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The Multiparameter r-Whitney Numbers

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Abstract. In this paper, we define the multiparameter r-Whitney numbers of the first and second kind. The recurrence relations, generating functions , explicit formulas of these numbers and some combinatorial identities are derived. Some relations between these numbers and generalized Stirling numbers of the first and second kind, Lah numbers, C-numbers and harmonic numbers are deduced. Furthermore, some interesting special cases are given. Finally matrix representation for these relations are given.

1. Introduction

Throughout this article we use the following notations:
The generalized falling factorial associated with $\bar{\alpha} = (\alpha_0, \ \alpha_1, \cdots, \alpha_{n-1})$ is defined by

$$(x; \bar{\alpha})_n = \prod_{i=0}^{n-1} (x - \alpha_i).$$

If $\alpha_i = i$, we have the falling factorial $(x)_n = \prod_{i=0}^{n-1}(x - i)$ and

$$(\alpha_i)_k = \prod_{j=0, j\neq i}^{k} (\alpha_i - \alpha_j)$$

The r-Whitney numbers of the first and second kind were introduced, respectively, by Mező [13] as

$$m^n (x)_n = \sum_{k=0}^{n} w_{m,r}(n, k)(mx + r)^k,$$

and

$$(mx + r)^n = \sum_{k=0}^{n} W_{m,r}(n, k)m^k (x)_k,$$

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if $r = 1$, the r-Whitney numbers are reduced to the Whitney numbers, see Dowling [7] and Benoumhani [2]. El-Desouky [8] defined the multiparameter non-central Stirling numbers of the first and second kind, respectively, by

$$(x)_n = \sum_{k=0}^{n} s(n,k;\overline{\boldsymbol{\alpha}})(x;\overline{\boldsymbol{\alpha}})_k,$$

and

$$(x;\overline{\boldsymbol{\alpha}})_n = \sum_{k=0}^{n} S(n,k;\overline{\boldsymbol{\alpha}})(x)_k.$$

In this paper, we define the multiparameter r-Whitney numbers of the first and second kind, recurrence relations, generating function and explicit formulas are obtained. Some important special cases are investigated. Moreover, some relations between these numbers and Stirling numbers, the generalized Stirling numbers, Lah numbers, C-numbers and the generalized harmonic numbers are derived. Finally, we give a matrix representation for some of these relations.

2. The Multiparameter r-Whitney Numbers of the First Kind

Let x be a real number, n nonnegative integer, and $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ where $\alpha_i,\ i = 0, 1, \cdots, n-1$ are real numbers.

Definition 2.1. *The multiparameter r-Whitney numbers of the first kind $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ with parameter $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ are defined by*

$$m^n (x)_n = \sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})(mx + r;\bar{\boldsymbol{\alpha}})_k, \tag{1}$$

where $w_{m,r}(0,0;\bar{\boldsymbol{\alpha}}) = 1$ and $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = 0$ for $k > n$.

Theorem 2.2. *The multiparameter r-Whitney numbers of the first kind $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ satisfy the recurrence relation*

$$w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = w_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}}) + (\alpha_k - r - m(n-1))w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}}), \tag{2}$$

for $k \geq 1$, and $w_{m,r}(n,0;\bar{\boldsymbol{\alpha}}) = \prod_{i=0}^{n-1}(\alpha_0 - r - im) = (\alpha_0 - r|m)_n$.

Proof. Since $m^n(x)_n = m^{n-1}(x)_{n-1}(mx + r - \alpha_k - r + \alpha_k - m(n-1))$, then we have

$$\begin{aligned}
&\textstyle\sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})(mx+r;\overline{\alpha})_k \\
&= (mx + r - \alpha_k)\sum_{k=0}^{n-1} w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})(mx+r;\bar{\boldsymbol{\alpha}})_k + (\alpha_k - r - m(n-1))\sum_{k=0}^{n-1} w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})(mx+r;\bar{\boldsymbol{\alpha}})_k \\
&= \sum_{k=1}^{n} w_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}})(mx+r;\bar{\boldsymbol{\alpha}})_k + (\alpha_k - r - m(n-1))\sum_{k=0}^{n-1} w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})(mx+r;\bar{\boldsymbol{\alpha}})_k.
\end{aligned}$$

Equating the coefficients of $(mx + r;\bar{\boldsymbol{\alpha}})_k$ on both sides, we get (2). For $k = 0$ it is easy to prove that $w_{m,r}(n,0;\bar{\boldsymbol{\alpha}}) = \prod_{i=0}^{n-1}(\alpha_0 - r - im)$. □

Theorem 2.3. *The multiparameter r-Whitney numbers of the first kind have the exponential generating function*

$$\varphi_k(t;\bar{\boldsymbol{\alpha}}) = \sum_{n=0}^{\infty} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\frac{t^n}{n!} = m^k \sum_{i=0}^{k} \frac{(1+mt)^{\frac{\alpha_i - r}{m}}}{(\alpha_i)_k}. \tag{3}$$

Proof. Since the exponential generating function of $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ is defined as

$$\varphi_k(t;\bar{\boldsymbol{\alpha}}) = \sum_{n=k}^{\infty} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\frac{t^n}{n!}, \tag{4}$$

where $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = 0$ for $n < k$. If $k = 0$ we get

$$\varphi_0(t;\bar{\boldsymbol{\alpha}}) = \sum_{n=0}^{\infty} w_{m,r}(n,0;\bar{\boldsymbol{\alpha}})\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\alpha_0 - r; m)_n\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\frac{\alpha_0 - r}{m})_n\frac{(mt)^n}{n!},$$

let $\beta_i = \frac{\alpha_i - r}{m}, i = 0, 1, \cdots, n-1$ then

$$\varphi_0(t;\bar{\boldsymbol{\alpha}}) = \sum_{n=0}^{\infty}(\beta_0)_n\frac{(mt)^n}{n!} = (1+mt)^{\beta_0}.$$

