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Some Properties of (m, C)-Isometric Operators

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Abstract. In this paper, we study if T is an (m, C)-isometric operator and CT^*C commutes with T, then T^* is an (m, C)-isometric operator. We also give local spectral properties and spectral relations of (m, C)-isometric operators, such as property (β), decomposability, the single-valued extension property and Dunford's boundedness. We also investigate perturbation of (m, C)-isometric operators by nilpotent operators and by algebraic operators and give some properties.

1. Introduction

We denote by $\mathcal{H}$ a complex separable Hilbert space endowed with the inner product $\langle ., . \rangle$ and by $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T), \sigma_p(T), \sigma_{ap}(T), \sigma_{su}(T), \sigma_{comp}(T), \sigma_r(T), \sigma_c(T), \sigma_e(T), \sigma_{le}(T), \sigma_{re}(T), \sigma_{se}(T), \sigma_{es}(T), \sigma_b(T), \sigma_\omega(T)$, for the spectrum, the point spectrum, the approximate point spectrum, the surjective spectrum, the compression spectrum, the residual spectrum, the continuous spectrum, the essential spectrum, the left essential spectrum, the right essential spectrum, the semi-regular spectrum, the essentially semi-regular spectrum, Browder spectrum, and Weyl spectrum of T, respectively.

In [1], S. R. Garcia and M. Putinar investigated complex symmetric operators. In [2–4], M. Chō, E. Ko and J. Lee investigated m-complex symmetric operators. In [5], S. Jung, E. Ko, M. Lee and J. Lee defined the (m, C)-isometric operator as follows: an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be (m, C)-isometric if there exists some conjugation C such that $\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}CT^{m-j}C = 0$ for some positive integer m.

Let $\Lambda_m(T) := \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}CT^{m-j}C$, then T is an (m, C)-isometric operator with conjugation C if and only if $\Lambda_m(T) = 0$, note that

$$T^*\Lambda_m(T)CTC - \Lambda_m(T) = \Lambda_{m+1}(T). \quad (1)$$

Hence, if $\Lambda_m(T) = 0$, then $\Lambda_n(T) = 0$ for all $n \geq m$. Moreover, it is clear that T is an (m, C)-isometric operator if and only if CTC is an (m, C)-isometric operator. In [5], S. Jung, E. Ko, M. Lee and J. Lee have given some properties and we will give some other properties of (m, C)-isometric operators.

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An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the single-valued extension property (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in H$, the local resolvent set $\rho_T(x)$of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to H$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The local spectrum of T at x is given by $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$. We define the local spectral subspace of $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be decomposable if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{H} = \mathcal{X} + \mathcal{Y}$, $\sigma(T \mid x) \subset \overline{U}$ and $\sigma(T \mid y) \subset \overline{V}$. It is well-known that

Decomposable ⇒ Bishop's property (β). Decomposable ⇒ Dunford's property (C) ⇒ SVEP.

In this paper, we study if T is an (m, C)-isometric operator and CT^*C commutes with T, then T^* is an (m, C)-isometric operator. We also give local spectral properties and spectral relations of (m, C)-isometric operators, such as property (β), decomposability, the single-valued extension property and Dunford's boundedness. We also investigate perturbation of (m, C)-isometric operators by nilpotent operators and by algebraic operators and give some properties.

2. Some properties of $\Lambda_m(T)$

In this section we will give some properties of (m, C)-isometric operators.

Theorem 1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator, where C is a conjugation on $\mathcal{H}$. If T commutes with CT^*C, then T^* is an (m, C)-isometric operator.*

Proof. Since T is an (m, C)-isometric operator, we obtain $\Lambda_m(T) = 0$, so

$$\begin{aligned}
\Lambda_m(T^*) &= \sum_{j=0}^{m}(-1)^j\tbinom{m}{j}(T^*)^{*m-j}C(T^*)^{m-j}C \\
&= \sum_{j=0}^{m}(-1)^j\tbinom{m}{j}(T)^{m-j}C(T^*)^{m-j}C \\
&= \sum_{j=0}^{m}(-1)^j\tbinom{m}{j}C(T^*)^{m-j}C(T)^{m-j} = \Lambda_m(T)^* = 0.
\end{aligned}$$

Hence, $\Lambda_m(T^*)$ is an (m, C)-isometric operator. □

Theorem 2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator and C be a conjugation on $\mathcal{H}$. If T^* has property (β), then T has property (β).*

Proof. Suppose T^* has property (β). Let sequence $\{f_n\} : G \to \mathcal{H}$ be $\mathcal{H}$-valued analytic functions on G such that $\lim_{n\to\infty} \|(T-\lambda)f_n(\lambda)\| = 0$ on compact subset of G. Then $\lim_{n\to\infty} \|(T^k - \lambda^k)f_n(\lambda)\| = 0$ and $\lim_{n\to\infty} \|(CT^kC - \overline{\lambda}^k)f_n(\lambda)\| = 0$ on compact subset of G for some $k \in N$). Since T is an (m, C)-isometric operator, then $\Lambda_m(T) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}C(T)^{m-j}C = 0$. Then

$$\begin{aligned}\lim_{n\to\infty}\|\Lambda_m(T)Cf_n(\lambda)\| &= \lim_{n\to\infty}\|[\sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}C(T)^{m-j}C]Cf_n(\lambda)\| \\ &= \lim_{n\to\infty}\|[\sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}\overline{\lambda}^{m-j}]Cf_n(\lambda)\| \\ &= \lim_{n\to\infty}\|(T^*\overline{\lambda}-1)^mCf_n(\lambda)\| \\ &= 0\end{aligned}$$

on compact subset of G.

