^L} \le \gamma_{ij}^{h^M} \le \gamma_{ij}^{h^U} \le 1$.

(4) When $1 \le i \le n$, we can obtain $\gamma_{ii}^{t^L} = w_{1ii}\beta_{1ii}^{h_{1q}{}^L} + w_{2ii}\beta_{2ii}^{h_{2q}{}^L} + \ldots + w_{kii}\beta_{kii}^{h_{kq}{}^L} = w_1 \cdot 0.5 + w_2 \cdot 0.5 + \ldots + w_k \cdot 0.5 = 0.5$
Similarly, we can obtain $\gamma_{ii}^{t^M} = 0.5$ and $\gamma_{ii}^{t^U} = 0.5$, when $1 \le i \le n$.

Property 1 The THFPROWA operator satisfies the following properties.

(1) (Boundedness) Let $\tilde{A}_y (y = 1, 2, \ldots, k)$ be a THFPR where $\tilde{A}_y = \left(\tilde{a}_{yij}\right)_{n\times n}$ and $\tilde{a}_{yij} = \left\{\left(\gamma_{yij}^{h_y{}^L}, \gamma_{yij}^{h_y{}^M}, \gamma_{yij}^{h_y{}^U}\right) \middle| h_y = 1, 2, \ldots, \#\tilde{a}_{yij}\right\}$. If $\tilde{P} = THFPROWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k\right)$ where $\tilde{P} = \left(\tilde{p}_{ij}\right)_{n\times n} = \left(\left\{\left(\gamma_{ij}^{t^L}, \gamma_{ij}^{t^M}, \gamma_{ij}^{t^U}\right) \middle| t = 1, 2, \ldots, \#\tilde{p}_{ij}\right\}\right)_{n\times n}$ then $\min_{y=1}^{k} \left\{\min\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\} \le \gamma_{ij}^{t^L} \le \max_{y=1}^{k} \left\{\max\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\}$, $\min_{y=1}^{k} \left\{\min\left\{\gamma_{yij}^{h_y{}^M}\right\}\right\} \le \gamma_{ij}^{t^M} \le \max_{y=1}^{k} \left\{\max\left\{\gamma_{yij}^{h_y{}^M}\right\}\right\}$, and $\min_{y=1}^{k}$ {min $\left\{\gamma_{yij}^{h_y{}^U}\right\}\right\}. \le \gamma_{ij}^{t^U} \le \max_{y=1}^{k} \left\{\max\left\{\gamma_{yij}^{h_y{}^U}\right\}\right\}$, for any $t \in \left\{1, 2, \ldots, \#\tilde{p}_{ij}\right\}$.

Proof.

If $\gamma_{ij}^{t^L} = w_{1ij}\beta_{1ij}^{h_{1q}{}^L} + w_{2ij}\beta_{2ij}^{h_{2q}{}^L} + \ldots + w_{kij}\beta_{kij}^{h_{kq}{}^L}$, where $\beta_{zij}^{h_{zq}{}^L} \in \left\{\beta_{zij}^{h_z{}^L} \middle| h_z = 1, 2, \ldots, \#\tilde{a}_{zij}\right\}$, then

$\gamma_{ij}^{t^L} = w_{1ij}\beta_{1ij}^{h_{1q}{}^L} + w_{2ij}\beta_{2ij}^{h_{2q}{}^L} + \ldots + w_{kij}\beta_{kij}^{h_{kq}{}^L} \le w_{1ij}\max\left\{\beta_{1ij}^{h_1{}^L}\right\} + w_{2ij}\max\left\{\beta_{2ij}^{h_2{}^L}\right\} + \ldots + w_{kij}\max\left\{\beta_{kij}^{h_k{}^L}\right\} \le$ $w_{1ij}\max\left\{\max\left\{\beta_{1ij}^{h_1{}^L}\right\}, \max\left\{\beta_{2ij}^{h_2{}^L}\right\}, \ldots, \max\left\{\beta_{kij}^{h_k{}^L}\right\}\right\} + w_{2ij}\max\left\{\max\left\{\beta_{1ij}^{h_1{}^L}\right\}, \max\left\{\beta_{2ij}^{h_2{}^L}\right\}, \ldots, \max\left\{\beta_{kij}^{h_k{}^L}\right\}\right\} + \ldots +$ $w_{kij}\max\left\{\max\left\{\beta_{1ij}^{h_1{}^L}\right\}, \max\left\{\beta_{2ij}^{h_2{}^L}\right\}, \ldots, \max\left\{\beta_{kij}^{h_k{}^L}\right\}\right\} = \left(w_{1ij} + w_{2ij} + \ldots + w_{kij}\right)\max_{y=1}^{k}\left\{\max\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\} = \max_{y=1}^{k}$ $\left\{\max\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\}$.

Similarly, we can obtain $\gamma_{ij}^{t^L} \ge \min_{y=1}^{k}\left\{\min\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\}$, which implies that $\min_{y=1}^{k}\left\{\min\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\} \le \gamma_{ij}^{t^L} \le \max_{y=1}^{k}\left\{\max\left\{\gamma_{yij}^{h_y{}^L}\right\}\right\}$. The conditions of $\gamma_{ij}^{t^L}$ are the same as those of $\gamma_{ij}^{t^M}$ and $\gamma_{ij}^{t^U}$.