Differentiating both sides of (4) with respect to t, we have

$$\acute{\varphi}_k(t;\bar{\boldsymbol{\alpha}}) = \sum_{n=k}^{\infty} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\frac{t^{n-1}}{(n-1)!}, \tag{5}$$

using (2) gives

$$\begin{aligned}\acute{\varphi}_k(t;\bar{\boldsymbol{\alpha}}) &= \sum_{n=k}^{\infty} w_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}})\frac{t^{n-1}}{(n-1)!} + m(\frac{\alpha_k - r}{m})\sum_{n=k}^{\infty} w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})\frac{t^{n-1}}{(n-1)!}\\ &- mt\sum_{n=k}^{\infty} w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})\frac{t^{n-2}}{(n-2)!}.\end{aligned}$$

Using (4), and (5), we have

$$\acute{\varphi}_k(t;\bar{\boldsymbol{\alpha}}) = \varphi_{k-1}(t;\bar{\boldsymbol{\alpha}}) - m\beta_k\varphi_k(t;\bar{\boldsymbol{\alpha}}) - mt\acute{\varphi}_k(t;\bar{\boldsymbol{\alpha}}).$$

Hence,

$$\acute{\varphi}_k(t;\bar{\boldsymbol{\alpha}}) + \frac{m\beta_k}{1+mt}\varphi_k(t;\bar{\boldsymbol{\alpha}}) = \frac{1}{1+mt}\varphi_{k-1}(t;\bar{\boldsymbol{\alpha}}).$$

Solving this difference-differential equation, we obtain (3). □

Theorem 2.4. *The multiparameter r-Whitney numbers of the first kind have the explicit formula*

$$w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = m^{n+k}\sum_{j=0}^{n}\frac{(-1)^{n-j}n!}{j!}\sum_{\ell_1+\ell_2+\cdots+\ell_j=n}\frac{1}{\ell_1\ell_2\cdots\ell_j}\sum_{i=0}^{k}\Big(\frac{\alpha_i - r}{m}\Big)^j\frac{1}{(\alpha_i)_k}. \tag{6}$$

Proof. From (3) and setting $\beta_i = \frac{\alpha_i - r}{m}$

$$\begin{aligned}\varphi_k(t;\bar{\boldsymbol{\alpha}}) &= \sum_{n=0}^{\infty} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\frac{t^n}{n!} = m^k\sum_{i=0}^{k}\frac{(1+mt)^{\beta_i}}{(\alpha_i)_k} = m^k\sum_{i=0}^{k}\frac{1}{(\alpha_i)_k}\exp(\beta_i\log(1+mt))\\ &= m^k\sum_{i=0}^{k}\frac{1}{(\alpha_i)_k}\sum_{j=0}^{\infty}\frac{(\beta_i\log(1+mt))^j}{j!} = m^k\sum_{i=0}^{k}\frac{1}{(\alpha_i)_k}\sum_{j=0}^{\infty}\frac{\beta_i^j}{j!}\Big(\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell-1}(mt)^\ell}{\ell}\Big)^j,\end{aligned}$$

using Cauchy's rule of multiplication of infinite series, we get

$$\begin{aligned}\varphi_k(t;\bar{\boldsymbol{\alpha}}) &= m^k\sum_{i=0}^{k}\frac{1}{(\alpha_i)_k}\sum_{j=0}^{\infty}\frac{\beta_i^j}{j!}\sum_{n=j}^{\infty}\sum_{\ell_1+\ell_2+\cdots+\ell_j=n}^{\infty}\frac{1}{\ell_1\ell_2\cdots\ell_j}(-1)^{n-j}(mt)^n\\ &= m^k\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}\frac{(-1)^{n-j}}{j!}\sum_{\ell_1+\ell_2+\cdots+\ell_j=n}^{\infty}\frac{1}{\ell_1\ell_2\cdots\ell_j}\sum_{i=0}^{k}\Big(\frac{\alpha_i-r}{m}\Big)^j\frac{1}{(\alpha_i)_k}\Big)(mt)^n,\end{aligned}$$

hence

$$\sum_{n=0}^{\infty}w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\frac{t^n}{n!} = m^k\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}\frac{(-1)^{n-j}}{j!}\sum_{\ell_1+\ell_2+\cdots+\ell_j=n}^{\infty}\frac{1}{\ell_1\ell_2\cdots\ell_j}\sum_{i=0}^{k}\Big(\frac{\alpha_i-r}{m}\Big)^j\frac{1}{(\alpha_i)_k}\Big)(mt)^n. \tag{7}$$

Equating the coefficients of (7), we get (6). □

Theorem 2.5. *The multiparameter r-Whitney numbers of the first kind have the following interesting explicit formula*

$$w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = \sum_{i_1+i_2+i_n=k,\ i_j\in\{0,1\}}\binom{i_1+\alpha_{i_1}-r}{1-i_1}\binom{i_2+\alpha_{i_1+i_2}-r-m}{1-i_2}\cdots\binom{i_n+\alpha_{i_1+\cdots+i_n}-r-m(n-1)}{1-i_n}. \tag{8}$$

Proof. For k=0, we have

$$w_{m,r}(n,0;\bar{\boldsymbol{\alpha}}) = (\alpha_0-r)(\alpha_0-r-m)\cdots(\alpha_0-r-m(n-1)).$$

Also, if $i_n\in\{0,1\}$, we get

$$\begin{aligned}w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) &= \sum_{i_1+i_2+\cdots+i_{n-1}=k-1}\binom{i_1+\alpha_{i_1}-r}{1-i_1}\cdots\binom{i_{n-1}+\alpha_{i_1+\cdots+i_{n-1}}-r-m(n-2)}{1-i_{n-1}}\\ &+ \sum_{i_1+i_2+\cdots+i_{n-1}=k}(\alpha_{i_1+\cdots+i_{n-1}}-r-m(n-1))\binom{i_1+\alpha_{i_1}-r}{1-i_1}\cdots\binom{i_{n-1}+\alpha_{i_1+\cdots+i_{n-1}}-r-m(n-2)}{1-i_{n-1}}\\ &= w_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}})+(\alpha_k-r-m(n-1))w_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}}),\end{aligned}$$

where $i_1+\cdots+i_{n-1}=k$. By virtue of (2), this completes the proof. □

Moreover, setting $m=1$ and $r=0$ in (8), we get the explicit formula of the multiparameter Stirling numbers of the first kind introduced in [5], also Setting $m=1,\ r=0$ and $\alpha_i=0$ in this equation we obtain the explicit formula of the Stirling numbers of the first kind.

