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Convergence and Stability of the One-leg θ Method for Stochastic Differential Equations with Piecewise Continuous Arguments

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Abstract. The equivalent relation is established here about the stability of stochastic differential equations with piecewise continuous arguments(SDEPCAs) and that of the one-leg θ method applied to the SDEPCAs. Firstly, the convergence of the one-leg θ method to SDEPCAs under the global Lipschitz condition is proved. Secondly, it is proved that the SDEPCAs are pth($p \in (0,1)$) moment exponentially stable if and only if the one-leg θ method is pth moment exponentially stable for some sufficiently small step-size. Thirdly, the corollaries that the pth moment exponential stability of the SDEPCAs (the one-leg θ method) implies the almost sure exponential stability of the SDEPCAs (the one-leg θ method) are given. Finally, numerical simulations are provided to illustrate the theoretical results.

1. Introduction

In this paper, we consider the following SDEPCAs

$$dx(t) = \mu(x(t), x([t]))dt + \sigma(x(t), x([t]))dB(t) \tag{1}$$

on $t \geq 0$, with initial value $x(0) = \xi \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^d)$, $B(t)$ is an r-dimensional Brownian motion. Here $x(t) = (x_1(t), x_2(t), ..., x_d(t)) \in \mathbb{R}^d$, $\mu : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$, $\mathbb{E}|\xi|^p < \infty$ for all $p > 0$. $[t]$ denotes the integer part of t. The argument $[t]$ has intervals of constancy. Since there is in general no explicit solution to an SDEPCA, numerical solutions are required in practice. The motivation of this paper is to establish the equivalent relation of the almost sure and the small-moment exponential stability between the SDEPCAs and the one-leg θ method applied to SDEPCAs.

At present, there are some results on the equivalence of stability between the stochastic differential equations(SDEs) and the numerical approximations. Higham et al. [3] proved that the mean square exponential stability of the SDEs is equivalent to that of the numerical method under the finite-time convergence condition, and the corresponding second-moment Lyapunov exponent bounds can be taken

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to be arbitrarily close. Afterwards, in [4], it was shown that the Euler method preserves both the almost sure and the small moment exponential stability on the scalar-noise SDEs which are linear or satisfy the linear growth condition. It is also proved that the backward Euler method maintains the almost sure exponential stability under the one-sided Lipschitz condition. Instead of Lyapunov functions, Mao [9] used the discrete approximation to investigate whether the almost sure exponential stability of the SDEs is shared with that of a numerical method or not. There are also several works on the stability of stochastic delay differential equations(SDDEs). Using continuous and discrete semimartingale convergence theorems, Wu [17] presented that the Euler method preserved the almost sure exponential stability of SDDEs with the local Lipschitz and the linear growth conditions, while it didn't hold when the SDDEs satisfy the one-sided Lipschitz condition. The author also showed that the backward Euler method preserves the almost sure exponential stability of SDDEs with the local Lipschitz and the one-sided Lipschitz conditions in [17]. In 2011, conditions under which the Euler method shared the almost sure exponential stability of the exact solution for SDDEs were given in [18], where the nonnegative semimartingale convergence theorem was employed.

The deterministic differential equation with piecewise continuous arguments(EPCAs) is formulated as

$$\begin{cases} x'(t) = f(t, x(t), x(\alpha(t))), \ t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{2}$$

where $\alpha(t)$ has intervals of constancy. The solutions of (2) are determined by a finite set of initial data, rather than by an initial function, as in the case of general functional differential equations(FDEs). In fact, the EPCAs have the structure of continuous dynamical systems within intervals of certain lengths. Therefore, the EPCAs are with properties of both the differential and the difference equations. The theory of EPCAs is firstly studied by Winner[15]. Subsequently, a number of works on EPCAs have been presented (see e.g. [16],[1], [2],[7],[12],[13] and references therein).

In recent years, Song and Zhang et al. [14] considered the convergence in probability of the Euler-Maruyama method for the SDEPCAs under Khasminskii-type condition. Subsequently, they presented several conditions under which the Euler method is convergent to the SDEPCAs in mean square in [19]. In 2014, Li [6] devoted to the existence and exponential stability of the solutions for stochastic cellular neural networks with piecewise constant argument, and some sufficient conditions were given for the existence and uniqueness of the equilibrium point for the addressed neural networks. Recently, Milošević [10] studied the convergence and stability of the Euler-Maruyama method for retarded SDEPCAs in mean square under the global Lipschitz condition. However, there is so far no theory on the equivalent relation of the almost sure and the small-moment exponential stability between SDEPCAs and their numerical methods. The aim of this paper is to study whether the one-leg θ method can preserve the stochastic stability (the pth moment exponential stability or the almost sure exponential stability) of the SDEPCAs (1) or not.

Compared with [10], there are several differences to highlight in this paper. On the one hand, we use the discrete solutions to obtain the results on stability, while [10] adopts the continuous-time approximation. In general, only the discrete solutions are computed by the numerical methods but not the continuous-time approximate solutions. Therefore, it is much more useful if the theory is based on the discrete solutions. On the other hand, many inequalities used in [10] for the 2nd moment do not work for small-moment. Hence, we develop this paper to deal with the small-moment case. Moreover, in [10], Milošević regards the SDEPCAs as functional differential equations and employs the techniques in the functional differential equations to get the lemmas and theorems. However, we resort to the constant intervals of $[t]$, which reveals the feature of the SDEPCAs, to develop the results in this paper.

An outline of this paper is as follows. Section 2 discusses some preliminary theory on the analytical solution. Section 3 proves the convergence of the one-leg θ method on SDEPCAs under the global-Lipschitz condition. In Section 4, we show that the SDEPCAs is pth($p \in (0, 1)$) moment exponentially stable if and only if the one-leg θ method is pth moment exponentially stable for some sufficiently small step-size, and the pth moment exponential stability of both the SDEPCAs and the one-leg θ method can derive the almost sure exponential stability of the SDEPCAs and the one-leg θ method. The mean-reverting Ornstein-Uhlenbeck process is taken as an example to verify the theoretical analysis in Section 5.

2. Preliminary notations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a completed probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $B(t)$ be an r-dimensional Brownian motion defined on this probability space. Throughout this paper, we assume $p \in (0, 1)$. We use the notation $\|x\| := (x_1^2 + x_2^2 + \ldots + x_d^2)^{\frac{1}{2}}$ for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. For $a, b \in \mathbb{R}$, we use $a \vee b$ and $a \wedge b$ for $\max(a, b)$ and $\min(a, b)$, respectively. Equation (1) is equivalent to the following stochastic integral equation

$$x(t) = \xi + \int_0^t \mu(x(s), x([s]))ds + \int_0^t \sigma(x(s), x([s]))dB(s), \quad t \geq 0. \tag{3}$$

It is helpful for us to introduce an inequality used in the rest of this paper.

Lemma 2.1. *([5]) Suppose that $a_i (i = 1, 2, .., n)$ are complex numbers, for any $p > 0$, then there exists C_p such that*

$$\Big(\sum_{i=1}^n |a_i|\Big)^p \leq C_p \sum_{i=1}^n |a_i|^p, \tag{4}$$

where $C_p = \begin{cases} 1, & 0 < p \leq 1, \\ n^{p-1}, & p > 1. \end{cases}$

Let us give the definition of the solution for Eq. (1).