(2) (Monotonicity) Let $\tilde{P} = \left(\tilde{p}_{ij}\right)_{n\times n} = \left(\left\{\left(\gamma_{ij}^{t^L}, \gamma_{ij}^{t^M}, \gamma_{ij}^{t^U}\right) \middle| t = 1, 2, \ldots, \#\tilde{p}_{ij}\right\}\right)_{n\times n}$, $\tilde{P}' = \left(\tilde{p}'_{ij}\right)_{n\times n}$ be two THFPRs that are aggregated from the THFPRs $\tilde{A}_y = \left(\tilde{a}_{yij}\right)_{n\times n}$ $(y = 1, 2, \ldots, k)$ and $\tilde{A}'_y = \left(\tilde{a}'_{yij}\right)_{n\times n}$ $(y = 1, 2, \ldots, k)$ by using the THRPROWA operator. Then. we obtain

$\tilde{p}_{ij} \le \tilde{p}'_{ij}$, for any $i, j \in \{1, 2, \ldots, n\}$, if $\tilde{a}_{yij} \le \tilde{a}'_{yij}$ and $\#\tilde{a}_{yij} = \#\tilde{a}'_{yij}$ for any $y = 1, 2, \ldots, k$, and $i, j \in \{1, 2, \ldots, n\}$.

(3) (Commutativity) If $\tilde{A}_y = \left(\tilde{a}_{yij}\right)_{n\times n} (y = 1, 2, \ldots, k)$ and $\tilde{A}'_y = \left(\tilde{a}'_{yij}\right)_{n\times n} (y = 1, 2, \ldots, k)$ are two THFPRs. Then

$THFPROWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k\right) = THFPROWA\left(\tilde{A}'_1, \tilde{A}'_2, \ldots, \tilde{A}'_k\right)$ (or $\tilde{P} = \tilde{P}'$)

4. Consistency in the THFPRs

Consistency is an important property in PRs because the lack of consistency can lead to inconsistent conclusions [47]. This section firstly presents the operational laws of the THFPRs, then describes the consistency of the THFPRs, and lastly proposes two consistency tests.

Definition 4.1 [49] If $R = \left(r_{ij}\right)_{n\times n}$ be a TFPR, where $r_{ij} = \left(r_{ij}^L, r_{ij}^M, r_{ij}^U\right)$, then R is defined as an additive consistent matrix if

$r_{ij}^M + r_{jk}^M + r_{ki}^M = 3/2;\ r_{ij}^L + r_{jk}^L + r_{ki}^U = 3/2;\ r_{ij}^U + r_{jk}^U + r_{ki}^L = 3/2, \forall i, j, k \in \{1, 2, \ldots, n\}$.

Definition 4.2 Let $\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times n}$ be a THFPR where $\tilde{a}_{ij} = \left\{\left(\gamma_{ij}^{h^L}, \gamma_{ij}^{h^M}, \gamma_{ij}^{h^U}\right) \middle| h = 1, 2, \ldots, \#\tilde{a}_{ij}\right\}$. Then the consistency measure matrix is $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n} = \left(\left(\bar{\gamma}_{ij}^L, \bar{\gamma}_{ij}^M, \bar{\gamma}_{ij}^U\right)\right)_{n\times n}$, and $\bar{a}_{ij}$ is defined as follows:

$$\bar{a}_{ij} = Cm\left(\tilde{a}_{ij}\right) = \left(\left(\sum_{h=1}^{\#\tilde{a}_{ij}} \gamma_{ij}^{h^L}\right)\Big/\#\tilde{a}_{ij}, \left(\sum_{h=1}^{\#\tilde{a}_{ij}} \gamma_{ij}^{h^M}\right)\Big/\#\tilde{a}_{ij}, \left(\sum_{h=1}^{\#\tilde{a}_{ij}} \gamma_{ij}^{h^U}\right)\Big/\#\tilde{a}_{ij}\right) \quad (1)$$

Definition 4.3 Let $\bar{a}^L_{ij}+\bar{a}^L_{jl}+\bar{a}^U_{li}=\frac{3}{2}$; be a matrix that transformed from a THFPR $\bar{a}^M_{ij}+\bar{a}^M_{jl}+\bar{a}^M_{li}=\frac{3}{2}$; using Eq. (1), then $\tilde{A}$ is defined as a consist matrix if

$\bar{a}^L_{ij}+\bar{a}^L_{jl}+\bar{a}^U_{li}=3/2;\bar{a}^M_{ij}+\bar{a}^M_{jl}+\bar{a}^M_{li}=3/2;\bar{a}^U_{ij}+\bar{a}^U_{jl}+\bar{a}^L_{li}=3/2$ for any $i<j<l$.

Theorem 3 If $\tilde{P}=\left(\tilde{p}_{ij}\right)_{n\times n}$ is aggregated from consistent THFPRs $\tilde{A}_z=\left(\tilde{a}_{zij}\right)_{n\times n}(z=1,2,\ldots,k)$ for $i,j\in\{1,2,\ldots,n\}$, by using THFPRWA operator, where $THFPRWA\left(\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_z\right)=\overset{k}{\underset{z=1}{\oplus}}\left(w_z\tilde{A}_z\right)$, then $\tilde{A}_z=\left(\tilde{a}_{zij}\right)_{n\times n}=\left(\left\{\left(\gamma^{h_z{}^L}_{zij},\gamma^{h_z{}^M}_{zij},\gamma^{h_z{}^U}_{zij}\right)\middle| h_z=1,2,\ldots,\#\tilde{a}_{zij}\right\}\right)_{n\times n}$; $W=(w_1,w_2,\ldots,w_k)^T$ is the weight vector of $\tilde{A}_z$ with $w_z\in[0,1]$ and $\sum\limits_{z=1}^{k}w_z=1$; then $\tilde{P}=\left(\tilde{p}_{ij}\right)_{n\times n}$ remains a consistent THFPR.

Proof.