Remark: From (6) and (8), we obtain the following new combinatorial identity

$$\begin{aligned}&m^{n+k}\sum_{j=0}^{n}\frac{(-1)^{n-j}n!}{j!}\sum_{\ell_1+\ell_2+\cdots+\ell_j=n}^{\infty}\frac{1}{\ell_1\ell_2\cdots\ell_j}\sum_{i=0}^{k}\Big(\frac{\alpha_i-r}{m}\Big)^j\frac{1}{(\alpha_i)_k} =\\ &\sum_{i_1+i_2+i_n=k,\ i_j\in\{0,1\}}\binom{i_1+\alpha_{i_1}-r}{1-i_1}\cdots\binom{i_n+\alpha_{i_1+\cdots+i_n}-r-m(n-1)}{1-i_n}.\end{aligned}$$

Corollary 2.6. *A new explicit expression for the r-Whitney numbers of the first kind is given by*

$$w_{m,r}(n,k) = \sum_{i_1+i_2+\cdots+i_n=k,\ i_j\in\{0,1\}}\binom{i_1-r}{1-i_1}\binom{i_2-r-m}{1-i_2}\cdots\binom{i_n-r-m(n-1)}{1-i_n}.$$

Proof. The proof follows directly by setting $\alpha_i=0$ in (8). □

Remark: Wagner [16] proved that

$$w_{m,r}(n,k) = \sum_{c_0+c_1+\cdots+c_{n-1}=n-k,\ c_j\in\{0,1\}} r^{c_0}(r+m)^{c_1}\cdots(r+(n-1)m)^{c_{n-1}}.$$

So, we have the combinatorial identity

$$\sum_{\substack{c_0+c_1+\cdots+c_{n-1}=n-k\\ c_j\in\{0,1\}}} r^{c_0}(r+m)^{c_1}\cdots(r+(n-1)m)^{c_{n-1}} = \sum_{\substack{i_1+i_2+\cdots+i_n=k\\ i_j\in\{0,1\}}} \binom{i_1-r}{1-i_1}\binom{i_2-r-m}{1-i_2}\cdots\binom{i_n-r-m(n-1)}{1-i_n}.$$

2.1. Special cases

From Eq. (1) we obtain the following special cases:

1	$m=1$ and $r=0$	$w_{1,0}(n,k;\bar{\boldsymbol{\alpha}}) = s(n,k;\bar{\boldsymbol{\alpha}})$, the multiparameter non-central Stirling numbers of the first kind, see El-Desouky [8].
2	$m=1,\ r=0$ and $\alpha_i=0$	$w_{1,0}(n,k;\mathbf{0}) = s(n,k)$, the usual Stirling numbers of the first kind, see Stirling [15] .
3	$m=1,\ r=0$ and $\alpha_i=\alpha$	$w_{1,0}(n,k;\boldsymbol{\alpha}) = s_\alpha(n,k)$, the noncentral Stirling numbers of the first kind, see Koutras [10].
4	$m=1,\ r=0$ and $\alpha_i=i$	$w_{1,0}(n,k;\boldsymbol{i}) = C(n,k,1)$, the C-numbers when $r=1$, see [6] and [8].
5	$\alpha_i=0$	$w_{m,r}(n,k;\mathbf{0}) = w_{m,r}(n,k)$, the r-Whitney numbers of the first kind, see Mező [13].
6	$\alpha_i=0$ and $r=1$	$w_{m,1}(n,k;\mathbf{0}) = w_m(n,k)$, the Whitney numbers of the first kind, see Benoumhani [2].
7	$\alpha_i=0$ and $m=1$	$w_{1,r}(n,k;\mathbf{0}) = s_r(n,k)$, the r-Stirling numbers of the first kind, see Broder [3].
8	$\alpha_i=a$ and $r=0$	$w_{m,0}(n,k;\boldsymbol{a}) = \tilde{w}_{m,a}(n,k)$, the noncentral Whitney numbers of the first kind, see Mangontarum [12].
9	$m=1,\ r=0$ and $\alpha_i=\frac{-r-i\alpha}{\beta}$	$w_{1,0}(n,k;\bar{\boldsymbol{\alpha}}) = \beta^{k-n}S(n,k;\alpha,\beta,r)$, the generalized Stirling-type pair, see Hsu and Shiue [9].
10	$m=1,\ r=0$ and $\alpha_i=\frac{r+i\beta}{\alpha}$	$w_{1,r}(n,k;\boldsymbol{i\alpha}) = \alpha^{k-n}S(n,k;\beta,\alpha,-r)$, the generalized Stirling-type pair, see Hsu and Shiue [9].
11	$m=1,\ r=0,\ \alpha_i=0$ and then multiplying the given equation by $(-\alpha)^n$	$w_{1,0}(n,k;\mathbf{0}) = (-\alpha)^{k-n}\begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)}$, the translated Whitney number of the first kind see [1].