Definition 2.2. *[19] An $\mathbb{R}^d$-valued stochastic process $\{x(t)\}_{t\geq 0}$ is called a solution of SDEPCAs* (1) *on* $[0, \infty)$, *if it has the following properties:*

- *$\{x(t)\}_{t\geq 0}$ is continuous on $[0, \infty)$ and $\mathcal{F}_t$ adapted;*
- *$\{\mu(x(t), x([t]))\}_{t\geq 0} \in \mathcal{L}^1([0, \infty), \mathbb{R}^d)$ and $\{\sigma(x(t), x([t]))\}_{t\geq 0} \in \mathcal{L}^2([0, \infty), \mathbb{R}^{d\times r})$;*
- *Equation* (3) *is satisfied on each interval $[k, k + 1) \subset [0, \infty)$ with integral end points almost surely.*

A solution $\{x(t)\}_{t\geq 0}$ is said to be unique if any other solution $\{\bar{x}(t)\}_{t\geq 0}$ is indistinguishable from $\{x(t)\}_{t\geq 0}$, that is

$$\mathbb{P}\{x(t) = \bar{x}(t), \text{ for any } t \in [0, \infty)\} = 1.$$

Remark 2.3. *Denote the solution of equation* (1) *by $x(t; 0, \xi)$. Note from* (3) *that for any integer $\bar{k} \in [0, \infty)$,*

$$x(t) = x(\bar{k}) + \int_{\bar{k}}^t \mu(x(s), x([s]))ds + \int_{\bar{k}}^t \sigma(x(s), x([s]))dB(s), \quad t \geq \bar{k}. \tag{5}$$

This is an SDEPCA on $[\bar{k}, \infty)$ with initial value $x(\bar{k}) = x(\bar{k}; 0, \xi)$, whose solution is denoted by $x(t; \bar{k}, x(\bar{k}; 0, \xi))$. Therefore, the solution of (1) *has the following semigroup property*

$$x(t; 0, \xi) = x(t; \bar{k}, x(\bar{k}; 0, \xi)), \qquad 0 \leq \bar{k} \leq t < \infty. \tag{6}$$

In order to investigate whether the numerical solution solved by the one-leg θ method shares with the almost sure exponential stability of (1) or not, we impose the global Lipschitz condition on the coefficients of (1).

Assumption 2.4. *Assume that there exists a positive constant K such that for any $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$, the coefficients μ and σ satisfy*

$$\|\mu(x_1, y_1) - \mu(x_2, y_2)\|^2 \vee \|\sigma(x_1, y_1) - \sigma(x_2, y_2)\|^2 \leq K(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2). \tag{7}$$

Furthermore, assume $\mu(0, 0) = 0$ and $\sigma(0, 0) = 0$.

We should point out that equation (1) has the trivial solution $x(t) \equiv 0$ corresponding to the initial value $\xi = 0$.

Remark 2.5. *Assumption 2.4 implies that there exists a unique global solution $x(t)$ to equation (1) on $t \geq 0$(see [19]).*

Lemma 2.6. *Let Assumption 2.4 hold. For any $p \in (0,1)$, there exists $H(t,p,K)$ dependent on $t,\ p,\ K$ such that the solution $x(t)$ of SDEPCAs satisfies*

$$\mathbb{E}\|x(t)\|^p \leq H(t,p,K)\mathbb{E}\|\xi\|^p, \quad t \geq 0. \tag{8}$$

Proof. Since the solution of equation (1) satisfies the integral equation (3), by conditional expectation and (4), we have

$$\begin{aligned}\mathbb{E}(\|x(t)\|^2|\xi) &\leq 3\mathbb{E}(\|\xi\|^2|\xi) + 3\mathbb{E}(\|\int_0^t \mu(x(s),x([s]))ds\|^2|\xi) + 3\mathbb{E}(\|\int_0^t \sigma(x(s),x([s]))dB(s)\|^2|\xi) \\ &\leq 3\|\xi\|^2 + 3t\int_0^t \mathbb{E}(\|\mu(x(s),x([s]))\|^2|\xi)ds + 3\int_0^t \mathbb{E}(\|\sigma(x(s),x([s]))\|^2|\xi)ds \\ &\leq 3\|\xi\|^2 + 3(t+1)K\int_0^t \mathbb{E}\Big((\|x(s)\|^2 + \|x([s])\|^2)|\xi\Big)ds.\end{aligned}$$

Hence

$$\sup_{0\leq t\leq t_1} \mathbb{E}(\|x(t)\|^2|\xi) \leq 3\|\xi\|^2 + 6(t_1+1)K\int_0^{t_1} \sup_{0\leq r\leq s} \mathbb{E}(\|x(r)\|^2|\xi)ds. \tag{9}$$

Using the Gronwall inequality, we obtain

$$\sup_{0\leq t\leq t_1} \mathbb{E}(\|x(t)\|^2|\xi) \leq 3e^{6Kt_1(t_1+1)}\|\xi\|^2 \tag{10}$$

for any $t_1 \geq t$. Therefore, for arbitrary $t \geq 0$, $x(t)$ yields

$$\mathbb{E}\|x(t)\|^2 = \mathbb{E}\Big(\mathbb{E}(\|x(t)\|^2|\xi)\Big) \leq 3e^{6Kt(t+1)}\mathbb{E}\|\xi\|^2. \tag{11}$$

According to the Hölder inequality, for any $p \in (0,1)$, we have

$$\mathbb{E}\|x(t)\|^p = \mathbb{E}\Big(\mathbb{E}(\|x(t)\|^p|\xi)\Big) \leq \mathbb{E}\Big(3^{\frac{p}{2}}e^{3pKt(t+1)}\|\xi\|^p\Big) = H(t,p,K)\mathbb{E}\|\xi\|^p, \tag{12}$$

where $H(t,p,K) = 3^{\frac{p}{2}}e^{3pKt(t+1)}$. The lemma is proved. □

For any integer $\bar{k} \in [0,\infty)$, if we regard $\{x(t)\}_{t\geq\bar{k}}$ as the solution of (1) on $t \geq \bar{k}$ with initial value $x(\bar{k})$ at $t = \bar{k}$, then, by time-homogeneity, $x(t)$ satisfies

$$\mathbb{E}\|x(t)\|^p \leq H(t-\bar{k},p,K)\mathbb{E}\|x(\bar{k})\|^p, \quad t \geq \bar{k}. \tag{13}$$

3. Convergence of the one-leg θ method

In this section, we consider the one-leg θ method to (1) on $[0,T]$. The main result is to obtain the convergence of the one-leg θ scheme under Assumption 2.4 on $[0,T]$.

Let $h = \frac{1}{m}$ be given step-size with integer $m > 1$. Grid points t_n are defined as $t_n = nh,\ n = 0,1,\cdots$. For simplicity, let $T = Nh$, $N \in \mathbb{N}^+$. The one-leg θ method to (1) is defined as

$$y_{n+1} = y_n + h\mu(y^h((n+\theta)h), y^h([(n+\theta)h])) + \sigma(y_n, y_{[nh]m})\Delta B_n, \quad n = 0,1,2,... \tag{14}$$

where $y_0 = x(0) = \xi$, $\Delta B_n = B(t_{n+1}) - B(t_n)$, $\theta \in [0, 1]$, and y_n is the approximation to $x(t_n)$ at t_n. Moreover, $y^h(t)$ with $t \geq 0$ is defined by the linear interpolation, i.e.

$$y^h(t) = \frac{t - nh}{h} y_{n+1} + \frac{(n+1)h - t}{h} y_n, \quad nh \leq t < (n+1)h, \quad n = 0, 1, 2, \cdots .$$

As we know, for arbitrary $n = 0, 1, 2, \cdots$, there exist $k \in \mathbb{N}$ and $l = 0, 1, 2, \cdots, m-1$ such that $n = km + l$. Hence (14) can be written as

$$y_{km+l+1} = y_{km+l} + h\mu(\theta y_{km+l+1} + (1-\theta) y_{km+l}, y_{km}) + \sigma(y_{km+l}, y_{km})\Delta B_{km+l}. \tag{15}$$

Remark 3.1. *If $t_n = nh \in [0, T]$, $n = 0, 1, 2, \cdots, N$, then $nh = (km + l)h = k + lh$, $l = 0, 1, \cdots, m-1$. Because $lh < 1$, hence $k \leq T$.*

Lemma 3.2. *Under Assumption 2.4, if $K^{\frac{1}{2}}\theta h < 1$, then the one-leg θ method* (15) *can be solved uniquely for y_{km+l+1} with probability* 1.