Let $\bar{A}_z=\left(\bar{a}_{zij}\right)_{n\times n}$, $\bar{a}_{zij}=Cm\left(\tilde{a}_{zij}\right)=\frac{1}{\#\tilde{a}_{zij}}\left(\sum\limits_{h_z=1}^{\#\tilde{a}_{zij}}\gamma^{h_z{}^L}_{zij},\sum\limits_{h_z=1}^{\#\tilde{a}_{zij}}\gamma^{h_z{}^M}_{zij},\sum\limits_{h_z=1}^{\#\tilde{a}_{zij}}\gamma^{h_z{}^U}_{zij}\right)$, $\bar{P}=\left(\bar{p}_{ij}\right)_{n\times n}$ where $\bar{p}_{ij}=Cm\left(\tilde{p}_{ij}\right)=\frac{1}{\#\tilde{p}_{ij}}\left(\sum\limits_{H=1}^{\#\tilde{p}_{ij}}\gamma^{H^L}_{ij},\sum\limits_{H=1}^{\#\tilde{p}_{ij}}\gamma^{H^M}_{ij},\sum\limits_{H=1}^{\#\tilde{p}_{ij}}\gamma^{H^U}_{ij}\right)$; $\#\tilde{p}_{ij}=\prod\limits_{z=1}^{k}\#\tilde{a}_{zij}$ is the number of TFNs in $\tilde{p}_{ij}$, and $\#\tilde{a}_{zij}$ is the number of TFNs in $\tilde{a}_{zij}$. Then we obtain

$$\bar{p}^L_{ij}=\frac{1}{\#\tilde{p}_{ij}}\sum_{H=1}^{\#\tilde{p}_{ij}}\gamma^{H^L}_{ij}=\frac{1}{\#\tilde{p}_{ij}}\left(w_1\frac{\#\tilde{p}_{ij}}{\#\tilde{a}_{1ij}}\sum_{T_{ij1}=1}^{\#\tilde{a}_{1ij}}\gamma^{T_{ij1}{}^L}_{1ij}+w_2\frac{\#\tilde{p}_{ij}}{\#\tilde{a}_{2ij}}\sum_{T_{ij2}=1}^{\#\tilde{a}_{2ij}}\gamma^{T_{ij2}{}^L}_{2ij}+\ldots+w_k\frac{\#\tilde{p}_{ij}}{\#\tilde{a}_{kij}}\sum_{T_{ijk}=1}^{\#\tilde{a}_{kij}}\gamma^{T_{ijk}{}^L}_{kij}\right)=\frac{1}{\#\tilde{p}_{ij}}\sum_{z=1}^{k}\left(w_z\frac{\#\tilde{p}_{ij}}{\#\tilde{a}_{zij}}\sum_{T_{ijz}=1}^{\#\tilde{a}_{zij}}\gamma^{T_{ijz}{}^L}_{zij}\right)=\sum_{z=1}^{k}\left(\frac{w_z}{\#\tilde{a}_{zij}}\sum_{T_{ijz}=1}^{\#\tilde{a}_{zij}}\gamma^{T_{ijz}{}^L}_{zij}\right)$$

Furthermore, we obtain

$$\bar{p}^L_{ij}+\bar{p}^L_{jl}+\bar{p}^U_{li}=\sum_{z=1}^{k}\left(\frac{w_z}{\#\tilde{a}_{zij}}\sum_{T_{zij}=1}^{\#\tilde{a}_{zij}}\gamma^{T_{zij}{}^L}_{zij}\right)+\sum_{z=1}^{k}\left(\frac{w_z}{\#\tilde{a}_{zji}}\sum_{T_{zjl}=1}^{\#\tilde{a}_{zji}}\gamma^{T_{zjl}{}^L}_{zjl}\right)+\sum_{z=1}^{k}\left(\frac{w_z}{\#\tilde{a}_{zli}}\sum_{Tzli=1}^{\#\tilde{a}_{zli}}\gamma^{T_{zli}{}^U}_{zli}\right)=\sum_{z=1}^{k}w_z\left(\frac{1}{\#\tilde{a}_{zij}}\sum_{T_{zij}=1}^{\#\tilde{a}_{zij}}\gamma^{T_{zij}{}^L}_{zij}+\frac{1}{\#\tilde{a}_{zji}}\sum_{T_{zjl}=1}^{\#\tilde{a}_{zji}}\gamma^{T_{zjl}{}^L}_{zjl}+\frac{1}{\#\tilde{a}_{zli}}\sum_{Tzli=1}^{\#\tilde{a}_{zli}}\gamma^{T_{zli}{}^U}_{zli}\right)=\sum_{z=1}^{k}w_z\left(\bar{a}^L_{zij}+\bar{a}^L_{zjl}+\bar{a}^U_{zli}\right)=\sum_{z=1}^{k}\frac{3}{2}w_z=\frac{3}{2}$$

Similarly, we can obtain $\bar{p}^M_{ij}+\bar{p}^M_{jl}+\bar{p}^M_{li}=\frac{3}{2}$ and $\bar{p}^U_{ij}+\bar{p}^U_{jl}+\bar{p}^L_{li}=\frac{3}{2}$.

Thus, we can see that the THFPR $\tilde{P}=\left(\tilde{p}_{ij}\right)_{n\times n}$ is consistent.

Following [5], we provide the properties of THFPR as follows.

Property 2: A consistent THFPR $\tilde{A}$ satisfies weak transitivity as follows:

If $\bar{a}_{ij}\geq(0.5,0.5,0.5)$ and $\bar{a}_{jl}\geq(0.5,0.5,0.5)$, then $\bar{a}_{il}\geq(0.5,0.5,0.5)$ for any $i<j<l\in\{1,\ldots,n\}$.