Remark: It is worth noting that $(mx+r;\bar{\boldsymbol{\alpha}})_n = (mx+r-\alpha_0)(mx+r-\alpha_1)\cdots(mx+r-\alpha_{n-1})$, hence

$$(mx+r;\bar{\boldsymbol{\alpha}})_n = m^n\left(x-\frac{\alpha_0-r}{m}\right)\left(x-\frac{\alpha_1-r}{m}\right)\cdots\left(x-\frac{\alpha_{n-1}-r}{m}\right),$$

where $\beta_i=\frac{\alpha_i-r}{m}$, we have $(mx+r;\bar{\boldsymbol{\alpha}})_n = m^n(x;\bar{\boldsymbol{\beta}})_n$, $\bar{\boldsymbol{\beta}}=(\beta_0,\beta_1,\cdots,\beta_{n-1})$.
Therefore the multiparameter r-Whitney numbers of the first kind can be written as

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})m^k(x;\bar{\boldsymbol{\beta}})_k. \quad (9)$$

2.2. Some relations

1. We determine the relation between the Stirling numbers of the first kind and the multiparameter r-Whitney numbers of the first kind. From (9), we have

$$m^n (x)_n = \sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) m^k (x;\bar{\boldsymbol{\beta}})_k,$$

hence,

$$\begin{aligned} m^n \sum_{i=0}^{n} s(n,i)x^i &= \sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) m^k \sum_{i=0}^{k} s_{\bar{\beta}}(k,i)x^i \\ &= \sum_{i=0}^{n} \Big(\sum_{k=i}^{n} m^k w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) s_{\bar{\beta}}(k,i) \Big) x^i. \end{aligned}$$

Equating coefficient of x^i on both sides, we get

$$s(n,i) = \sum_{k=i}^{n} m^{-(n-k)} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) s_{\bar{\beta}}(k,i), \tag{10}$$

where $s_{\bar{\beta}}(k,i)$ are the generalized Stirling numbers of the first kind see [6].

2. Also $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ can be expressed in terms of $s(n,k)$ and $S_{\bar{\beta}}(k,i)$ as

$$\begin{aligned} \sum_{i=0}^{n} w_{m,r}(n,i;\bar{\boldsymbol{\alpha}}) m^i (x;\bar{\boldsymbol{\beta}})_i &= m^n \sum_{k=0}^{n} s(n,k)x^k = m^n \sum_{k=0}^{n} s(n,k) \sum_{i=0}^{k} S_{\bar{\beta}}(k,i)(x;\bar{\boldsymbol{\beta}})_i \\ &= m^n \sum_{i=0}^{n} \Big(\sum_{k=i}^{n} s(n,k) S_{\bar{\beta}}(k,i) \Big) (x;\bar{\boldsymbol{\beta}})_i. \end{aligned}$$

Equating the coefficients of $(x;\bar{\boldsymbol{\beta}})_i$ on both sides, we get

$$m^{n-i} \sum_{k=i}^{n} s(n,k) S_{\bar{\beta}}(k,i) = w_{m,r}(n,i;\overline{\alpha}). \tag{11}$$

where $S_{\bar{\beta}}(k,i)$ are the generalized Stirling numbers of the second kind, see Comtet [6].

3. Similarly we can express $S_{\bar{\beta}}(n,i)$ in terms of $S(n,k)$ and $w_{m,r}(k,i;\bar{\boldsymbol{\alpha}})$ as

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k,$$

then

$$\begin{aligned} \sum_{i=0}^{n} S_{\bar{\beta}}(n,i)(x;\bar{\boldsymbol{\beta}})_i &= \sum_{k=0}^{n} S(n,k) m^{-k} \sum_{i=0}^{k} w_{m,r}(k,i;\bar{\boldsymbol{\alpha}}) m^i (x;\bar{\boldsymbol{\beta}})_i \\ &= \sum_{i=0}^{n} \Big(\sum_{k=i}^{n} m^{i-k} S(n,k) w_{m,r}(k,i;\bar{\boldsymbol{\alpha}}) \Big) (x;\bar{\boldsymbol{\beta}})_i. \end{aligned}$$

Equating coefficient of $(x;\bar{\boldsymbol{\beta}})_i$ on both sides, we get

$$S_{\bar{\beta}}(n,i) = \sum_{k=i}^{n} m^{i-k} S(n,k) w_{m,r}(k,i;\bar{\boldsymbol{\alpha}}). \tag{12}$$

4. From [11], [14] we have

$$s(n,i)=\sum_{k=i}^{n}(-1)^{i}L(n,k)s(k,i),$$

where $L(n,k)$ denote the Lah numbers, and substituting in (10), we get

$$\sum_{k=i}^{n}(-1)^{i}L(n,k)s(k,i)-m^{-(n-k)}w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})s_{\bar{\beta}}(k,i)=0.$$

5. From El-desouky [8] we have

$$s(n,i)=r^{-i}\sum_{k=i}^{n}C(n,k,r)s(k,i),$$

where $C(n,k,r)$ denotes the C-numbers, and substituting in (10), we get

$$\sum_{k=i}^{n}m^{-(n-k)}w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})s_{\bar{\beta}}(k,i)-r^{-i}C(n,k,r)s(k,i)=0.$$

2.3. Matrix representation

1. Eq. (10) can be represented in a matrix form as

$$\widehat{\boldsymbol{s}}=\boldsymbol{w}_{m,r}(\bar{\boldsymbol{\alpha}})\widehat{\boldsymbol{s}_{\bar{\beta}}},\tag{13}$$

where $\widehat{s}(n,i)=m^{n}s(n,i)$ and $\widehat{s_{\bar{\beta}}}(k,i)=m^{k}s_{\bar{\beta}}(k,i)$ and $\boldsymbol{s},\boldsymbol{w}_{m,r}(\bar{\boldsymbol{\alpha}})$ and $\boldsymbol{s}_{\bar{\beta}}$ are $n\times n$ lower triangle matrices whose entries are, respectively, the Stirling number of the first kind, the multiparameter r-Whitney numbers of the first kind and the generalized Stirling numbers of the first kind.
For example if $0\le n,k,i\le 3$, hence (13) can be written as

$$\begin{pmatrix}1&0&0&0\\0&m&0&0\\0&-m^{2}&m^{2}&0\\0&2m^{3}&-3m^{3}&m^{3}\end{pmatrix}=\begin{pmatrix}1&0&0&0\\ \alpha_0-r&1&0&0\\(\alpha_0-r-m)_1&\alpha_0+\alpha_1-2r-m&1&0\\(\alpha_0-r-m)_2&w_{m,r}(3,1;\bar{\boldsymbol{\alpha}})&w_{m,r}(3,2;\bar{\boldsymbol{\alpha}})&1\end{pmatrix}$$

$$\cdot\begin{pmatrix}1&0&0&0\\-(\alpha_0-r)&m&0&0\\(\alpha_0-r)(\alpha_1-r)&-m(\alpha_0+\alpha_1-2r)&m^{2}&0\\-(\alpha_0-r)(\alpha_1-r)(\alpha_2-r)&\widehat{s_{\bar{\beta}}}(3,1)&-m^{2}(\alpha_0+\alpha_1+\alpha_2-3r)&m^{3}\end{pmatrix}$$

where $w_{m,r}(3,1;\bar{\boldsymbol{\alpha}})=(\alpha_0-r)(\alpha_0-r-m)+(\alpha_1-r-2m)(\alpha_0+\alpha_1-2r-m)$, $w_{m,r}(3,2;\bar{\boldsymbol{\alpha}})=\alpha_0+\alpha_1+\alpha_2-3r-3m$ and $\widehat{s_{\bar{\beta}}}(3,1)=m((\alpha_0-r)(\alpha_1+\alpha_2-2r)+(\alpha_1-r)(\alpha_2-r))$.