Proof. Writing (15) as $F(y_{km+l+1}) = y_{km+l+1}$, $l = 0, 1, 2, \cdots, m-1$ and using (7), we have

$$\begin{aligned} \|F(x) - F(y)\| &= h\|\mu(\theta x + (1-\theta)a, b) - \mu(\theta y + (1-\theta)a, b)\| \\ &\leq hK^{\frac{1}{2}}\theta\|x - y\|. \end{aligned}$$

Due to $K^{\frac{1}{2}}\theta h < 1$ and Banach contraction mapping theorem[11], the one-leg θ method has unique solution. □

In order to get the convergence theorem, several key lemmas are presented.

Lemma 3.3. *Let Assumption 2.4 hold. If h satisfies $12Kh\theta^2 < 1$, then, for any $p \in (0, 1)$, we have*

$$\sup_{0 \leq t_{km+l} \leq T} \mathbb{E}\|y_{km+l}\|^p \leq \bar{H}(T, p, K)\mathbb{E}\|\xi\|^p \quad for\ any\ T > 0. \tag{16}$$

where $k \in \mathbb{N}$, $l = 0, 1, 2, \cdots, m-1$, $\bar{H}(T, p, K) = 6^{p(T+1)/2}(1 + 2K)^{p(T+1)/2}e^{9Kp(T+1)}$.

Proof. For convenience, we let

$$\begin{aligned} z_1(t) &= \sum_{km+l=0}^{\infty} y_{km+l} 1_{[t_{km+l}, t_{km+l+1})}(t), \\ z_2(t) &= \sum_{km+l=0}^{\infty} y_{km+l+1} 1_{[t_{km+l}, t_{km+l+1})}(t), \\ z_3(t) &= \sum_{k=0}^{\infty} y_{km} 1_{[t_{km}, t_{(k+1)m})}(t) \end{aligned}$$

for $k \in \mathbb{N}$, $l = 0, 1, 2, \cdots, m-1$.
According to (15), we obtain

$$\begin{aligned} y_{km+l+1} &= y_{km+l} + h\mu(\theta y_{km+l+1} + (1-\theta) y_{km+l}, y_{km}) + \sigma(y_{km+l}, y_{km})\Delta B_{km+l} \\ &= y_{km+l} + \int_{t_{km+l}}^{t_{km+l+1}} \mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))ds + \int_{t_{km+l}}^{t_{km+l+1}} \sigma(z_1(s), z_3(s))dB(s) \\ &= y_{km} + \int_{t_{km}}^{t_{km+l+1}} \mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))ds + \int_{t_{km}}^{t_{km+l+1}} \sigma(z_1(s), z_3(s))dB(s). \end{aligned} \tag{17}$$

Hence

$$\begin{aligned}\mathbb{E}(\|y_{km+l+1}\|^2|\xi) \leq & 3\mathbb{E}(\|y_{km}\|^2|\xi) + 3\mathbb{E}\Big(\|\int_{t_{km}}^{t_{km+l+1}} \sigma(z_1(s), z_3(s))dB(s)\|^2|\xi\Big) \\ & + 3\mathbb{E}\Big(\|\int_{t_{km}}^{t_{km+l+1}} \mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))ds\|^2|\xi\Big) \\ \leq & 3\mathbb{E}(\|y_{km}\|^2|\xi) + 3\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\|\sigma(z_1(s), z_3(s))\|^2|\xi\Big)ds \\ & + 3(l+1)h\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\|\mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))\|^2|\xi\Big)ds.\end{aligned}$$

By Assumption 2.4 and $(a+b)^2 \leq 2(a^2+b^2)$, it is easy to derive that

$$\begin{aligned}\mathbb{E}\Big(\|y_{km+l+1}\|^2|\xi\Big) \leq & 3\mathbb{E}(\|y_{km}\|^2|\xi) + 3K\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big((\|z_1(s)\|^2 + \|z_3(s)\|^2)|\xi\Big)ds \\ & + 3(l+1)Kh\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\Big(2\Big(\theta^2\|z_2(s)\|^2 + (1-\theta)^2\|z_1(s)\|^2\Big) + \|z_3(s)\|^2\Big)|\xi\Big)ds \\ = & 3\mathbb{E}(\|y_{km}\|^2|\xi) + 3K\Big(2(1-\theta)^2(l+1)h + 1\Big)\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}(\|z_1(s)\|^2|\xi)ds \\ & + 6\theta^2K(l+1)h\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}(\|z_2(s)\|^2|\xi)ds \\ & + 3K\Big(1 + (l+1)h\Big)\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}(\|z_3(s)\|^2|\xi)ds.\end{aligned} \tag{18}$$

Because of $z_2(t_{km+l}) = z_1(t_{km+l+1})$, we have

$$\int_{t_{km}}^{t_{km+l+1}} \|z_2(s)\|^2 ds = \int_{t_{km+1}}^{t_{km+l+1}} \|z_1(s)\|^2 ds + h\|y_{km+l+1}\|^2.$$

According to $(l+1)h \leq mh = 1$ and $\theta^2 + (1-\theta)^2 \leq 1$ $(\theta \in [0,1])$, (18) yields

$$\begin{aligned}\mathbb{E}(\|y_{km+l+1}\|^2|\xi) \leq & 3(1+2K)\mathbb{E}(\|y_{km}\|^2|\xi) + 6\theta^2Kh\mathbb{E}(\|y_{km+l+1}\|^2|\xi) \\ & + 3K\Big(1 + 2\theta^2 + 2(1-\theta)^2\Big)\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}(\|z_1(s)\|^2|\xi)ds \\ \leq & 3(1+2K)\mathbb{E}(\|y_{km}\|^2|\xi) + 6\theta^2Kh\mathbb{E}(\|y_{km+l+1}\|^2|\xi) + 9Kh\sum_{j=0}^{l} \mathbb{E}(\|y_{km+j}\|^2|\xi).\end{aligned}$$

By $12\theta^2Kh < 1$, for all $l = 0, 1, \cdots, m-1$, we obtain

$$\mathbb{E}(\|y_{km+l+1}\|^2|\xi) \leq 6(1+2K)\mathbb{E}(\|y_{km}\|^2|\xi) + 18Kh\sum_{j=0}^{l} \mathbb{E}(\|y_{km+j}\|^2|\xi). \tag{19}$$

According to the Gronwall inequality, we have

$$\mathbb{E}(\|y_{km+l+1}\|^2|\xi) \leq 6(1+2K)e^{18K}\mathbb{E}(\|y_{km}\|^2|\xi) \qquad l = 0, 1, \cdots, m-1. \tag{20}$$

If $l = m-1$, then

$$\mathbb{E}(\|y_{(k+1)m}\|^2|\xi) \leq 6(1+2K)e^{18K}\mathbb{E}(\|y_{km}\|^2|\xi). \tag{21}$$

Hence (20) satisfies that

$$\begin{aligned}\mathbb{E}(\|y_{km+l+1}\|^2|\xi) \leq& 6(1+2K)e^{18K}\mathbb{E}(\|y_{km}\|^2|\xi)\\ \leq& 6^2(1+2K)^2e^{36K}\mathbb{E}(\|y_{(k-1)m}\|^2|\xi)\\ \leq& \dots\\ \leq& 6^{k+1}(1+2K)^{(k+1)}e^{18K(k+1)}\mathbb{E}(\|\xi\|^2|\xi)\\ \leq& 6^{T+1}(1+2K)^{T+1}e^{18K(T+1)}\|\xi\|^2.\end{aligned} \tag{22}$$