This condition can be interpreted by considering that if x_i is preferred to x_j and x_j is preferred to x_l, then x_i should be preferred to x_l. Weak transitivity is a necessary condition if DMs are logical and rational. Therefore, a consistent THFPR should satisfy the above mentioned condition at the minimum.

Proof.

Let $\bar{A}=\left(\bar{a}_{ij}\right)_{n\times n}$ be a consistency measure matrix of the THFPR $\tilde{A}=\left(\tilde{a}_{ij}\right)_{n\times n}$ and $\tilde{A}$ be a consistency THFPR, and then $\bar{a}^L_{ij}+\bar{a}^L_{jl}=\bar{a}^L_{il}+0.5$. If $\bar{a}^L_{ij}\geq 0.5$ and $\bar{a}^L_{jl}\geq 0.5$, then we can determine that $\bar{a}^L_{il}=\bar{a}^L_{ij}+\bar{a}^L_{jl}-0.5\geq 0.5+0.5-0.5=0.5$

Similarly, we obtain $\bar{a}^M_{il}\geq 0.5$ and $\bar{a}^U_{il}\geq 0.5$ from $\bar{a}^M_{ij}+\bar{a}^M_{jl}=\bar{a}^M_{il}+0.5$ and $\bar{a}^U_{ij}+\bar{a}^U_{jl}=\bar{a}^U_{il}+0.5$, correspondingly. If $\bar{a}_{ij}\geq(0.5,0.5,0.5)$ and $\bar{a}_{jl}\geq(0.5,0.5,0.5)$, then $\bar{a}_{il}\geq(0.5,0.5,0.5)$.

Property 3: A consistency THFPR $\tilde{A}$ satisfies the max-min transitivity as follows: if $\bar{a}_{ij}\geq(0.5,0.5,0.5)$and $\bar{a}_{jl}\geq(0.5,0.5,0.5)$, then $\bar{a}_{il}\geq\min\left(\bar{a}_{ij},\bar{a}_{jl}\right)$ for any $i<j<l\in\{1,\ldots,n\}$.

This type of transitivity has the following interpretation: if an alternative x_i is preferred to x_j with the value a_{ij} and x_j is preferred to x_l with a_{jl}, then the preference value that is gained from a direct comparison between x_i and x_l should be equal or greater than the minimum partial values of a_{ij} and a_{jl}. A restricted max-min transitivity is a necessary requirement for characterising the consistency of a THFPR.

Proof.

Let $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ be a consistency measure matrix of the THFPR $\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times n}$ and $\tilde{A}$ be a consistency THFPR, and then $\bar{a}_{ij}^{L} + \bar{a}_{jl}^{L} = \bar{a}_{il}^{L} + 0.5$, if $\bar{a}_{ij}^{L} \geq 0.5$ and $\bar{a}_{jl}^{L} \geq 0.5$. we can determine that $\bar{a}_{il}^{L} = \bar{a}_{ij}^{L} + \bar{a}_{jl}^{L} - 0.5 \geq \min\left(\bar{a}_{ij}^{L}, \bar{a}_{jl}^{L}\right) + \min\left(\bar{a}_{ij}^{L}, \bar{a}_{jl}^{L}\right) - 0.5 \geq \min\left(\bar{a}_{ij}^{L}, \bar{a}_{jl}^{L}\right) + 0.5 - 0.5 \geq \min\left(\bar{a}_{ij}^{L}, \bar{a}_{jl}^{L}\right)$

Similarly, we obtain $\bar{a}_{il}^{M} \geq \min\left(\bar{a}_{ij}^{M}, \bar{a}_{jl}^{M}\right)$ and $\bar{a}_{il}^{U} \geq \min\left(\bar{a}_{ij}^{U}, \bar{a}_{jl}^{U}\right)$ from $\bar{a}_{ij}^{M} + \bar{a}_{jl}^{M} = \bar{a}_{il}^{M} + 0.5$ and $\bar{a}_{ij}^{U} + \bar{a}_{jl}^{U} = \bar{a}_{il}^{U} + 0.5$. Then, we can draw the following conclusion.

From Definition 4.1, the additive consistency is found to be strict. The consistency measurement defined in Definition 4.3 enriches the theory of THFPR which is too strict to be satisfied under realistic environment. We provide the following definition of another method for measuring consistency in order to overcome this difficulty. This method provides a parameter for DMs to select the more flexible method than that in Definition 4.3.

Definition 4.4 Let $\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times n}$ be a THFPR where $\tilde{a}_{ij} = \left\{\left(\gamma_{ij}^{h^L}, \gamma_{ij}^{h^M}, \gamma_{ij}^{h^U}\right) \middle| h = 1, 2, \ldots, \#\tilde{a}_{ij}\right\}$, and $\bar{A} = \left(\bar{a}_{ij}\right)_{n\times n}$ be the consistency measure matrix where $\bar{a}_{ij} = \left(\bar{a}_{ij}^{L}, \bar{a}_{ij}^{M}, \bar{a}_{ij}^{U}\right)$. Then $\hat{A} = \left(\hat{a}_{ij}\right)_{n\times n}$ with $\hat{a}_{ij} = \left(\hat{a}_{ij}^{L}, \hat{a}_{ij}^{M}, \hat{a}_{ij}^{U}\right)$ is defined as a perfect consistent THFPR if

$$\hat{a}_{ij} = \begin{cases} \left(\frac{1}{j-i-1}\sum\limits_{l=i+1}^{j-1}\left(\bar{a}_{il}^{U} + \bar{a}_{lj}^{U} - 0.5\right), \frac{1}{j-i-1}\sum\limits_{l=i+1}^{j-1}\left(\bar{a}_{il}^{M} + \bar{a}_{lj}^{M} - 0.5\right), \frac{1}{j-i-1}\sum\limits_{l=i+1}^{j-1}\left(\bar{a}_{il}^{L} + \bar{a}_{lj}^{L} - 0.5\right)\right), i+1 < j \\ \left(\bar{a}_{ij}^{L}, \bar{a}_{ij}^{M}, \bar{a}_{ij}^{U}\right), i+1 = j \\ (0.5, 0.5, 0.5), i = j \\ \left(1 - \bar{a}_{ji}^{U}, 1 - \bar{a}_{ji}^{M}, 1 - \bar{a}_{ji}^{L}\right), i > j \end{cases}$$