2. Eq. (11) can be represented in a matrix form as

$$\widehat{\boldsymbol{s}}\,\boldsymbol{S}_{\bar{\beta}}=\widehat{\boldsymbol{w}}_{m,r}(\bar{\boldsymbol{\alpha}}),\tag{14}$$

where $\widehat{\boldsymbol{w}}_{m,r}(n,i;\bar{\boldsymbol{\alpha}})=m^{i}\boldsymbol{w}_{m,r}(n,i;\bar{\boldsymbol{\alpha}})$ and $\boldsymbol{S}_{\bar{\beta}}$ is $n\times n$ lower triangle matrix whose entries is the generalized Stirling number of the second kind.
For example if $0\le n,k,i\le 3$, hence (14) can be written as

$$\begin{pmatrix}1&0&0&0\\0&m&0&0\\0&-m^{2}&m^{2}&0\\0&2m^{3}&-3m^{3}&m^{3}\end{pmatrix}\begin{pmatrix}1&0&0&0\\ \frac{\alpha_0-r}{m}&1&0&0\\ \frac{(\alpha_0-r)^2}{m^2}&\frac{\alpha_0+\alpha_1-2r}{m}&1&0\\ \frac{(\alpha_0-r)^3}{m^3}&\frac{(\alpha_0-r)^2+(\alpha_0-r)(\alpha_1-r)+(\alpha_1-r)^2}{m^2}&\frac{\alpha_0+\alpha_1+\alpha_2-3r}{m}&1\end{pmatrix}=$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_0 - r & m & 0 & 0 \\ (\alpha_0 - r - m)_1 & m(\alpha_0 + \alpha_1 - 2r - m) & m^2 & 0 \\ (\alpha_0 - r - m)_2 & m w_{m,r}(3,1;\bar{\boldsymbol{\alpha}}) & m^2 w_{m,r}(3,2;\bar{\boldsymbol{\alpha}}) & m^3 \end{pmatrix}$$

3. Eq. (12) can be represented in a matrix form as

$$\boldsymbol{S}_{\bar{\beta}} = \boldsymbol{S}\,\widehat{\boldsymbol{w}}_{m,r}(\bar{\boldsymbol{\alpha}}), \tag{15}$$

where $\widehat{\boldsymbol{w}}_{m,r}(k,i;\bar{\boldsymbol{\alpha}}) = m^{i-k}\boldsymbol{w}_{m,r}(k,i;\bar{\boldsymbol{\alpha}})$ and $\boldsymbol{S}$ is $n \times n$ lower triangle matrix whose entries is the Stirling number of the second kind.
For example if $0 \le n,k,i \le 3$, hence (15) can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\alpha_0-r}{m} & 1 & 0 & 0 \\ \frac{(\alpha_0-r)^2}{m^2} & \frac{\alpha_0+\alpha_1-2r}{m} & 1 & 0 \\ \frac{(\alpha_0-r)^3}{m^3} & \frac{(\alpha_0-r)^2+(\alpha_0-r)(\alpha_1-r)+(\alpha_1-r)^2}{m^2} & \frac{\alpha_0+\alpha_1+\alpha_2-3r}{m} & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\alpha_0-r}{m} & 1 & 0 & 0 \\ \frac{(\alpha_0-r-m)_1}{m^2} & \frac{\alpha_0+\alpha_1-2r-m}{m} & 1 & 0 \\ \frac{(\alpha_0-r-m)_2}{m^3} & \widehat{w}_{m,r}(3,1;\bar{\boldsymbol{\alpha}}) & \widehat{w}_{m,r}(3,2;\bar{\boldsymbol{\alpha}}) & 1 \end{pmatrix}$$

where $\widehat{w}_{m,r}(3,1;\bar{\boldsymbol{\alpha}}) = \frac{(\alpha_0-r)(\alpha_0-r-m)+(\alpha_1-r-2m)(\alpha_0+\alpha_1-2r-m)}{m^2}$, $\widehat{w}_{m,r}(3,2;\bar{\boldsymbol{\alpha}}) = \frac{\alpha_0+\alpha_1+\alpha_2-3r-3m}{m}$.

3. The Multiparameter r-Whitney Numbers of the Second Kind

Definition 3.1. *The multiparameter r-Whitney numbers of the second kind $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ with parameter $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ are defined by*

$$(mx + r;\bar{\boldsymbol{\alpha}})_n = \sum_{k=0}^{n} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) m^k (x)_k, \tag{16}$$

where $W_{m,r}(0,0;\bar{\boldsymbol{\alpha}}) = 1$ and $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = 0$ for $k > n$.

Theorem 3.2. *The multiparameter r-Whitney numbers of the second kind $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ satisfy the recurrence relation*

$$W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = W_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}}) + (r + mk - \alpha_{n-1}) W_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}}), \tag{17}$$

for $k \ge 1$.