Choose an open set $G_1 \subsetneq G$ such that $0 \notin G_1$, the we obtain $\lim_{n\to\infty}\|(T^* - \frac{1}{\overline{\lambda}})^mCf_n(\lambda)\| = 0$ on G_1. Set $\lambda = a+bi$, for $a, b \in R$. Then $\frac{1}{\overline{\lambda}} = \frac{1}{a-bi} = \frac{1}{a^2+b^2}\lambda = \gamma\lambda$, where $\gamma = \frac{1}{a^2+b^2}$ is positive real number. So $\lim_{n\to\infty}\|(T^* - \gamma\lambda)^mCf_n(\lambda)\| = 0$ for a positive real number γ, set $\mu = \gamma\lambda$. Define $g : \gamma G_1 \to G_1$ by $g(\mu) = \frac{1}{\gamma}\mu$, then we have $\lim_{n\to\infty}\|(T^* - \mu)^mCf_n(g(\mu))\| = 0$ on γG_1. Since T^* has property (β), we have $\lim_{n\to\infty}\|Cf_n(g(\mu))\| = 0$ on γG_1 and so $\lim_{n\to\infty}\|Cf_n(\lambda)\| = 0$ on G_1. Moreover, since C is isometric, $\lim_{n\to\infty}\|f_n(\lambda)\| = 0$ on G_1. Since $\lim_{n\to\infty}\|f_n(\lambda)\| = 0$ on G_1 and G_1 is an open set of domain G of f_n. It follows that $\lim_{n\to\infty}\|f_n(\lambda)\| = 0$ on G. Hence T has the property (β). □

Corollary 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator, where C be a conjugation on $\mathcal{H}$. If T^* has property (β), if and only if T is decomposable.*

Proof. *Since the converse implication holds by ([6], Theorem 1.2.29 and Theorem 2.2.5). From Theorem 3, we can obtain that T has property (β) and so T is decomposable from [6]. Hence, we conclude that T^* has property (β), if and only if T is decomposable.* □

Corollary 4. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator, where C be a conjugation on $\mathcal{H}$. If T^* is hyponormal, then T is decomposable.*

Proof. *since T^* is hyponormal, then T has the property (β). From Corollary 3, we obtain T is decomposable.* □

Corollary 5. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator, where C be a conjugation on $\mathcal{H}$. If T^* has property (β) and $\sigma(T)$ has nonempty interior. Then T has a nontrivial invariant subspace.*

Proof. *If T^* has property (β), then T is decomposable from Corollary 4. So in this case T has the property (β) by Theorem 3, since $\sigma(T)$ has nonempty interior, we obtain this result from ([7], Theorem 2.1).* □

Lemma 6. *Let C be a conjugation on $\mathcal{H}$. Assume $T \in \mathcal{L}(\mathcal{H})$ is an (m, C)-isometric operator. If T^* has the single-valued extension property, then CTC has the single-valued extension property.*

Proof. *Suppose T^* has the single-valued extension property. Let $f : G \to H$ be an analytic function such that $(CTC - \lambda)f(\lambda) \equiv 0$ on G, where G is a domain of f. Then $(CT^kC - \lambda^k)f(\lambda) \equiv 0$ on G, for some $k \in N$. Since T is an (m, C)-isometric operator, it follows that*

$$\begin{aligned}0 &= \Lambda_m(T)f(\lambda) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}C(T)^{m-j}Cf(\lambda) \\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}\lambda^{m-j}f(\lambda) = (T^*\lambda - 1)^m f(\lambda).\end{aligned}$$

Since $\lambda \neq 0$, then $(T^* - \frac{1}{\lambda})^m f(\lambda) = 0$. Let $\mu = \frac{1}{\lambda}, \lambda = \frac{1}{\mu} = g(\mu)$ and $g : G \to G$. Then $(T^* - \mu)^m f(g(\mu)) = 0$. Since T^* has the single-valued extension property, then $f(g(\mu)) = 0$. We obtain $f(\lambda) = 0$, hence CTC has the single-valued extension property. □

Corollary 7. *Let C be a conjugation on $\mathcal{H}$. Assume $T \in \mathcal{L}(\mathcal{H})$ is an (m, C)-isometric operator. If T^* has the single-valued extension property, then $\sigma_{T^*}(x) \subseteq \sigma_T(Cx)$.*