Then $\tilde{A} = \left(\tilde{a}_{ij}\right)_{n\times n}$ is defined as an acceptable consistent matrix if $d\left(\bar{A}, \hat{A}\right) < \theta_0$, where $d\left(\bar{A}, \hat{A}\right)$ is the distance between $\bar{A}$ and $\hat{A}$. The parameter θ_0 represents the consistency level of the THFPR $\tilde{A}$. Obviously, $0 \leq \theta_0 \leq 1$, $\tilde{A}$ is a completely consistent THFPR where $\theta_0 = 0$, $\tilde{A}$ has no consistency where $\theta_0 = 1$. Without loss of generality, we select θ_0=0.1 for this study.

5. Multi-criteria Group Decision-making Approach with Triangular Hesitant Fuzzy Information

This section introduces an MCDM-THFPR model to help a group of DMs evaluate and rank several alternatives. The proposed method based on THFPRs is described in Section 5.1, and a systematic MCDM-THFPR model is discussed in Section 5.2.

5.1. MCDM-THFPR method based on THFPRs

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of n alternatives and $C = \{c_1, c_2, \ldots, c_k\}$ be a collection of k criteria with the weight vector $W = (w_1, w_2, \ldots, w_k)^T$, $w_z \in [0, 1]$, where $z = 1, 2, \ldots, k$, and let $\sum\limits_{z=1}^{k} w_z = 1$. d experts $\{n_1, n_2, \ldots, n_d\}$ provide their preferences $\tilde{\gamma}_{zij}^{h}$ $(h = 1, 2, ..., d)$ which indicates the hth DM provides the preference degree of alternative x_i over the alternative x_j under the criterion z.

Criteria are divided into benefit criteria and cost criteria in a multi-criteria group decision-making problem. To solve this problem, we formulate the following transformation:

$$\tilde{a}_{zij} = \begin{cases} \left\{\left(\gamma_{zij}^{h^L}, \gamma_{zij}^{h^M}, \gamma_{zij}^{h^U}\right) \middle| h = 1, 2, \ldots, \#\tilde{a}_{zij}\right\}, forthebenefitattribute \\ \left\{\left(1 - \gamma_{zij}^{h^U}, 1 - \gamma_{zij}^{h^M}, 1 - \gamma_{zij}^{h^L}\right) \middle| h = 1, 2, \ldots, \#\tilde{a}_{zij}\right\}, forthe\cos tattribute \end{cases} \tag{2}$$

where $\tilde{A}_z = \left(\tilde{a}_{zij}\right)_{n\times n} (z = 1, 2, \ldots, k)$ is a collection of THFPRs.

On the basis of the above mentioned analysis, we propose the following decision-making steps where the original matrices are the THFPRs:

Step 1. Determine the original THFPR matrix $\tilde{A}_z = \left(\tilde{a}_{zij}\right)_{n\times n}(z = 1,2,\ldots,k)$ using Definition 3.1 and Eq. (2).

Step 2. Determine the consistency measure matrix $\bar{A}_z$ using Eq. (1). Construct a perfect consistent THFPR $\hat{A} = \left(\hat{a}_{ij}\right)_{n\times n}$ on the basis of Definition 4.4, and then calculate the deviation $d\left(\bar{A},\hat{A}\right)$ using the Hamming distance represented as follows:

$$d\left(\bar{A},\hat{A}\right) = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left|\bar{a}_{ij}^{h^L} - \hat{a}_{ij}^{h^L}\right| + \left|\bar{a}_{ij}^{h^M} - \hat{a}_{ij}^{h^M}\right| + \left|\bar{a}_{ij}^{h^U} - \hat{a}_{ij}^{h^U}\right|\right)\Big/3\right)\Bigg/(n-1)(n-2), i<j$$

If $d\left(\bar{A},\hat{A}\right) < \theta_0$, then proceed to the next step; otherwise, revert to Step 1 and adopt the inconsistent original THFPRs.

Step 3. Aggregate the THFPRs $\tilde{A}_z(z = 1,2,\ldots,k)$ using the THFPRWA operator with the weight vector $W = (w_1, w_2, \ldots, w_k)^T$ of several criteria or the THFPROWA operator with the unknown weight vector. Then we can obtain a triangular hesitant fuzzy matrix as follows:

$$\tilde{P} = \left(\tilde{p}_{ij}\right)_{n\times n} = \left(\left\{\left(\gamma_{ij}^{h\,L}, \gamma_{ij}^{h\,M}, \gamma_{ij}^{h\,U}\right)\middle| i,j = 1,2,\ldots,n; h = 1,2,\ldots,\#\tilde{p}_{ij}\right\}\right)_{n\times n}$$

The matrix $\tilde{P}$ includes all the triangular hesitant fuzzy preference information under each criterion.