Proof. Since $(mx + r;\bar{\boldsymbol{\alpha}})_n = (mx - mk + r - \alpha_{n-1} + mk)(mx + r;\bar{\boldsymbol{\alpha}})_{n-1}$, using (16) we have

$$\begin{aligned} \textstyle\sum_{k=0}^{n} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) m^k (x)_k \\ &= m(x-k)\sum_{k=0}^{n-1} W_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})\, m^k (x)_k + (r + mk - \alpha_{n-1})\sum_{k=0}^{n-1} W_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})\, m^k (x)_k \\ &= \sum_{k=1}^{n} W_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}})\, m^k (x)_k + (r + mk - \alpha_{n-1})\sum_{k=0}^{n-1} W_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}})\, m^k (x)_k. \end{aligned}$$

Equating the coefficient of $(x)_k$ on both sides, we get (17). □

Theorem 3.3. *The multiparameter r-Whitney numbers of the second kind have the interesting explicit formula*

$$W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = \sum_{I_{n-1}=n-k} \binom{r-\alpha_0}{i_0}\binom{r-\alpha_1+(1-i_0)m)}{i_1}\cdots\binom{r-\alpha_{n-1}+(n-1-i_0-i_1-\cdots-i_{n-2})m}{i_{n-1}}, \quad (18)$$

where $i_j \in \{0,1\}$, $j \in \{0,1,\cdots,n-1\}$, *and* $I_{n-1} = i_0 + i_1 + \cdots + i_{n-1}$.

Proof. For $k = 0$ (18), gives

$$W_{m,r}(n,0;\bar{\boldsymbol{\alpha}}) = (r-\alpha_0)(r-\alpha_1)\cdots(r-\alpha_{n-1}),$$

which is agrees with the definition of $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$.
Also, if $i_{n-1} \in \{0,1\}$, we have that

$$\begin{aligned} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) &= \sum_{I_{n-2}=(n-1)-(k-1)} \binom{r-\alpha_0}{i_0}\binom{r-\alpha_1+(1-i_0)m}{i_1}\cdots\binom{r-\alpha_{n-2}+(n-2-i_0-\cdots-i_{n-3})m}{i_{n-2}} \\ &+ \sum_{I_{n-2}=(n-1)-k} [r-\alpha_{n-1}+(n-1-i_0-i_1-\cdots-i_{n-2})m] \\ &\times \binom{r-\alpha_0}{i_0}\cdots\binom{r-\alpha_{n-2}+(n-2-i_0-\cdots-i_{n-3})m}{i_{n-2}}, \end{aligned}$$

i.e.,

$$W_{m,r}(n,k;\bar{\boldsymbol{\alpha}}) = W_{m,r}(n-1,k-1;\bar{\boldsymbol{\alpha}}) + (r-\alpha_{n-1}+mk)W_{m,r}(n-1,k;\bar{\boldsymbol{\alpha}}).$$

Therefore, by (17) and induction we get the desired result. □

Moreover, setting $m = 1$ and $r = 0$ in (18), we get the explicit formula of the multiparameter Stirling numbers of the second kind introduced in [5], also Setting $m = 1$, $r = 0$ and $\alpha_i = 0$ in this equation we obtain the explicit formula of the Stirling numbers of the second kind.

3.1. Special cases

From Eq. (16) we have the following special cases:

1	$m = 1$ and $r = 0$	$W_{1,0}(n,k;\bar{\boldsymbol{\alpha}}) = S(n,k;\bar{\boldsymbol{\alpha}}))$, the multiparameter non-central Stirling numbers of the second kind, see El-Desouky [8].
2	$m = 1$, $r = 0$ and $\alpha_i = 0$	$W_{1,0}(n,k;\mathbf{0}) = S(n,k)$, the usual Stirling numbers of the second kind, see Stirling [15] .
3	$m = 1$, $r = 0$ and $\alpha_i = \alpha$	$W_{1,0}(n,k;\boldsymbol{\alpha}) = S_\alpha(n,k)$, the noncentral Stirling numbers of the second kind, see Koutras [10].
4	$m = 1$, $r = 0$ and $\alpha_i = i$	$W_{1,0}(n,k;\boldsymbol{i}) = C(n,k;1)$, the C-numbers where $r = 1$, see [6] and [8].
5	$\alpha_i = 0$	$W_{m,r}(n,k;\mathbf{0}) = W_{m,r}(n,k)$, the r-Whitney numbers of the second kind, see Mező [13].
6	$\alpha_i = 0$ and $r = 1$	$W_{m,1}(n,k;\mathbf{0}) = W_m(n,k)$, the Whitney numbers of the second kind, see Benoumhani [2].
7	$\alpha_i = 0$ and $m = 1$	$W_{1,r}(n,k;\mathbf{0}) = S_r(n,k)$, the r-Stirling numbers of the second kind, see Broder [3].
8	$\alpha_i = a$ and $r = 0$	$W_{m,r}(n,k;\boldsymbol{a}) = \tilde{W}_{m,a}(n,k)$, the The noncentral Whitney numbers of the second kind, see Mangontarum [12].
9	Setting $m = 1$, $r = 0$, $\alpha_i = 0$ and then multiplying the given equation by $(\alpha)^n$	$w_{1,0}(n,k;\mathbf{0}) = (\alpha)^{k-n}\left\{ {n \atop k} \right\}^{(\alpha)}$, the translated Whitney numbers of the second kind see [1].

Corollary 3.4. *A new explicit expression for the r-Whitney numbers of the second kind is given by*

$$W_{m,r}(n,k) = \sum_{I_{n-1}=n-k} \binom{r}{i_0}\binom{r+(1-i_0)m}{i_1}\cdots\binom{r+(n-1-i_0-i_1-\cdots-i_{n-2})m}{i_{n-1}}.$$

Proof. The proof follows directly by setting $\alpha_i = 0$ in (18). □

Remark: Mező [13] proved that

$$W_{m,r}(n,k) = \sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i}(mi+r)^n.$$

So, we have the combinatorial identity

$$\sum_{i=0}^{k} \binom{k}{i}(-1)^{k-i}(mi+r)^n = \sum_{I_{n-1}=n-k} \binom{r}{i_0}\binom{r+(1-i_0)m}{i_1}\cdots\binom{r+(n-1-i_0-i_1-\cdots-i_{n-2})m}{i_{n-1}}.$$