Proof. *Since T^* has the single-valued extension property, so does CTC. Moreover, by [6] we know $\sigma_{T^*}(x) \subseteq \sigma_{CTC}(x)$ for all $x \in \mathcal{H}$. Let $\lambda_0 \in \rho_{CTC}(x)$, then there is an $\mathcal{H}$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0, such that $(CTC - \lambda)f(\lambda) \equiv x$, for every $\lambda \in D$. Therefore $(T - \lambda)Cf(\overline{\lambda}) \equiv Cx$, for every $\lambda \in D^*$. Since $Cf(\overline{\lambda})$ is analytic on D^*, we get $\overline{\lambda_0} \in \rho_T(Cx)$, so $\lambda_0 \in \rho_T(Cx)^*$. Hence $\rho_{CTC}(x) \subseteq \rho_T(Cx)^*$.*

Conversely, assume that $\lambda_0 \in \rho_T(Cx)^*$, then there exist an $\mathcal{H}$-valued analytic function $f(\lambda)$ in a neighborhood D of λ_0, such that $(T - \lambda)f(\lambda) \equiv Cx$, for every $\lambda \in D$. Then we have $(CT - \overline{\lambda}C)f(\lambda) = (CTC - \overline{\lambda})Cf(\lambda) \equiv x$, for every $\lambda \in D$, that means that $(CTC - \lambda)Cf(\overline{\lambda}) \equiv x$, for every $\lambda \in D^*$, therefore $\overline{\lambda_0} \in \rho_{CTC}(x)$, then $\rho_T(Cx)^* \subseteq \rho_{CTC}(x)$. Hence, $\rho_T(Cx)^* = \rho_{CTC}(x)$. Since $(\mathbb{C}\backslash G)^* = \mathbb{C}\backslash G^*$ for any subset G of $\mathbb{C}$. We have $\sigma_T(Cx)^* = \sigma_{CTC}(x)$, since $\sigma_{T^*}(x) \subseteq \sigma_{CTC}(x)$, hence $\sigma_{T^*}(x) \subseteq \sigma_T(Cx)^*$. □

Let us recall that we assume that T has the single-valued extension property, if there exists a constant K such that for every $x, y \in \mathcal{H}$ with $\sigma_T(x) \cap \sigma_T(y) = \emptyset$, we have $\|x\| \leq K\|x + y\|$, where K is independent of x and y, we say that an operator T satisfies Dunford's boundedness condition (B).

Corollary 8. *Let $T \in \mathcal{L}(H)$ be an (m, C)-isometric operator and let C be a conjugation on $\mathcal{H}$. If T^* has the single-valued extension property and Dunford's boundedness condition (B), then T has Dunford's boundedness condition (B).*

Proof. The proof is similar to ([3], Theorem 4.12). □

Lemma 9. *([5], Corollary 3.11) Let C be a conjugation on $\mathcal{H}$. Assume $T \in \mathcal{L}(\mathcal{H})$ is an (m, C)-isometric operator. If T^* has the single-valued extension property, then the following properties hold:*
(i) $\sigma(T) = \sigma_{su}(T) = \sigma_{ap}(T) = \sigma_{se}(T)$.
(ii) $\sigma_{es}(T) = \sigma_b(T) = \sigma_\omega(T) = \sigma_e(T)$.
(iii) $H_0(T - \lambda) = H_T(\{\lambda\})$ and $H_0(T^* - \lambda) = H_{T^*}(\{\lambda\})$ for all $\lambda \in \mathbb{C}$.
(iv) T is biquasitriangular.

Theorem 10. *Let C be a conjugation on $\mathcal{H}$. Assume $T \in \mathcal{L}(\mathcal{H})$ is an (m, C)-isometric operator. If T commutes with CT^*C, then the following properties hold:*
(i) $\sigma_{ap}(T) = \frac{1}{\sigma_{ap}(T^*)^*}$, $\sigma_p(T) = \frac{1}{\sigma_p(T^*)^*}$, $\sigma_{comp}(T) = \frac{1}{\sigma_{comp}(T^*)^*}$, $\sigma_{su}(T) = \frac{1}{\sigma_{su}(T^*)^*}$.
(ii) $\sigma_{le}(T) = \frac{1}{\sigma_{le}(T^*)^*}$, $\sigma_{re}(T) = \frac{1}{\sigma_{re}(T^*)^*}$.
(iii) If T^* has the single-valued extension property, then
$\sigma(T) = \sigma_{su}(T) = \frac{1}{\sigma_{su}(T^*)^*} = \sigma_{ap}(T) = \frac{1}{\sigma_{ap}(T^*)^*} = \sigma_{se}(T)$.