Step 4. Obtain consistency measure matrix $\bar{P}$ of the triangular hesitant fuzzy matrix $\tilde{P}$ using Eq. (1), construct a perfect consistent THFPR $\hat{P} = \left(\hat{p}_{ij}\right)_{n\times n}$ on the basis of Definition 4.4 and calculate the deviation $d\left(\bar{P},\hat{P}\right)$. If $d\left(\bar{P},\hat{P}\right) < \theta_0$, then proceed to the next step; otherwise, return to Step 1 and adopt the inconsistent original THFPRs.

Step 5. Determine the score function matrix $s\left(\tilde{P}\right)$ on the basis of Definition 2.5, calculate the sum of each line $s\left(\tilde{P}\right)$ and obtain a set $\sigma(X) = \{\sigma(x_i) | i = 1,2,\ldots,n\}$, where $x_i = \sum_{t=1}^{n} s\left(\tilde{p}_{it}\right), t \neq i$.

Step 6. Rank the alternatives $X = \{x_1, x_2, \ldots, x_n\}$ in descending order of $\sigma(X)$.

5.2. The MCDM-THFPR model

This section describes the model of MCDM-THFPR, as depicted in Figure 1.

Subsection 5.1 and Figure 1 display several advantages of the proposed model. Firstly, this model presents the THFPR which helps DMs express their ideas effectively and adequately. Secondly, we present two operators for the situation in which the weight of criteria is determined to adapt to different situations. Thirdly, DMs can select the appropriate parameter to represent the preferences of DMs. However, this model disregards the situation in which the preference values are incomplete.

6. Numerical Example

6.1. Background

If an enterprise desires to select the most appropriate business partner among four selected companies $X = \{x_1, x_2, x_3, x_4\}$. A group of experts is invited to provide preference degrees of every two companies using TFNs. The benefit-type criteria in decision-making include the quality of the product c_1, the financial situation of the company c_2 and the cost-type criterion is the price of the product c_3.

Complete and accurate evaluation values for each region under each criterion is difficult to provide. Thus, we use the THFPR to represent peoples preferences which can embody different peoples demands and permit people to provide null values when they lack knowledge. Then, original preference values under a criterion are in the form of THFNs. Each THFN is composed of three TFNs which are the preference values provided by a groups of people.

In this study, we use an illustration on basis of THFPRs which applies the THFPRWA (THFPROWA) operator to provide people additional choices to express their preferences. This illustration is described in Sections 6.2, where the comparisons with other previous methods are stated at the end of Section 6.2.

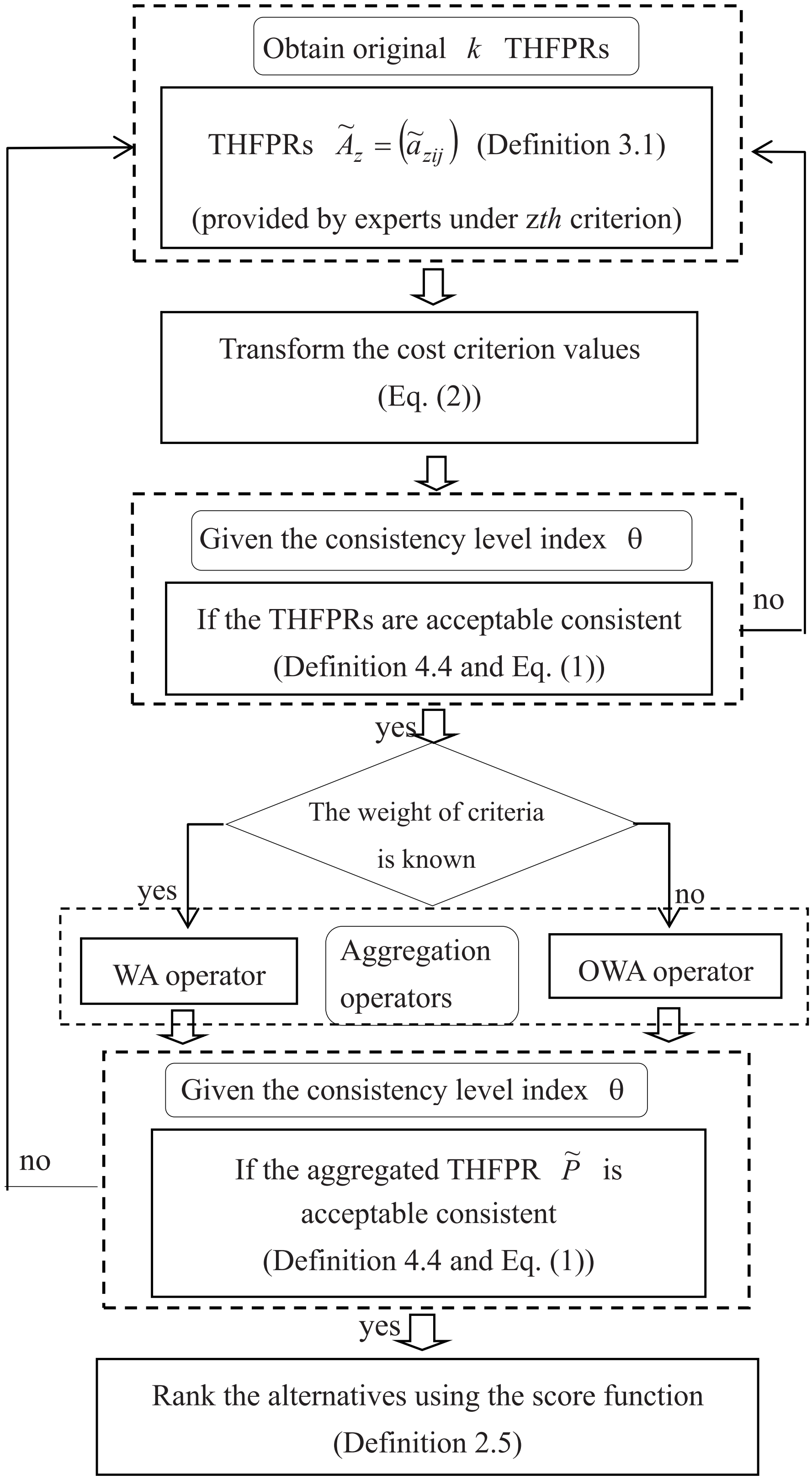

Figure 1: Model of MCDM-THFPR.