3.2. Some relations

1. We determine the relation between generalized Stirling numbers of the first kind $s_{\bar{\beta}}(n,i)$ and multiparameter r-Whitney numbers of the second kind $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$. From (16), we have

$$m^n(x;\bar{\boldsymbol{\beta}})_n = \sum_{k=0}^{n} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})m^k(x)_k,$$

hence

$$m^n \sum_{i=0}^{n} s_{\bar{\beta}}(n,i)x^i = \sum_{k=0}^{n} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})m^k \sum_{i=0}^{k} s(k,i)x^i = \sum_{i=0}^{n}\Big(\sum_{k=i}^{n} m^k W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})s(k,i)\Big)x^i.$$

Equating the coefficients of x^i on both sides, we get

$$m^n s_{\bar{\beta}}(n,i) = \sum_{k=i}^{n} m^k W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})s(k,i). \tag{19}$$

2. Also $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ can be expressed in terms of $S(k,i)$ and $s_{\bar{\beta}}(n,k)$ as

$$(x;\bar{\boldsymbol{\beta}})_n = \sum_{k=0}^{n} s_{\bar{\beta}}(n,k)x^k = \sum_{k=0}^{n} s_{\bar{\beta}}(n,k)\sum_{i=0}^{k} S(k,i)(x)_i,$$

hence

$$m^{-n}\sum_{i=0}^{n} W_{m,r}(n,i;\bar{\boldsymbol{\alpha}})m^i(x)_i = \sum_{i=0}^{n}\Big(\sum_{k=i}^{n} s_{\bar{\beta}}(n,k)S(k,i)\Big)(x)_i.$$

Equating the coefficients of $(x)_i$ on both sides, we get

$$W_{m,r}(n,i;\bar{\boldsymbol{\alpha}}) = m^{n-i}\sum_{k=i}^{n} s_{\bar{\beta}}(n,k)S(k,i). \tag{20}$$

3. Similarly, we can express $S(n,i)$ in terms of $S_{\bar{\beta}}(n,k)$ and $W_{m,r}(k,i;\bar{\boldsymbol{\alpha}})$ as

$$x^n = \sum_{k=0}^{n} S_{\bar{\beta}}(n,k)(x;\bar{\boldsymbol{\beta}})_k,$$

then

$$\sum_{i=0}^{n} S(n,i)(x)_i = \sum_{k=0}^{n} S_{\bar{\beta}}(n,k)m^{-k}\sum_{i=0}^{k} W_{m,r}(k,i;\bar{\boldsymbol{\alpha}})m^i(x)_i = \sum_{i=0}^{n}\Big(\sum_{k=i}^{n} m^{i-k}S_{\bar{\beta}}(n,k)W_{m,r}(k,i;\bar{\boldsymbol{\alpha}})\Big)(x)_i.$$

Equating the coefficients of $(x)_i$ on both sides, we get

$$S(n,i) = \sum_{k=i}^{n} m^{i-k}S_{\bar{\beta}}(n,k)W_{m,r}(k,i;\bar{\boldsymbol{\alpha}}). \tag{21}$$

4. From (1) and (16), we get the orthogonal relation of $w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ and $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ as

$$\begin{aligned} m^n(x)_n &= \sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})(mx+r;\bar{\boldsymbol{\alpha}})_k = \sum_{k=0}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\Big(\sum_{i=0}^{k} W_{m,r}(k,i;\bar{\boldsymbol{\alpha}})m^i(x)_i\Big) \\ &= \sum_{i=0}^{n}\Big(\sum_{k=i}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})W_{m,r}(k,i;\bar{\boldsymbol{\alpha}})\Big)m^i(x)_i, \end{aligned}$$

hence

$$\sum_{k=i}^{n} w_{m,r}(n,k;\bar{\boldsymbol{\alpha}})W_{m,r}(k,i;\bar{\boldsymbol{\alpha}}) = \delta_{ni},$$

where δ_{ni} is Kronecker's delta.

5. Cakić [4] defined the generalized harmonic numbers as

$$H_n(k;\overline{\boldsymbol{\alpha}}) = \sum_{i=0}^{n-1}\frac{1}{(\alpha_i)^k}.$$

From (16) we have

$$\begin{aligned} (mx+r;\overline{\boldsymbol{\alpha}})_n &= \sum_{k=0}^{n} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})m^k(x)_k = \sum_{k=0}^{\infty} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})\sum_{j=0}^{k} w_{m,r}(k,j)(mx+r)^j \\ &= \sum_{j=0}^{\infty}\Big(\sum_{k=j}^{\infty} W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})w_{m,r}(k,j)\Big)(mx+r)^j. \end{aligned} \tag{22}$$

Also, we have

$$\begin{aligned} (mx+r;\overline{\boldsymbol{\alpha}})_n &= \prod_{i=0}^{n}((mx+r)-\alpha_i) = \prod_{i=0}^{n}(-\alpha_i)(1-\frac{mx+r}{\alpha_i}) = \prod_{i=0}^{n}(-\alpha_i).\exp\Big(\sum_{i=0}^{n-1}\log(1-\frac{mx+r}{\alpha_i})\Big) \\ &= \prod_{i=0}^{n}(-\alpha_i).\exp\Big(-\sum_{i=0}^{n-1}\sum_{k=1}^{\infty}\frac{1}{k}\Big(\frac{mx+r}{\alpha_i}\Big)^k\Big) = \prod_{i=0}^{n}(-\alpha_i).\exp\Big(-\sum_{k=1}^{\infty}\frac{(mx+r)^k}{k}\sum_{i=0}^{n-1}\Big(\frac{1}{\alpha_i}\Big)^k\Big) \\ &= \prod_{i=0}^{n}(-\alpha_i).\exp\Big(-\sum_{k=1}^{\infty}\frac{(mx+r)^k}{k}H_n(k;\overline{\boldsymbol{\alpha}})\Big) = \prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\Big(\sum_{k=1}^{\infty}\frac{(mx+r)^k}{k}H_n(k;\overline{\boldsymbol{\alpha}})\Big)^\ell, \end{aligned}$$

using Cauchy rule product of series, this lead to

$$\begin{aligned}(mx+r;\overline{\boldsymbol{\alpha}})_n &= \prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\sum_{j=\ell}^{\infty}\Big(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\overline{\boldsymbol{\alpha}})\Big)(mx+r)^j\\ &= \prod_{i=0}^{n}(-\alpha_i)\sum_{j=0}^{\infty}\Big(\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\Big(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\overline{\boldsymbol{\alpha}})\Big)(mx+r)^j.\end{aligned} \tag{23}$$