Proof. *(i) From ([5], Theorem 3.4 (i)), we know that $\sigma_{ap}(T) \subseteq \frac{1}{\sigma_{ap}(T^*)^*}$, we need only to prove $\frac{1}{\sigma_{ap}(T^*)^*} \subseteq \sigma_{ap}(T)$. Let $\lambda \in \sigma_{ap}(T^*)$, if $\{x_n\}$ is a sequence of unit vectors such that $\lim_{n\to\infty}(T^* - \lambda)x_n = 0$, then $\lim_{n\to\infty}(CT^*C - \overline{\lambda})Cx_n = 0$ and $\lim_{n\to\infty}(C(T^*)^kC - (\overline{\lambda})^k)Cx_n = 0$ for every $k \in N$. Since $T \in \mathcal{L}(\mathcal{H})$ is an (m, C)-isometric operator and T commutes with CT^*C, we obtain $\Lambda_m(T^*) = 0$ from Theorem 1. Then*

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \Lambda_m(T^*)Cx_n = \lim_{n\to\infty}\Big(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}(T)^{m-j}C(T^*)^{m-j}C\Big)Cx_n \\
&= \lim_{n\to\infty}\Big(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}(T)^{m-j}\overline{\lambda}^{m-j}\Big)Cx_n = \lim_{n\to\infty}(T\overline{\lambda} - 1)^m Cx_n.
\end{aligned}
$$

From ([5], Lemma 3.1 (ii)), we know that $\overline{\lambda} \neq 0$, we can obtain $\lim_{n\to\infty}[(T - \frac{1}{\overline{\lambda}})^m Cx_n] = 0$. If $\lim_{n\to\infty}\frac{(T-\frac{1}{\overline{\lambda}})^{m-1}Cx_n}{\|(T-\frac{1}{\overline{\lambda}})^{m-1}Cx_n\|} \neq 0$, then $\frac{1}{\overline{\lambda}} \in \sigma_{ap}(T)$ and $\lambda \in \frac{1}{\sigma_{ap}(T)^*}$, otherwise, $\lim_{n\to\infty}(T - \frac{1}{\overline{\lambda}})^{m-1}Cx_n = 0$. If $\lim_{n\to\infty}\frac{(T-\frac{1}{\overline{\lambda}})^{m-2}Cx_n}{\|(T-\frac{1}{\overline{\lambda}})^{m-2}Cx_n\|} \neq 0$, then $\frac{1}{\overline{\lambda}} \in \sigma_{ap}(T)$

and $\lambda \in \frac{1}{\sigma_{ap}(T)^*}$, *otherwise,* $\lim_{n\to\infty}(T - \frac{1}{\overline{\lambda}})^{m-2}Cx_n = 0$. *By induction, we have* $\lim_{n\to\infty}(T - \frac{1}{\overline{\lambda}})Cx_n = 0$, *so we have* $\frac{1}{\overline{\lambda}} \in \sigma_{ap}(T)$ *and* $\lambda \in \frac{1}{\sigma_{ap}(T)^*}$, *therefore* $\frac{1}{\sigma_{ap}(T^*)^*} \subseteq \sigma_{ap}(T)$. *Hence,* $\sigma_{ap}(T) = \frac{1}{\sigma_{ap}(T^*)^*}$.

For $\sigma_p(T) = \frac{1}{\sigma_p(T^*)^*}$, *the proof is similar to the* $\sigma_{ap}(T) = \frac{1}{\sigma_{ap}(T^*)^*}$.

For any $S \in \mathcal{L}(\mathcal{H})$, *we have* $\sigma_{comp}(S)^* = \sigma_p(S^*)$, *it's easy to get* $\sigma_{comp}(T) = \frac{1}{\sigma_{comp}(T^*)^*}$.

For any $S \in \mathcal{L}(\mathcal{H})$, *we have* $\sigma_{su}(S)^* = \sigma_{ap}(S^*)$, *it's easy to get* $\sigma_{su}(T) = \frac{1}{\sigma_{su}(T^*)^*}$.

(ii) If $\lambda \in \sigma_{le}(T)$, *then there exist a sequence* $\{x_n\}$ *of unit vectors in* $\mathcal{H}$ *such that* $\{x_n\}$ *weakly converges to 0 and* $\lim_{n\to\infty}\|(T-\lambda)x_n\| = 0$ *for any* $T \in \mathcal{L}(\mathcal{H})$, *then we have* $\lim_{n\to\infty}\|(CT^kC - \overline{\lambda}^k)Cx_n\| = 0$. *Since* $T \in \mathcal{L}(\mathcal{H})$ *is an* (m, C)*-isometric operator, so* $\Lambda_m(T) = 0$, *it follows that*

$$\begin{aligned} 0 &= \lim_{n\to\infty}\|\Lambda_m(T)Cx_n\| = \lim_{n\to\infty}\|[(\sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}CT^{m-j}C)Cx_n]\| \\ &= \lim_{n\to\infty}\|(\sum_{j=0}^{m}(-1)^j\binom{m}{j}(T^*)^{m-j}\overline{\lambda}^{m-j})Cx_n\| = \lim_{n\to\infty}\|(T^*\overline{\lambda} - 1)^mCx_n\| \\ &= \lim_{n\to\infty}\|(T - \frac{1}{\overline{\lambda}})^mCx_n\|. \end{aligned}$$

Moreover, since $\{x_n\}$ *weakly converges to 0,* $\{Cx_n\}$ *weakly converges to 0. Hence,* $\sigma_{le}(T) \subseteq \frac{1}{\sigma_{le}(T^*)^*}$.