6.2. Application of the THFPR in the MCDM based on the proposed model

The original THFPRs $\tilde{A}_z$ are stated as follows ($\tilde{A}_z = \left(\tilde{a}_{zij}\right)_{4\times 4}$ where $z = 1,2,3$, $i, j = 1,2,3,4$), and $\tilde{a}_{ij}$ represents the preference values of the ith company over the jth company under zth criterion provided by a group of people. Each person can select to provide a preference value in the TFN form. In this study, we use the THFPRWA operator for the THFPRs when the weight vector is set as $W = (1/3, 1/3, 1/3)^T$. The decision steps of this method are as follows.

Step 1. The original group decision-making information is defined as follows.

The preference information is under criterion c_1.

$$\tilde{A}_1 = \left(\tilde{a}_{1ij}\right)_{4\times 4} = \begin{bmatrix} \{(0.5,0.5,0.5)\} & \{(0.1,0.2,0.2),(0.2,0.3,0.5),(0.3,0.4,0.5)\} & \{(0.2,0.5,0.6),(0.4,0.5,0.7)\} & \{(0,0.2,0.6),(0.1,0.35,0.65),(0.2,0.65,0.7)\} \\ \{(0.5,0.6,0.7),(0.5,0.7,0.8),(0.8,0.8,0.9)\} & \{(0.5,0.5,0.5)\} & \{(0.55,0.6,0.65),(0.65,0.8,0.85)\} & \{(0.3,0.5,0.65),(0.5,0.7,0.85)\} \\ \{(0.3,0.6,0.6),(0.4,0.5,0.8)\} & \{(0.15,0.2,0.35),(0.35,0.4,0.45)\} & \{(0.5,0.5,0.5)\} & \{(0.2,0.25,0.35),(0.3,0.45,0.5),(0.4,0.5,0.65)\} \\ \{(0.3,0.35,0.8),(0.35,0.65,0.9),(0.4,0.8,1)\} & \{(0.15,0.3,0.5),(0.35,0.5,0.7)\} & \{(0.35,0.5,0.6),(0.5,0.55,0.7),(0.65,0.75,0.8)\} & \{(0.5,0.5,0.5)\} \end{bmatrix}$$

The preference information is under criterion c_2.

$$\tilde{A}_2 = \left(\tilde{a}_{2ij}\right)_{4\times 4} = \begin{bmatrix} \{(0.5,0.5,0.5)\} & \{(0.1,0.35,0.4),(0.2,0.45,0.55),(0.3,0.7,0.85)\} & \{(0.2,0.5,0.6),(0.4,0.7,0.8)\} & \{(0,0.4,0.85),(0,0.55,0.9),(0,0.85,0.95)\} \\ \{(0.15,0.3,0.7),(0.45,0.55,0.8),(0.6,0.65,0.9)\} & \{(0.5,0.5,0.5)\} & \{(0.6,0.6,0.6)\} & \{(0.15,0.5,0.65),(0.45,0.7,0.95)\} \\ \{(0.2,0.3,0.6),(0.4,0.5,0.8)\} & \{(0.4,0,4,0,4)\} & \{(0.5,0.5,0.5)\} & \{(0.1,0.35,0.55),(0.2,0.55,0.75),(0.3,0.6,0.8)\} \\ \{(0.05,0.15,1),(0.1,0.45,1),(0.15,0.6,1)\} & \{(0.05,0.3,0.55),(0.35,0.5,0.85)\} & \{(0.2,0.4,0.7),(0.25,0.45,0.8),(0.45,0.65,0.9)\} & \{(0.5,0.5,0.5)\} \end{bmatrix}$$

The preference information is under criterion c_3.

$$\tilde{A}_3 = \left(\tilde{a}_{3ij}\right)_{4\times 4} = \begin{bmatrix} \{(0.5,0.5,0.5)\} & \{(0.3,0.6,0.8)\} & \{(0.1,0.25,0.65),(0.2,0.55,0.75)\} & \{(0,0.4,0.9),(0.1,0.6,0.9)\} \\ \{(0.2,0.4,0.6)\} & \{(0.5,0.5,0.5)\} & \{(0.2,0.25,0.3),(0.25,0.3,0.4),(0.3,0.35,0.5)\} & \{(0.1,0.3,0.55),(0.2,0.5,0.65)\} \\ \{(0.25,0.45,0.8),(0.35,0.75,0.9)\} & \{(0.5,0.65,0.7),(0.6,0.7,0.75),(0.7,0.75,0.8)\} & \{(0.5,0.5,0.5)\} & \{(0.3,0.5,0.65),(0.5,0.7,0.75)\} \\ \{(0.1,0.4,0.9),(0.1,0.6,1)\} & \{(0.35,0.5,0.8),(0.45,0.7,0.9)\} & \{(0.25,0.3,0.5),(0.35,0.5,0.7)\} & \{(0.5,0.5,0.5)\} \end{bmatrix}$$

The matrix $\tilde{A}_3$ can be converted by using Eq. (2).