From (22) and (23) we have the following identity

$$\sum_{k=j}^{\infty}W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})w_{m,r}(k,j)=\prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\Big(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\overline{\boldsymbol{\alpha}})\Big).$$

3.3. Matrix representation

1. Eq. (19) can be represented in a matrix form as

$$\widehat{\boldsymbol{s}_{\bar\beta}}=\boldsymbol{W}_{m,r}(\bar{\boldsymbol{\alpha}})\,\widehat{\boldsymbol{s}}, \tag{24}$$

where $\boldsymbol{W}_{m,r}(\bar{\boldsymbol{\alpha}})$ is a matrix whose entries are the multiparameter r-Whitney numbers of the second kind.

Remark: It is worth noting that (24)comes directly by multiplying Eq. (13) by $W_{m,r}(\bar{\boldsymbol{\alpha}})$ from left.

For example if $0\le n,k,i\le 3$, Eq.(24) can be written as

$$\begin{pmatrix}1&0&0&0\\-(\alpha_0-r)&m&0&0\\(\alpha_0-r)(\alpha_1-r)&-m(\alpha_0+\alpha_1-2r)&m^2&0\\-(\alpha_0-r)(\alpha_1-r)(\alpha_2-r)&\widehat{s_\beta}(3,1)&-m^2(\alpha_0+\alpha_1+\alpha_2-3r)&m^3\end{pmatrix}=$$

$$\begin{pmatrix}1&0&0&0\\r-\alpha_0&1&0&0\\(r-\alpha_0)(r-\alpha_1)&2r+m-\alpha_0-\alpha_1&1&0\\W_{m,r}(3,0;\bar{\boldsymbol{\alpha}})&W_{m,r}(3,1;\bar{\boldsymbol{\alpha}})&W_{m,r}(3,2;\bar{\boldsymbol{\alpha}})&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&m&0&0\\0&-m^2&m^2&0\\0&2m^3&-3m^3&m^3\end{pmatrix}$$

where $W_{m,r}(3,0;\bar{\boldsymbol{\alpha}})=(r-\alpha_0)(r-\alpha_1)(r-\alpha_2)$, $W_{m,r}(3,1;\bar{\boldsymbol{\alpha}})=(r-\alpha_0)(r-\alpha_1)+(r+m-\alpha_2)(2r+m-\alpha_0-\alpha_1)$ and $W_{m,r}(3,2;\bar{\boldsymbol{\alpha}})=3r+3m-\alpha_0-\alpha_1-\alpha_2$.

2. Eq. (20) can be represented in a matrix form as

$$\widehat{\boldsymbol{s}_{\bar\beta}}\,\boldsymbol{S}=\widehat{\boldsymbol{W}}_{m,r}(\bar{\boldsymbol{\alpha}}), \tag{25}$$

where $\widehat{W}_{m,r}(n,i;\bar{\boldsymbol{\alpha}})=m^iW_{m,r}(n,i;\bar{\boldsymbol{\alpha}})$.

For example if $0\le n,k,i\le 3$, Eq.(25) can be written as

$$\begin{pmatrix}1&0&0&0\\-(\alpha_0-r)&m&0&0\\(\alpha_0-r)(\alpha_1-r)&-m(\alpha_0+\alpha_1-2r)&m^2&0\\-(\alpha_0-r)(\alpha_1-r)(\alpha_2-r)&\widehat{s_\beta}(3,1)&-m^2(\alpha_0+\alpha_1+\alpha_2-3r)&m^3\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&3&1\end{pmatrix}=$$

$$\begin{pmatrix}1&0&0&0\\r-\alpha_0&m&0&0\\(r-\alpha_0)(r-\alpha_1)&m(2r+m-\alpha_0-\alpha_1)&m^2&0\\W_{m,r}(3,0;\bar{\boldsymbol{\alpha}})&mW_{m,r}(3,1;\bar{\boldsymbol{\alpha}})&m^2W_{m,r}(3,2;\bar{\boldsymbol{\alpha}})&m^3\end{pmatrix}$$

3. Eq. (21) can be represented in a matrix form as

$$\boldsymbol{S}_{\bar{\beta}}\ \widehat{\boldsymbol{W}}_{m,r}(\bar{\boldsymbol{\alpha}}) = \boldsymbol{S}\ , \tag{26}$$

where $\widehat{W}_{m,r}(k,i;\bar{\boldsymbol{\alpha}}) = m^{i-k}W_{m,r}(k,i;\bar{\boldsymbol{\alpha}})$.
Remark: It is worth noting that (26) comes directly by multiplying Eq. (15) by $W_{m,r}(\bar{\boldsymbol{\alpha}})$ from right. For example if $0 \le n,k,i \le 3$, Eq. (26) can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\alpha_0-r}{m} & 1 & 0 & 0 \\ \frac{(\alpha_0-r)^2}{m^2} & \frac{\alpha_0+\alpha_1-2r}{m} & 1 & 0 \\ \frac{(\alpha_0-r)^3}{m^3} & \frac{(\alpha_0-r)^2+(\alpha_0-r)(\alpha_1-r)+(\alpha_1-r)^2}{m^2} & \frac{\alpha_0+\alpha_1+\alpha_2-3r}{m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{r-\alpha_0}{m} & 1 & 0 & 0 \\ \frac{(r-\alpha_0)(r-\alpha_1)}{m^2} & \frac{(2r+m-\alpha_0-\alpha_1)}{m} & 1 & 0 \\ \frac{W_{m,r}(3,0;\bar{\alpha})}{m^3} & \frac{W_{m,r}(3,1;\bar{\alpha})}{m^2} & \frac{W_{m,r}(3,2;\bar{\alpha})}{m} & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix}$$

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