Conversely, $\lambda \in \sigma_{le}(T^*)$, *then there exists a sequence*$\{x_n\}$ *of unit vectors in* $\mathcal{H}$ *such that* $\{x_n\}$ *weakly converges to 0 and* $\lim_{n\to\infty}\|(T^* - \lambda)x_n\| = 0$ *for any* $T \in \mathcal{L}(\mathcal{H})$, *then we have* $\lim_{n\to\infty}\|(C(T^*)^kC - \overline{\lambda}^k)Cx_n\| = 0$. *Since* $T \in \mathcal{L}(\mathcal{H})$ *is an* (m, C)*-isometric operator. From Theorem 1, we obtain* $\Lambda_m(T^*) = 0$. *It follows that*

$$\begin{aligned} 0 &= \lim_{n\to\infty}\|\Lambda_m(T^*)Cx_n\| = \lim_{n\to\infty}\|(\sum_{j=0}^{m}(-1)^j\binom{m}{j}(T)^{m-j}C(T^*)^{m-j}C)Cx_n\| \\ &= \lim_{n\to\infty}\|(\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}\overline{\lambda}^{m-j})Cx_n\| = \lim_{n\to\infty}\|(T\overline{\lambda} - 1)^mCx_n\|. \end{aligned}$$

We obtain $\lim_{n\to\infty}\|(T - \frac{1}{\overline{\lambda}})^mCx_n\| = 0$. *Moreover, since* $\{x_n\}$ *weakly converges to 0,* $\{Cx_n\}$ *weakly converges to 0. Therefore,* $\frac{1}{\sigma_{le}(T^*)^*} \subseteq \sigma_{le}(T)$. *Hence* $\sigma_{le}(T) = \frac{1}{\sigma_{le}(T^*)^*}$.

For $\sigma_{re}(T) = \frac{1}{\sigma_{re}(T^*)^*}$, *since for any* $S \in \mathcal{L}(\mathcal{H})$, $\sigma_{re}(S)^* = \sigma_{le}(S^*)$. *It is easy to get* $\sigma_{re}(T) = \frac{1}{\sigma_{re}(T^*)^*}$.

(iii) From Lemma 9 and (i), we obtain $\sigma(T) = \sigma_{su}(T) = \frac{1}{\sigma_{su}(T^*)^*} = \sigma_{ap}(T) = \frac{1}{\sigma_{ap}(T^*)^*} = \sigma_{se}(T)$. □

3. Perturbation of (m, C)-isometric operators by nilpotent operators

In this section, we provide some properties of perturbation of (m, C)-isometric operators by nilpotent operators, we need some preliminaries.

Recall that an operator $N \in \mathcal{L}(\mathcal{H})$ is said to be nilpotent of order n, if $N^n = 0$ and $N^{n-1} \neq 0$ for some positive integer n. Let G be a commutative group and denote its operation by $+$. Given a sequence $a = (a_n)_{n\geq 0}$ in G , the difference sequence $Da = (Da)_{n\geq 0}$ is defined by $(Da)_{n\geq 0} := a_{n+1} - a_n$. The powers of D are defined recursively by $D^0a := a, D^{k+1}a = D(D^ka)$. It is easy to show that

$$(D^ka)_n = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}a_{i+n}, \tag{2}$$

for all $k \geq 0$ and $i \geq 0$ integers. A sequence a in a group G is called an arithmetic progression of order $h = 0, 1, 2, \ldots$ if $D^{h+1}a = 0$. Equivalently,

$$\sum_{i=0}^{h+1}(-1)^{h+1-i}\binom{h+1}{i}a_{i+n} = 0, \tag{3}$$

for $n = 0, 1, 2, \ldots$. It is well known that the sequence a in G is an arithmetic progression of order h if and only if there exists a polynomial $p(n)$ in n, with coefficients in G and of degree less than or equal to h, such that $p(n) = a_n$, for every $n = 0, 1, 2, \ldots$; that is, there are $\gamma_h, \gamma_{h-1}, \cdots, \gamma_1, \gamma_0 \in G$, which depend only on G, such that, for every $n = 0, 1, 2, \ldots,$

$$a_n = p(n) = \sum_{i=0}^{h}\gamma_i n^i \tag{4}$$

We say that the sequence a is an arithmetic progression of strict order $h = 0, 1, 2, \ldots,$ if $h = 0$ or if it is of order $h > 0$ but is not of order $h - 1$; that is, the polynomial p of (4) has degree h. Moreover, a sequence a in a group G is an arithmetic progression of order h if and only if, for all $n \geq 0$,

$$a_n = p(n) = \sum_{k=0}^{h}(-1)^{h-k}\frac{n(n-1)...\overbrace{(n-k)}...(n-h)}{k!(h-k)!}a_k; \tag{5}$$

that is,

$$a_n = \sum_{k=0}^{h}(-1)^{h-k}\binom{n}{k}\binom{n-k-1}{n-k}a_k. \tag{6}$$

Now we give a basic result about perturbation of (m, C)-isometric operators by nilpotent operators.