$$\tilde{A}'_3 = \left(a'_{3ij}\right)_{4\times 4} = \begin{bmatrix} \{(0.5,0.5,0.5)\} & \{(0.2,0.4,0.6)\} & \{(0.25,0.45,0.8),(0.35,0.75,0.9)\} & \{(0.1,0.4,0.9),(0.1,0.6,1)\} \\ \{(0.3,0.6,0.8)\} & \{(0.5,0.5,0.5)\} & \{(0.5,0.65,0.7),(0.6,0.7,0.75),(0.7,0.75,0.8)\} & \{(0.35,0.5,0.8),(0.45,0.7,0.9)\} \\ \{(0.1,0.25,0.65),(0.2,0.55,0.75)\} & \{(0.2,0.25,0.3),(0.25,0.3,0.4),(0.3,0.35,0.5)\} & \{(0.5,0.5,0.5)\} & \{(0.25,0.3,0.5),(0.35,0.5,0.7)\} \\ \{(0,0.4,0.9),(0.1,0.6,0.9)\} & \{(0.1,0.3,0.55),(0.2,0.5,0.65)\} & \{(0.3,0.5,0.65),(0.5,0.7,0.75)\} & \{(0.5,0.5,0.5)\} \end{bmatrix}$$

Step 2. We can obtain the original THFPRs $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{A}_3$ using Eq. (1) and Definition 4.4 to satisfy consistency with the deviation $d\left(\bar{A}_z, \hat{A}_z\right) = 0$ (the concrete results are omitted).

Step 3-4. Aggregate the THFPRs $\tilde{A}_z (z = 1,2,3)$ with $\tilde{P} = THFPRWA\left(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3\right) = \left(\tilde{p}_{ij}\right)_{n\times n}$ (the concrete results is omitted); we can find that $\tilde{P}$ satisfies consistency with the deviation $d\left(\bar{P}, \hat{P}\right) = 0$.

Step 5. Determine the score function $s\left(\tilde{P}\right)$ as follows:

$$s\left(\tilde{P}\right)=\begin{bmatrix} 0.5 & 0.36 & 0.52 & 0.45 \\ 0.64 & 0.5 & 0.66 & 0.59 \\ 0.48 & 0.34 & 0.5 & 0.43 \\ 0.55 & 0.41 & 0.57 & 0.5 \end{bmatrix}$$

Calculate the sum of each line of $s\left(\tilde{P}\right)$ and obtain $\sigma(x_1) = 1.33$, $\sigma(x_2) = 1.89$, $\sigma(x_3) = 1.25$ and $\sigma(x_4) = 1.53$.

Step 6. Owing to $\sigma(x_2) > \sigma(x_4) > \sigma(x_1) > \sigma(x_3)$, we can draw the conclusion that alternative x_2 is the optimal alternative.

If the weight vector of the three criteria is unknown, then we can use the THFPROWA operator instead and obtain $d\left(\bar{P},\hat{P}\right) = 0.031 < 0.1$ and the order $\sigma(x_2) > \sigma(x_4) > \sigma(x_1) > \sigma(x_3)$; this result is the same as that we achieved using the THFPRWA operator, thereby implying that x_2 is the most appropriate alternative for this enterprise.

The comparison of the results using our approach and two other previous methods [36, 50] is summarised in Table 1.

Table 1: Comparison of the three methods.

Method	Hesitant-IFPR	IVHFPR	Proposed MCDM-THFPR
$\sigma(x_1)$	1.91	[0.2675,0.625]	1.33
$\sigma(x_2)$	2.39	[0.4925,0.7]	1.89
$\sigma(x_3)$	1.71	[0.3375,0.545]	1.25
$\sigma(x_4)$	1.99	[0.335,0.6975]	1.53
Final ranking	$x_2 \succ x_4 \succ x_1 \succ x_3$	$x_2 \succ x_4 \succ x_1 \succ x_3$	$x_2 \succ x_4 \succ x_1 \succ x_3$

6.3. Discussions

From the results of the preceding section, we can draw the following discussions:

(1) The result of $d\left(\bar{P},\hat{P}\right)$ for the THFPRs suggests that the THFPRWA operator is advantageous because this result can ensure that the matrix aggregated from the consistent THFPRs is also consistent, and its process is simple. The THFPROWA operator is also advantageous because it can help DMs handle a situation with an unknown weight vector. Then we obtain a suggestion that, if the weight of criteria is unknown, then the DM can use the THFPROWA operator, or select the THFPRWA operator, in processing of evaluation alternatives.

(2) Rankings using the THFPRWA operator may be different from those using the THFPROWA operator. Therefore, we can select different operator based on the preferences of the DMs in terms of the weight vectors of criteria. These two types of operators provide additional choices for the DMs in the decision-making process.

(3) The consistency level can change with the parameter θ_0, in which DMs can select the appropriate θ_0 to control the consistency level for satisfying each member of the group satisfied with the final choice.

(4) Table 1 indicates that the ranking of four companies (alternatives) in our proposed MCDM-THFPR is the same as that in the two other previous methods, thereby implying the applicability and validity of our method. Furthermore, the proposed MCDM-THFPR can express further information about DMs and reduce information distortion.

7. Conclusions

This study defines the concept of THFPR and presents an MCDM-THFPR model for MCDM problems. The MCDM-THFPR model is composed of three aspects as follows: the aggregation method for THFPRs, the consistency measurements for THFPRs and the MCDM-THFPR to rank alternatives.

A case is proposed to verify the practicability and accuracy of the MCDM-THFPR model, and the comparison with previous methods is performed. From the discussions based on the results of the case, the MCDM-THFPR method is a novel, valid tool for DMs to assess and select an appropriate alternative during the process of decision-making process.

In our future work, theoretical problems, in which the preference values are incomplete or the type of preference values is a mix of several types of numbers, will be addressed in consideration of the interests of DMs.

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