For any $p \in (0,1)$, we have

$$\begin{aligned}\mathbb{E}\|y_{km+l}\|^p =& \mathbb{E}\Big(\mathbb{E}(\|y_{km+l}\|^p|\xi)\Big) \leq \mathbb{E}\Big(\mathbb{E}(\|y_{km+l}\|^2|\xi)\Big)^{\frac{p}{2}}\\ \leq& 6^{p(T+1)/2}(1+2K)^{p(T+1)/2}e^{9Kp(T+1)}\mathbb{E}\|\xi\|^p.\end{aligned} \tag{23}$$

The proof is now complete. □

Lemma 3.4. *Under Assumption 2.4, if h is sufficiently small for* $12Kh < 1$, *then the solution* $x(t)$ *of SDEPCAs* (1) *satisfies*

$$\mathbb{E}\|x(t)-x(t_{km+l})\|^2 \vee \mathbb{E}\|x(t)-x(t_{km+l+1})\|^2 \leq C(K,T)h\mathbb{E}\|\xi\|^2 \tag{24}$$

for any $0 \leq t_{km+l} \leq t < t_{km+l+1} \leq T$, *and* $C(K,T) = (1+12K)e^{6KT(T+1)}$.

Proof. To prove this lemma, we only need to prove the first part, and the proof of the second part is similar to that of the first part.

According to (3), for $0 \leq t_{km+l} \leq t < t_{km+l+1} \leq T$, we have

$$x(t)-x(t_{km+l}) = \int_{t_{km+l}}^{t} \mu(x(s),x([s]))ds + \int_{t_{km+l}}^{t} \sigma(x(s),x([s]))dB(s).$$

By Lemma 2.1, the Hölder inequality, and Assumption 2.4, it can be deduced

$$\begin{aligned}&\mathbb{E}(\|x(t)-x(t_{km+l})\|^2|\xi)\\ \leq& 2\mathbb{E}\Big(\|\int_{t_{km+l}}^{t}\mu(x(s),x([s]))ds\|^2|\xi\Big) + 2\mathbb{E}\Big(\|\int_{t_{km+l}}^{t}\sigma(x(s),x([s]))dB(s)\|^2|\xi\Big)\\ \leq& 2(t-t_{km+l})\int_{t_{km+l}}^{t}\mathbb{E}(\|\mu(x(s),x([s]))\|^2|\xi)ds + 2\int_{t_{km+l}}^{t}\mathbb{E}(\|\sigma(x(s),x([s]))\|^2|\xi)ds\\ \leq& 2(1+t-t_{km+l})K\int_{t_{km+l}}^{t}\mathbb{E}\Big((\|x(s)\|^2+\|x([s])\|^2)|\xi\Big)ds.\end{aligned} \tag{25}$$

Using the Inequality (10), we obtain

$$\begin{aligned}\mathbb{E}(\|x(t)-x(t_{km+l})\|^2|\xi) \leq& 12K(1+h)e^{6KT(T+1)}h\|\xi\|^2\\ \leq& (1+12K)e^{6KT(T+1)}h\|\xi\|^2.\end{aligned} \tag{26}$$

Therefore

$$\mathbb{E}\|x(t)-x(t_{km+l})\|^2 = \mathbb{E}\Big(\mathbb{E}(\|x(t)-x(t_{km+l})\|^2|\xi)\Big) \leq (1+12K)e^{6KT(T+1)}h\mathbb{E}\|\xi\|^2.$$

Similarly, we obtain

$$\mathbb{E}\|x(t)-x(t_{km+l+1})\|^2 \leq (1+12K)e^{6KT(T+1)}h\mathbb{E}\|\xi\|^2.$$

The proof is completed. □

Next, the convergence theorem of the one-leg θ method in the pth moment is given.

Theorem 3.5. *Under Assumption 2.4, if $p \in (0,1)$ and h is sufficiently small with $24K\theta^2 h < 1$, then the one-leg θ method is convergent and satisfies*

$$\sup_{0\le t_{km+l}\le T} \mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p \le \widetilde{C}(K,p,T)h^{p/2}\mathbb{E}\|\xi\|^p \quad for\ any\ T>0, \tag{27}$$

where $\widetilde{C}(K,p,T)$ is independent of h, $k \in \mathbb{N}$ and $l = 0,1,2,\cdots,m-1$.

Proof. In view of equations (5) and (17), for $k \in \mathbb{N}$ and $l = 0,1,2,\cdots,m-1$, the following expression holds

$$\begin{aligned} x(t_{km+l+1}) - y_{km+l+1} =& x(t_{km}) - y_{km} + \int_{t_{km}}^{t_{km+l+1}} \Big(\sigma(x(s),x([s])) - \sigma(z_1(s),z_3(s))\Big)dB(s) \\ &+ \int_{t_{km}}^{t_{km+l+1}} \Big(\mu(x(s),x([s])) - \mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))\Big)ds. \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) \le 3\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\ &+ 3\mathbb{E}\Big(\|\int_{t_{km}}^{t_{km+l+1}} \Big(\mu(x(s),x([s])) - \mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))\Big)ds\|^2|\xi\Big) \\ &+ 3\mathbb{E}\Big(\|\int_{t_{km}}^{t_{km+l+1}} \Big(\sigma(x(s),x([s])) - \sigma(z_1(s),z_3(s))\Big)dB(s)\|^2|\xi\Big). \end{aligned}$$

Due to the Hölder inequality, the property of the stochastic integral and Assumption 2.4, we show

$$\begin{aligned} &\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) \le 3\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\ &+ 3(l+1)h\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\|\mu(x(s),x([s])) - \mu(\theta z_2(s) + (1-\theta)z_1(s), z_3(s))\|^2|\xi\Big)ds \\ &+ 3\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\|\sigma(x(s),x([s])) - \sigma(z_1(s),z_3(s))\|^2|\xi\Big)ds \\ \le& 3\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) + 3K(l+1)h\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\|x([s]) - z_3(s)\|^2|\xi\Big)ds \\ &+ 3K(l+1)h\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big(\|x(s) - \theta z_2(s) - (1-\theta)z_1(s)\|^2|\xi\Big)ds \\ &+ 3K\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big((\|x(s) - z_1(s)\|^2 + \|x([s]) - z_3(s)\|^2)|\xi\Big)ds. \end{aligned} \tag{28}$$

Let

$$y_1(s) = \sum_{km+l=0}^{\infty} x(t_{km+l})1_{[t_{km+l},y_{km+l+1})}(s), \quad y_2(s) = \sum_{km+l=0}^{\infty} x(t_{km+l+1})1_{[t_{km+l},t_{km+l+1})}(s).$$