Theorem 11. *Let $T \in \mathcal{L}(\mathcal{H})$ be a strict (m, C)-isometric operator with conjugation C. If and only if there are $B_{m-1}, B_{m-2}, ..., B_1, B_0$ in $\mathcal{L}(\mathcal{H})$, which depend only on T and C, such that, for every n = 0, 1, 2 . . .,*

$$T^{*n}CT^nC = \sum_{i=0}^{m-1}B_i n^i \tag{7}$$

that is, the sequence $(T^{*n}CT^nC)_{n\geq 0}$ is an arithmetic progression of strict order $m - 1$ in $\mathcal{L}(\mathcal{H})$.

Proof. From ([8], Theorem 1). Since T is a strict (m, C)-isometric operator with conjugation C, then $\Lambda_m(T) = \sum_{j=0}^{m}\binom{m}{j}(-1)^j T^{*m-j}CT^{m-j}C = 0$. Let $m - j = k$, so $\Lambda_m(T) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}CT^kC = 0$. So we have

$$\begin{aligned} T^{*i}\Lambda_m(T)CT^iC &= T^{*i}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}CT^kC)CT^iC \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k+i}CT^{k+i}C = 0. \end{aligned}$$

Then we obtain

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k+i}CT^{k+i}C = 0. \tag{8}$$

Since $T \in \mathcal{L}(\mathcal{H})$ is a strict (m, C)-isometric operator with conjugation C, then $\Lambda_{m-1}(T) \neq 0$. We obtain

$$\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}T^{*k}CT^{k}C \neq 0. \tag{9}$$

By (3) the operator sequence $(T^{*n}CT^{n}C)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$, Therefore, from (4) we obtain that there is a polynomial $p(n)$ of degree $m-1$ in n, with coefficients in $\mathcal{L}(\mathcal{H})$ satisfying $p(n) = T^{*n}CT^{n}C$ for every $n = 0, 1, 2, ...,$

$$T^{*n}CT^{n}C = B_{m-1}n^{m-1} + B_{m-2}n^{m-2} + ... + B_{1}n + B_{0}.$$

Conversely, if $(T^{*n}CT^{n}C)_{n\geq 0}$ is an arithmetic progression of strict order $m-1$, then (9) and (10) hold, take $i = 0$, we obtain $\Lambda_{m}(T) = \sum_{j=0}^{m}\binom{m}{j}(-1)^{j}T^{*m-j}CT^{m-j}C = 0$ and $\Lambda_{m-1}(T) \neq 0$, hence $T \in \mathcal{L}(\mathcal{H})$ is a strict (m, C)-isometric operator. □

Theorem 12. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator and $N \in \mathcal{L}(\mathcal{H})$ be an $n-$nilpotent ($n \geq 1$integer) such that $TN = NT$. Then $T + N$ is a strict $(2n + m - 2, C)$-isometric operator.*

Proof. From ([8], Theorem 3). Let $R = T + N, k = 2n + m - 2$, then we need to prove that $\Lambda_{k}(R) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{i}R^{*k-i}CR^{k-i}C = 0$. Let $q = k - i, h = min\{q, n-1\}$, for

$$\begin{aligned}
R^{*q}CR^{q}C &= (T^{*} + N^{*})^{q}(CTC + CNC)^{q} \\
&= [\sum_{j=0}^{h}\binom{q}{j}N^{*j}T^{*q-j}][\sum_{r=0}^{h}\binom{q}{r}CT^{q-r}CCN^{r}C] \\
&= \sum_{j=0}^{h}\binom{q}{j}\binom{q}{r}N^{*j}T^{*q-j}CT^{q-r}CCN^{r}C \\
&= \sum_{0\leq j\leq r\leq h}\binom{q}{j}\binom{q}{r}N^{*j}T^{*r-j}(T^{*q-r}CT^{q-r}C)CN^{r}C \\
&+ \sum_{0\leq r\leq j\leq h}\binom{q}{j}\binom{q}{r}N^{*j}(T^{*q-j}CT^{q-j}C)CT^{j-r}CCN^{r}C
\end{aligned}$$

from (8) we obtain, for certain $B_{m-1}, B_{m-2}, ..., B_{1}, B_{0}$, we have

$$T^{*q-r}CT^{q-r}C = \sum_{w=0}^{m-1}B_{w}(q-j)^{w},$$

then we have

$$\begin{aligned}
&(T^{*} + N^{*})^{q}(CTC + CNC)^{q} \\
= &\sum_{0\leq j\leq r\leq h}\binom{q}{j}\binom{q}{r}N^{*j}T^{*r-j}\sum_{w=0}^{m-1}B_{w}(q-r)^{w}CN^{r}C \\
&+ \sum_{0\leq r\leq j\leq h}\binom{q}{j}\binom{q}{r}N^{*j}\sum_{w=0}^{m-1}B_{w}(q-j)^{w}CT^{j-r}CCN^{r}C.
\end{aligned}$$

Let

$$\begin{aligned}
H_{1} &= N^{*j}T^{*r-j}B_{w}CN^{r}C, \\
H_{2} &= N^{*j}B_{w}CT^{j-r}CCN^{r}C, \\
Q_{1} &= \binom{q}{j}\binom{q}{r}(q-r)^{w}, \\
Q_{2} &= \binom{q}{j}\binom{q}{r}(q-j)^{w}.
\end{aligned}$$

Note that $\binom{q}{j}$ and $\binom{q}{r}$ are real polynomials in q of degree less than or equal to $h \le n-1$, and $(q-r)^w$ and $(q-j)^w$ degree $w \le m-1$, hence Q_1 and Q_2 are real polynomials of degree less than an equal to $m-1+2(n-1) = 2n+m-3$. Consequently, we can write

$$(T^* + N^*)^q (CTC + CNC)^q = \sum_{0\le j\le r\le h}^{m-1} H_1 Q_1 + \sum_{0\le r\le j\le h}^{m-1} H_2 Q_2,$$

which is a polynomial of q of degree less than or equal to $2n+m-3$ with coefficients in $\mathcal{L}(\mathcal{H})$, by Theorem 11, the operator $T+N$ is a strict (2n+m-2,C)-isometric operator. □

Corollary 13. *Let $T \in \mathcal{L}(\mathcal{H})$ be a strict (1,C)-isometric operator and $N \in \mathcal{L}(\mathcal{H})$ be an $n-$nilpotent ($n \ge 1, n \in \mathbb{N}$) such that $TN = NT$, then $T+N$ is a strict ($2n-1$,C)-isometric operator.*

Proof. The proof is similar to ([8], Theorem 4). □

Theorem 14. *Let $T \in \mathcal{L}(\mathcal{H})$ be a strict (m, C)-isometric operator and $N \in \mathcal{L}(\mathcal{H})$ be an $n-$nilpotent ($n \ge 1, n \in \mathbb{N}$) such that $TN = NT$, let $R = T+N$, then the following arguments hold:*

(i) If T^* has the single-valued extension property, then R^* and R has the single-valued extension property.

(ii) If T has Dunford's property (C) and $\sigma_T(x) \subset \sigma_R(N^{n-1}x) \cap \sigma_R(x)$ for all $x \in \mathcal{H}$, then R has Dunford's property (C).

Proof. (i) Let G be an open set in $\mathbb{C}$ and let $f : G \to \mathcal{H}$ be an analytic function such that $(R^* - z)f(z) \equiv 0$ on G, which implies

$$(R^* - z)f(z) = (T^* + N^* - z)f(z) = (T^* - z)f(z) + N^* f(z) \equiv 0.$$

Since $N^{*n} = 0$ and $T^*N^* = N^*T^*$, it follows that $(T^* - z)N^{*n-1}f(z) = 0$. Since T^* has the single-valued extension property, we have $N^{*n-1}f(z) = 0$, moreover, since $(T^* - z)f(z) + N^* f(z) \equiv 0$, we can obtain $(T^* - z)N^{*n-2}f(z) = 0$. Since T^* has the single-valued extension property, we get $N^{*n-2}f(z) = 0$. By similar process, we obtain that $f(z) = 0$, hence R^* has the single-valued extension property. From Lemma 2, we can obtain T has the single-valued extension property, similarly, we get R has the single-valued extension property.

(ii) If T has Dunford's property (C) and $\sigma_T(x) \subset \sigma_R(N^{n-1}x)$ for all $x \in \mathcal{H}$, then it suffices to show that $\sigma_R(N^{n-1}x) \subset \sigma_T(x)$, assume $z_0 \in \rho_T(x)$, then there is an $\mathcal{H}$-valued analytic function $f(z)$ in an neighborhood D of z_0 such that $(T-z)f(z) = x$ for every $z \in D$, since $TN = NT$ and $N^n = 0$, it follows that $(R-z)N^{n-1}f(z) = (T-z)N^{n-1}f(z) = N^{n-1}x$ on D. Since $N^{n-1}f(z)$ is analytic on D, we get $z_0 \in \rho_R(N^{n-1}x)$, then $\rho_T(x) = \rho_R(N^{n-1}x)$ on D. For any $S \in \mathcal{L}(\mathcal{H})$, $\sigma_S(x) = \mathbb{C} \setminus \rho_S(x)$. So $\sigma_R(N^{n-1}x) \subset \sigma_T(x)$. Thus $\sigma_T(x) = \sigma_R(N^{n-1}x)$, therefore, we have $N^{n-1}H_R(F) = H_T(F)$. Since $N^{n-1}H_R(F) \subset H_T(F)$, it follows that $H_T(F) \subset N^{n-1}H_R(F)$, where F is closed subset of $\mathbb{C}$. Moreover, since $\sigma_T(x) \subset \sigma_R(x)$ for all $x \in \mathcal{H}$, it follows that $H_R(F) \subset H_T(F)$ and so $H_R(F) = H_T(F)$ is closed for each closed subset F of $\mathbb{C}$, hence R has Dunford's property (C), this completes the proof. □