According to Lemma 3.4 and $(l+1)h \le 1$, (28) yields

$$\begin{aligned}
&\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) \le 3\Big(1 + K\Big(1 + (l+1)h\Big)(l+1)h\Big)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\
&\quad + 12K\theta^2(l+1)h\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big((\|x(s) - y_2(s)\|^2 + \|y_2(s) - z_2(s)\|^2)|\xi\Big)ds \\
&\quad + 6K\Big(1 + 2(1-\theta)^2(l+1)h\Big)\int_{t_{km}}^{t_{km+l+1}} \mathbb{E}\Big((\|x(s) - y_1(s)\|^2 + \|y_1(s) - z_1(s)\|^2)|\xi\Big)ds \\
&\le 3\Big(1 + K\Big(1 + (l+1)h\Big)(l+1)h\Big)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\
&\quad + 12K\theta^2(l+1)h\sum_{j=0}^{l}\int_{t_{km+j}}^{t_{km+j+1}} \mathbb{E}(\|x(s) - x(t_{km+j+1})\|^2|\xi)ds \\
&\quad + 6K\Big(1 + 2(1-\theta)^2(l+1)h\Big)\sum_{j=0}^{l}\int_{t_{km+j}}^{t_{km+j+1}} \mathbb{E}(\|x(s) - x(t_{km+j})\|^2|\xi)ds \\
&\quad + 12K\theta^2(l+1)h^2\sum_{j=0}^{l} \mathbb{E}(\|x(t_{km+j+1}) - y_{km+j+1}\|^2|\xi) \\
&\quad + 6K\Big(1 + 2(1-\theta)^2(l+1)h\Big)h\sum_{j=0}^{l} \mathbb{E}(\|x(t_{km+j}) - y_{km+j}\|^2|\xi) \\
&\le 12K\theta^2 h\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) + 3(1+2K)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\
&\quad + 6K\Big(1 + 2\theta^2 + 2(1-\theta)^2\Big)C(K,T)\|\xi\|^2 h \\
&\quad + 6K\Big(1 + 2(1-\theta)^2 + 2\theta^2\Big)h\sum_{j=0}^{l} \mathbb{E}(\|x(t_{km+j}) - y_{km+j}\|^2|\xi).
\end{aligned}$$

Using $24K\theta^2 h < 1$ and $\theta^2 + (1-\theta)^2 \le 1$, we obtain

$$\begin{aligned}
&\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) \le 6(1+2K)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\
&+ 36KC(K,T)\|\xi\|^2 h + 36Kh\sum_{j=0}^{l} \mathbb{E}(\|x(t_{km+j}) - y_{km+j}\|^2|\xi).
\end{aligned} \tag{29}$$

Due to the Gronwall inequality, for $l = 0, 1, 2, \cdots, m-1$, we have

$$\begin{aligned}
&\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) \le (A_0\|\xi\|^2 h + B_0\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi))F_0 \\
&= A_0F_0\|\xi\|^2 h + (B_0F_0)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi).
\end{aligned} \tag{30}$$

where $A_0 = 36KC(K,T)$, $B_0 = 6(1+2K)$ and $F_0 = e^{36K}$. If $l = m-1$, then

$$\mathbb{E}(\|x(t_{(k+1)m}) - y_{(k+1)m}\|^2|\xi) \le A_0F_0\|\xi\|^2 h + (B_0F_0)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi). \tag{31}$$

Hence (29) yields

$$\begin{aligned}
&\mathbb{E}(\|x(t_{km+l+1}) - y_{km+l+1}\|^2|\xi) \le A_0F_0\|\xi\|^2 h + (B_0F_0)\mathbb{E}(\|x(t_{km}) - y_{km}\|^2|\xi) \\
&\le A_0F_0\|\xi\|^2 h + (B_0F_0)A_0F_0\|\xi\|^2 h + (B_0F_0)^2\mathbb{E}(\|x(t_{(k-1)m}) - y_{(k-1)m}\|^2|\xi) \\
&\le \cdots \\
&\le \frac{(B_0F_0)^{k+1} - 1}{B_0F_0 - 1}A_0F_0\|\xi\|^2 h \\
&\le \frac{(B_0F_0)^{T+1} - 1}{B_0F_0 - 1}A_0F_0\|\xi\|^2 h.
\end{aligned} \tag{32}$$

Using the Hölder inequality gives

$$\begin{aligned} &\sup_{0\le t_{km+l}\le T} \mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p = \sup_{0\le t_{km+l}\le T} \mathbb{E}\Big(\mathbb{E}(\|x(t_{km+l}) - y_{km+l}\|^p|\xi)\Big) \\ \le &\sup_{0\le t_{km+l}\le T} \mathbb{E}\Big(\mathbb{E}(\|x(t_{km+l}) - y_{km+l}\|^2|\xi)\Big)^{\frac{p}{2}} \le \tilde{C}(K,p,T)h^{\frac{p}{2}}\mathbb{E}\|\xi\|^p, \end{aligned} \tag{33}$$

where $p \in (0, 1)$ and $\tilde{C}(K,p,T) = \Big(\frac{(B_0F_0)^{T+1}-1}{B_0F_0-1}A_0F_0\Big)^{\frac{p}{2}}$. The proof is completed.
□

4. Stability of the one-leg θ method

In this section, we aim to investigate whether the one-leg θ method shares the exponential stability with SDEPCAs (1) in the sense of pth($p \in (0, 1)$) moment or not. Theorem 4.5 gives the positive answer. To begin with, two useful definitions are given.

Definition 4.1. *Let $p > 0$. The SDEPCA* (1) *is said to be exponentially stable in pth moment if there exists a pair of positive constants λ and M such that for any initial value ξ*

$$\mathbb{E}\|x(t)\|^p \le M\mathbb{E}\|\xi\|^p e^{-\lambda t} \quad \text{for all } t \ge 0, \tag{34}$$

where λ is the rate constant and M is the growth constant.

Since SDEPCAs (1) is time-homogeneous, (34) has the following more general form

$$\mathbb{E}\|x(t)\|^p \le M\mathbb{E}\|x(\bar{k})\|^p e^{-\lambda(t-\bar{k})} \quad \text{for all } t \ge \bar{k}, \tag{35}$$

where $\bar{k}$ is an integer.

Definition 4.2. *Let $p > 0$. For a given step size $h > 0$, the one-leg θ method is said to be exponentially stable in pth moment on SDEPCA* (1) *if there exists a pair of positive constants γ and L such that for any initial value ξ*

$$\mathbb{E}\|y_{km+l}\|^p \le L\mathbb{E}\|\xi\|^p e^{-\gamma(km+l)h}, \tag{36}$$

for all $k \in \mathbb{N}$, $l = 0, 1, 2, ..., m-1$, where γ is the rate constant and L is the growth constant.

In fact, for each $\bar{k} \in \mathbb{N}$, we know that if $t \in [\bar{k}, \bar{k}+1)$, then SDEPCAs (1) become SDEs. Hence, if the initial time $t_0 = \bar{k}$ and the initial value $x(\bar{k})$ of (1) are given, then SDEPCAs possess Markov property in $[\bar{k}, \bar{k}+1)$. The numerical solution $\{y_{km+l}\}_{km+l\ge\bar{k}m}$ is solved by the one-leg θ method on (1) with the initial value $x(t_{\bar{k}m}) = y_{\bar{k}m}$. By time-homogeneity, (36) yields the following general form

$$\mathbb{E}\|y_{km+l}\|^p \le L\mathbb{E}\|y_{\bar{k}m}\|^p e^{-\gamma(km+l-\bar{k}m)h} \tag{37}$$

for all $k \ge \bar{k}$, $l = 0, 1, 2, \cdots, m-1$.

Theorem 4.3. *Let Assumption 2.4 hold. Assume that the SDEPCAs* (1) *is pth($p \in (0, 1)$) moment exponentially stable and satisfies* (34)*. Then there exists $h^* > 0$ such that for every $0 < h < \min(h^*, \frac{1}{24K\theta^2})$, the one-leg θ method applied to SDEPCAs* (1) *is pth moment exponentially stable with rate constant $\gamma = \frac{1}{2}\lambda$ and growth constant $L = \bar{H}(K,p,T)e^{\frac{1}{2}\lambda T}$, where $T = 1 + [\frac{4}{\lambda}logM]$. And $\bar{H}(K,p,T)$ is given by Lemma 3.3.*

Proof. Let $24K\theta^2 h < 1$. It is not difficult to know $\frac{4}{\lambda}logM < T$. Hence,

$$Me^{-\lambda T} < e^{-\frac{3}{4}\lambda T}. \tag{38}$$

Due to $p \in (0, 1)$ and the Elementary Inequality (4), we have

$$\mathbb{E}\|y_{km+l}\|^p \le \mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p + \mathbb{E}\|x(t_{km+l})\|^p \tag{39}$$

for $k \in \mathbb{N}$, $l = 0, 1, 2, \cdots, m-1$. Using Theorem 3.5, we get

$$\sup_{0 \le t_{km+l} \le 2T} \mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p \le \widetilde{C}(K, p, 2T) h^{\frac{p}{2}} \mathbb{E}\|\xi\|^p. \tag{40}$$

According to (40) and (34), (39) yields that

$$\begin{aligned}\sup_{T \le t_{km+l} \le 2T} \mathbb{E}\|y_{km+l}\|^p \le & \sup_{T \le t_{km+l} \le 2T} \mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p + Me^{-\lambda T}\mathbb{E}\|\xi\|^p \\ \le & \Big(\widetilde{C}(K, p, 2T) h^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda T}\Big)\mathbb{E}\|\xi\|^p \\ = & R(h)\mathbb{E}\|\xi\|^p\end{aligned}$$

for $k \in \mathbb{N}$, $l = 0, 1, 2, \cdots, m-1$, where $R(h) = \widetilde{C}(K, p, 2T) h^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda T}$.

Since $R(0) = e^{-\frac{3}{4}\lambda T}$, there exists h^* such that

$$R(h^*) \le e^{-\frac{1}{2}\lambda T}.$$

Therefore, for every $0 < h < \min(h^*, \frac{1}{24K\theta^2})$, we have

$$\mathbb{E}\|y_{km+l}\|^p \le e^{-\frac{1}{2}\lambda T}\mathbb{E}\|\xi\|^p, \quad t_{km+l} \in [T, 2T] \tag{41}$$

for $k \in \mathbb{N}$, $l = 0, 1, 2, \cdots, m-1$.

Denote $\widetilde{x}(t)$ as the solution of SDEPCAS (1) with initial value $\widetilde{x}(T) = y_{Tm}$ for $t \ge T$. As the same procedure as Theorem 3.5, we can shift (27) to obtain

$$\sup_{T \le t_{km+l} \le 3T} \mathbb{E}\|\widetilde{x}(t_{km+l}) - y_{km+l}\|^p \le \widetilde{C}(K, p, 2T) h^{p/2}\mathbb{E}\|y_{Tm}\|^p, \tag{42}$$

and we may shift (35) to get

$$\mathbb{E}\|\widetilde{x}(t)\|^p \le M\mathbb{E}\|y_{Tm}\|^p e^{-\lambda(t-T)} \qquad for\ all\ t \ge T. \tag{43}$$

Hence, we obtain

$$\begin{aligned}\sup_{2T \le t_{km+l} \le 3T} \mathbb{E}\|y_{km+l}\|^p \le & \sup_{2T \le t_{km+l} \le 3T} \mathbb{E}\|\widetilde{x}(t_{km+l}) - y_{km+l}\|^p + Me^{-\lambda T}\mathbb{E}\|y_{Tm}\|^p \\ \le & \Big(\widetilde{C}(K, p, 2T) h^{\frac{p}{2}} + e^{-\frac{3}{4}\lambda T}\Big)\mathbb{E}\|y_{Tm}\|^p \\ = & R(h)\mathbb{E}\|y_{Tm}\|^p.\end{aligned}$$

Continuing the approach above and using $R(h) \le e^{-\frac{1}{2}\lambda T}$, we obtain

$$\sup_{(i+1)T \le t_{km+l} \le (i+2)T} \mathbb{E}\|y_{km+l}\|^p \le e^{-\frac{1}{2}\lambda T}\mathbb{E}\|y_{iTm}\|^p, \qquad i \ge 0.$$

Consequently

$$\begin{aligned}\sup_{(i+1)T \le t_{km+l} \le (i+2)T} \mathbb{E}\|y_{km+l}\|^p \le & e^{-\frac{1}{2}\lambda T} \sup_{iT \le t_{km+l} \le (i+1)T} \mathbb{E}\|y_{km+l}\|^p \\ \le & \cdots \\ \le & e^{-\frac{1}{2}\lambda (i+1)T} \sup_{0 \le t_{km+l} \le T} \mathbb{E}\|y_{km+l}\|^p.\end{aligned} \tag{44}$$

By Lemma 3.3, (44) becomes

$$\begin{aligned}\sup_{(i+1)T\le t_{km+l}\le(i+2)T}\mathbb{E}\|y_{km+l}\|^p \le& e^{-\frac{1}{2}\lambda(i+1)T}\bar{H}(K,p,T)\mathbb{E}\|\xi\|^p\\ =& e^{\frac{1}{2}\lambda T}\bar{H}(K,p,T)e^{-\frac{1}{2}\lambda(i+2)T}\mathbb{E}\|\xi\|^p\\ \le& e^{\frac{1}{2}\lambda T}\bar{H}(K,p,T)e^{-\frac{1}{2}\lambda(km+l)h}\mathbb{E}\|\xi\|^p,\end{aligned}\tag{45}$$

which implies the one-leg θ method is exponentially stable in pth moment on SDEPCAs (1) with rate constant $\gamma = \frac{1}{2}\lambda$ and growth constant $L = \bar{H}(K,p,T)e^{\frac{1}{2}\lambda T}$. Now the proof is complete. □

Theorem 4.4. *Under Assumption 2.4, if for a step size $h > 0$, the one-leg θ method is pth($p \in (0,1)$) moment exponentially stable with rate constant γ and growth constant L and satisfies* (36), *and if h satisfies* $\max\{12Kh,\ 24K\theta^2h\} < 1$ *and*

$$\bar{C}(K,p,2T)h^{\frac{p}{2}} + e^{-\frac{3}{4}\gamma T} \le e^{-\frac{1}{2}\gamma T},\tag{46}$$

where $T = 1 + [\frac{4}{\gamma}logL]$, then the SDEPCAs (1) *is pth($p \in (0,1)$) moment exponentially stable with rate constant $\lambda = \frac{1}{2}\gamma$ and growth constant $M = H(T,p,K)e^{\frac{1}{2}\gamma T}$, where $\bar{C}(K,p,2T) = \widetilde{C}(K,p,2T) + C(K,2T)^{\frac{p}{2}}$, and the constants $H(T,p,K)$, $C(K,2T)$ and $\widetilde{C}(K,p,2T)$ are given in Lemma 2.6, Lemma 3.4and Theorem 3.5, respectively.*

Proof. From the definition of T, it is easy to obtain

$$Le^{-\gamma T} \le e^{-\frac{3}{4}\gamma T}.$$

Note that for any $t \ge 0$, there exist integers k and $l = 0,1,2,\cdots,m-1$ such that $(km+l)h \le t < (km+l+1)h$. Using $p \in (0,1)$ and the Elementary Inequality (4), we have

$$\begin{aligned}\mathbb{E}\|x(t)\|^p =& \mathbb{E}\|x(t) - x(t_{km+l}) + x(t_{km+l}) - y_{km+l} + y_{km+l}\|^p\\ \le& \mathbb{E}\|x(t) - x(t_{km+l})\|^p + \mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p + \mathbb{E}\|y_{km+l}\|^p.\end{aligned}\tag{47}$$

According to Lemma 3.4 and Hölder inequality, the first term in (47) yields

$$\sup_{t\in[0,2T]}\mathbb{E}\|x(t) - x(t_{km+l})\|^p \le C(K,2T)^{\frac{p}{2}}h^{\frac{p}{2}}\mathbb{E}\|\xi\|^p.\tag{48}$$

Using (27), (36) and (48), we have

$$\begin{aligned}\sup_{t\in[T,2T]}\mathbb{E}\|x(t)\|^p \le& \sup_{t\in[T,2T]}\mathbb{E}\|x(t) - x(t_{km+l})\|^p + \sup_{km+l\in[mT,2mT]}\mathbb{E}\|y_{km+l}\|^p\\ &+ \sup_{t\in[T,2T]}\mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p\\ \le& \sup_{t\in[0,2T]}\mathbb{E}\|x(t) - x(t_{km+l})\|^p + \sup_{km+l\in[mT,2mT]}\mathbb{E}\|y_{km+l}\|^p\\ &+ \sup_{t\in[0,2T]}\mathbb{E}\|x(t_{km+l}) - y_{km+l}\|^p\\ \le& \Big(C(K,2T)^{\frac{p}{2}}h^{\frac{p}{2}} + \widetilde{C}(K,p,2T)h^{\frac{p}{2}} + Le^{-\gamma T}\Big)\mathbb{E}\|\xi\|^p\\ \le& \Big(\bar{C}(K,p,2T)h^{\frac{p}{2}} + e^{-\frac{3}{4}\gamma T}\Big)\mathbb{E}\|\xi\|^p.\end{aligned}\tag{49}$$

Using (46), we have

$$\begin{aligned}\sup_{t\in[T,2T]}\mathbb{E}\|x(t)\|^p \le& e^{-\frac{1}{2}\gamma T}\mathbb{E}\|\xi\|^p\\ \le& e^{-\frac{1}{2}\gamma T}\sup_{t\in[0,T]}\mathbb{E}\|x(t)\|^p.\end{aligned}\tag{50}$$

Let $\{\widehat{y}_{km+l}\}_{km+l\geq Tm}$ be the approximate solutions of SDEPCAs (1) obtained by the one-leg θ method on $t \geq T$ with initial value $x(T)$ at $t = T$. Then (37) becomes

$$\mathbb{E}\|\widehat{y}_{km+l}\|^p \leq Le^{[\gamma(km+l)h-T]}\mathbb{E}\|x(T)\|^p, \quad km+l \geq Tm. \tag{51}$$

Using the similar approach that produces (50), we obtain

$$\begin{aligned}\sup_{t\in[2T,3T]} \mathbb{E}\|x(t)\|^p \leq & \sup_{t\in[T,3T]} \mathbb{E}\|x(t)-x(t_{km+l})\|^p + \sup_{km+l\in[2Tm,3Tm]} \mathbb{E}\|\widehat{y}_{km+l}\|^p \\ & + \sup_{t\in[T,3T]} \mathbb{E}\|x(t_{km+l})-\widehat{y}_{km+l}\|^p \\ \leq & \Big(\bar{C}(K,p,2T)h^{\frac{p}{2}} + e^{-\frac{3}{4}\gamma T}\Big)\mathbb{E}\|x(T)\|^p \\ \leq & e^{-\frac{1}{2}\gamma T} \sup_{t\in[T,2T]} \mathbb{E}\|x(t)\|^p.\end{aligned} \tag{52}$$

Repeating this procedure, we can show that for any $i \geq 0$

$$\begin{aligned}\sup_{t\in[iT,(i+1)T]} \mathbb{E}\|x(t)\|^p \leq & e^{-\frac{1}{2}\gamma T} \sup_{t\in[(i-1)T,iT]} \mathbb{E}\|x(t)\|^p \\ \leq & \ldots \\ \leq & e^{-\frac{1}{2}\gamma iT} \sup_{t\in[0,T]} \mathbb{E}\|x(t)\|^p.\end{aligned} \tag{53}$$

Moreover, according to (8), for $t \in [iT,(i+1)T]$, (53) yields

$$\begin{aligned}\mathbb{E}\|x(t)\|^p \leq & e^{-\frac{1}{2}\gamma iT}H(T,p,K)\mathbb{E}\|\xi\|^p \\ = & e^{\frac{1}{2}\gamma T}H(T,p,K)e^{-\frac{1}{2}\gamma(i+1)T}\mathbb{E}\|\xi\|^p \\ \leq & e^{\frac{1}{2}\gamma T}H(T,p,K)e^{-\frac{1}{2}\gamma t}\mathbb{E}\|\xi\|^p.\end{aligned} \tag{54}$$

Let $\lambda = \frac{1}{2}\gamma$, $M = H(T,p,K)e^{\frac{1}{2}\gamma T}$, then the SDEPCAs is pth moment exponentially stable with rate constant λ and growth constant M. The proof is completed. □

Theorem 4.3 and Theorem 4.4 lead to the following necessary and sufficient theorem.

Theorem 4.5. *Under Assumption 2.4, the SDEPCAs is exponentially stable in pth($p \in (0,1)$) moment if and only if there exists $h^* > 0$ such that the one-leg θ method is exponentially stable in pth($p \in (0,1)$) moment with rate constant γ, growth constant L, step size $0 < h < h^*$, and $\bar{C}(K,p,2T)$ satisfying* (46) *with $T = 1 + [\frac{4}{\gamma}\log L]$.*

Theorem 4.6 below indicates that by taking h small enough, the rate constants λ for SDEPCAs and γ for the one-leg θ method can be arbitrarily close.

Theorem 4.6. *Let Assumption 2.4 hold. If SDEPCAs* (1) *is exponentially stable in pth($p \in (0,1)$) moment with rate constant λ, then given any $\varepsilon \in (0,\lambda)$ there exists h^* such that for all $h \in (0,h^*)$ the one-leg θ method is exponentially stable in pth($p \in (0,1)$) moment with rate constant $\gamma = \lambda - \varepsilon$. Conversely, if the one-leg θ method on* (1) *is exponentially stable in pth($p \in (0,1)$) moment with rate constant γ for some sufficiently small h, then for any given $\varepsilon \in (0,\gamma)$, the SDEPCAs* (1) *is exponentially stable in pth($p \in (0,1)$) moment with rate constant $\gamma - \varepsilon$.*

Proof. The proof of Theorem 4.6 is similar to that of Theorems 4.3 and 4.4. For given $\varepsilon \in (0,\lambda)$, choose $T = 1 + [\frac{2}{\varepsilon}logM]$ such that

$$Me^{-\lambda T} < e^{-(\lambda-0.5\varepsilon)T}.$$

As in the proof of Theorem 4.3, there exists some h^* such that

$$R(h) = \widetilde{C}(K, p, 2T)h^{\frac{p}{2}} + e^{-(\lambda-0.5\varepsilon)T} \leq e^{-(\lambda-\varepsilon)T},$$

for all $h \in (0, h^*)$. Here, $\widetilde{C}(K, p, 2T)$ depends on ε. Continuing as in the proof of Theorem 4.3, we observe that for each $h < h^*$ the one-leg θ methods are exponentially stable in pth moment with $\gamma = \lambda - \varepsilon$ and $L = \bar{H}(K, p, T)e^{(\lambda-\varepsilon)T}$.

To prove the converse, for any $\varepsilon \in (0, \gamma)$, we can choose $T = 1 + [\frac{2}{\varepsilon} logL]$ such that

$$Le^{-\gamma T} < e^{-(\gamma-0.5\varepsilon)T}.$$

Choosing $M = H(K, p, T)e^{(\gamma-\varepsilon)T}$ and continuing as in the proof of Theorem 4.4, we can obtain the second part of this theorem. □

Now, we are in position to consider the almost sure exponential stability of both equation (1) and the one-leg θ method.

Corollary 4.7. *Under Assumption 2.4, if the SDEPCAs is pth($p \in (0, 1)$) moment exponentially stable and satisfies* (34), *then the solution of SDEPCAs* (1) *is almost surely exponentially stable. That is*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\|x(t)\|) \leq -\frac{\lambda}{p} \quad a.s.. \tag{55}$$

Proof. The proof of this theorem is similar to that of Theorem 4.2 in [8]. □

Corollary 4.8. *Under Assumption 2.4, if the one-leg θ method is pth($p \in (0, 1)$) moment exponentially stable and satisfies* (36), *then the method is almost surely exponentially stable. That is*

$$\limsup_{km+l\to\infty} \frac{1}{(km+l)h} \log(\|y_{km+l}\|) \leq -\frac{\gamma}{p} \quad a.s.. \tag{56}$$

Proof. The proof of this theorem is similar to that of Theorem 4.2 in [9]. □

Remark 4.9. *(1) According to Theorems 4.5 and 4.6, under Assumption 2.4, if h is small enough, then the pth($p \in (0, 1)$) moment exponential stability of the one-leg θ method is equivalent to that of SEPCAs* (1)*. Therefore, it is feasible to investigate the exponential stability of SDEPCAs from carefully numerical simulations.*
(2) The two corollaries above show that the almost sure exponential stability of SDEPCAs can be derived by the pth($p \in (0, 1)$) moment exponential stability of SDEPCAs. Moreover, by Theorem 4.5, it is feasible to investigate the almost sure exponential stability of SDEPCAs from the pth($p \in (0, 1)$) moment exponential stability of the numerical approximations.

5. Numerical simulations

In this section, we focus on the following one-dimensional linear SDEPCAs

$$dx(t) = \Big(-5x(t) + x([t])\Big)dt + x([t])dB(t), \qquad t \geq 0, \tag{57}$$

with $x(0) = -1$. (57) has an explicit solution which is called the mean-reverting Ornstein-Uhlenbeck process (see Example 3.5.2 in [8]) and the solution can be expressed as

$$x(t) = \frac{1}{5}x(k) + \frac{4}{5}e^{-5(t-k)}x(k) + x(k)\int_k^t e^{-5(t-s)}dB(s), \tag{58}$$

step size	$\epsilon(1)$	$\epsilon(2)$	$\epsilon(3)$	$\epsilon(4)$	$\epsilon(5)$	$\epsilon(6)$	$\epsilon(7)$	$\epsilon(8)$
2^{-4}	0.1384	0.0919	0.0543	0.0308	0.0169	0.0092	0.0049	0.0026
2^{-5}	0.0962	0.0652	0.0390	0.0218	0.0122	0.0067	0.0036	0.0019
2^{-6}	0.0691	0.0459	0.0272	0.0155	0.0086	0.0047	0.0025	0.0013
2^{-7}	0.0496	0.0327	0.0194	0.0111	0.0062	0.0033	0.0018	0.0009

Table 1: The error for $p = 0.5$, $\theta = 0.5$ at times $T = 1, 2, ..., 8$.

for all $t \in [k, k + 1)$, where k is an integer. Applying the one-leg θ method to (57), we have

$$y_{km+l+1} = \frac{1 - 5(1-\theta)h}{1 + 5\theta h} y_{km+l} + \frac{h}{1 + 5\theta h} y_{km} + \frac{1}{1 + 5\theta h} y_{km} \Delta B_{km+l}, \tag{59}$$

where k is integer and $l = 0, 1, 2, \cdots, m - 1$.

Our simulations consist of two parts which are used to verify Theorem 3.5 and stability results in Theorem 4.5. The following settings are the same in each part. Let $\theta = 0.5$, $p = 0.5$. Moreover, the exact value of (58) at time t_{km+l} is not known because of the Itô's integral. To get $x(t_{km+l})$, we approximate the Itô's integral in (58) with a sum with 2^{16} summands for each interval $(k, k + 1]$, that is, the step size h_1 is 2^{-16} when computing the Itô's integral. In addition, a set of 50 blocks, each of which contains 100 outcomes($\omega_{ij} : 1 \le i \le 50, 1 \le j \le 100$), is simulated. We denote the numerical solution of the jth trajectory in the ith block by $y_{km+l}(\omega_{ij})$, and the exact solution of (57) in the jth trajectory and ith block by $x(t_{km+l}, \omega_{ij})$ at $t = t_{km+l}$.

Part 1: The convergence of the one-leg θ method is tested in this part. Let ϵ denote the error in pth moment, then by the law of large numbers, at the final integer time T, ϵ satisfies

$$\epsilon(T) = \mathbb{E}|x(T) - y_{Tm}|^p = \frac{1}{5000} \sum_{i=1}^{50} \sum_{j=1}^{100} |x(T, \omega_{ij}) - y_{Tm}(\omega_{ij})|^p. \tag{60}$$

There are 8 tests to compute the error in Table 1 with $T = 1, 2, ..., 8$ and $m = 2^4,\ 2^5,\ 2^6,\ 2^7$. It is obvious to see that the one-leg θ method converges to (57).

Part 2: In this part, we consider the exponential stability of both (57) and the numerical solution (59). Because of $x(0) = -1 < 0$, according to Lemma 4.3.2 in [8], we see that for $t \ge 0$

$$x(t) < 0, \qquad a.s. \tag{61}$$

Choose function $V(t, x) = |x|^p$, by (61), Theorem 4.4.4 in [8] and $p \in (0, 1)$, then there exist $\lambda = -2p$ and $M = 10$ such that

$$\mathbb{E}|x(t)|^p \le 10e^{-2pt}\mathbb{E}|x(k)|^p, \tag{62}$$

for all $t \in [k, k + 1]$. Figure 1 shows the pth moment exponential stability of Equation (57) and the one-leg θ method with $\theta = 0.5$ and $p = 0.5$. The solid line which is the reference line indicates the value of $10e^{-2pt}$ with $p = 0.5$. It is apparent to observe that $\mathbb{E}|x(t)|^p \le 10e^{-2pt}$ and $\mathbb{E}|y_{km+l}|^p \le 10e^{-2pt_{km+l}}$. We can also discover that the values of $\mathbb{E}|x(t_{km+l})|^p$ are quite close to that of $\mathbb{E}|y_{km+l}|^p$, which demonstrates that the rate constants λ and γ can be considerably close. Figure 1 illustrates Theorem 4.6.

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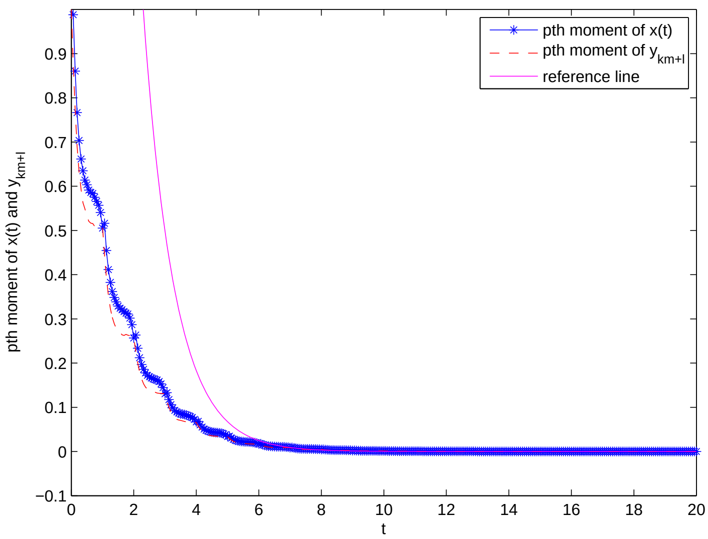

Figure 1: The pth moment exponential stability of (57) and the one-leg θ method with $\theta = 0.5$ and $p = 0.5$

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