Corollary 15. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator and $N \in \mathcal{L}(\mathcal{H})$ be an $n-$nilpotent ($n \ge 1$integer) such that $TN = NT$, let $R = T+N$ and if T^* has the single-valued extension property, then the following properties hold:*

(i) $\sigma(R) = \sigma_{su}(R) = \sigma_{ap}(R) = \sigma_{se}(R)$.

(ii) $\sigma_{es}(R) = \sigma_b(R) = \sigma_\omega(R) = \sigma_e(R)$.

(iii) $H_0(R-\lambda) = H_R(\{\lambda\})$ and $H_0(R^* - \lambda) = H_{R^*}(\{\lambda\})$ for all $\lambda \in \mathbb{C}$.

Proof. Since T^* has the single-valued extension property, it follows that R and R^* the single-valued extension property, hence it follows from Lemma 10. □

Proposition 16. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator and $N \in \mathcal{L}(\mathcal{H})$ be an $n-$nilpotent ($n \ge 1$integer) such that $TN = NT$, let $R = T+N$, if T^* has the single-valued extension property and T^* commutes with CTC, then the following properties hold:*

(i) $\sigma_{ap}(R) \subset \frac{1}{\sigma_{ap}(T^*)^*} \cup 0$, $\sigma_p(R) \subset \frac{1}{\sigma_p(T^*)^*} \cup 0$, $\sigma_{comp}(R) \subset \frac{1}{\sigma_{comp}(T^*)^*} \cup 0$, $\sigma_{su}(R) \subset \frac{1}{\sigma_{su}(T^*)^*} \cup 0$.

(ii) $\sigma_{le}(R) \subset \sigma_{le}(T)$, $\sigma_{re}(R^*)^* \subset \sigma_{re}(T^*)^*$.

Proof. *(i) Let $R = T + N$, where $T \in \mathcal{L}(\mathcal{H})$ is an (m,C)-isometric operator , $N^n = 0$ and $TN = NT$. It follows from Theorem 11 and ([6], Page256) that $\sigma_{ap}(R) = \sigma_{ap}(T) + \sigma_{ap}(N)$, it is easy to obtain $\sigma_{ap}(N) = 0$. Since $0 \notin \sigma_{ap}(T)$ from Theorem 11, we obtain $\sigma_{ap}(T) = \frac{1}{\sigma_{ap}(T^*)^*}$, therefore we have $\sigma_{ap}(R) \subset \frac{1}{\sigma_{ap}(T^*)^*} \cup 0$. By the similar method we get that $\sigma_p(R) = \frac{1}{\sigma_p(T^*)^*} \cup 0$. From the proof of Theorem 11, we can obtain $\sigma_{comp}(R) \subset \frac{1}{\sigma_{comp}(T^*)^*} \cup 0$, $\sigma_{su}(R) \subset \frac{1}{\sigma_{su}(T^*)^*} \cup 0$.*

(ii) The proof is similar to ([2], Proposition 16). If $\lambda \in \sigma_{le}(R)$, then there exists a sequence x_i of unit vectors in $\mathcal{H}$ such that x_i weakly converges to 0 and $\lim_{i\to\infty} \|(R - \lambda)x_i\| = 0$. Put $y_i = \frac{N^{n-1}x_i}{\|N^{n-1}x_i\|}$, for some $n \geq 1$,since $N^n = 0$ and $TN = NT$, it follows that

$$\begin{aligned}\lim_{i\to\infty} \|(T - \lambda)y_i\| &= \lim_{i\to\infty} \|(T - \lambda)\frac{N^{n-1}x_i}{\|N^{n-1}x_i\|}\| \\ &= \lim_{i\to\infty} \|N^{n-1}(T + N - \lambda)\frac{x_i}{\|N^{n-1}x_i\|}\| \\ &= \lim_{i\to\infty} \|N^{n-1}(R - \lambda)\frac{x_i}{\|N^{n-1}x_i\|}\|\end{aligned}$$

In addition, if x_i weakly converges to 0, then y_i weakly converges to 0. Therefore, $\lambda \in \sigma_{le}(T)$, hence $\sigma_{le}(R) \subset \sigma_{le}(T)$. Since $\sigma_{re}(S^*) = \sigma_{le}(S)^*$, for any $S \in \mathcal{L}(\mathcal{H})$, we have $\sigma_{re}(R^*)^* \subset \sigma_{re}(T^*)^*$. □

4. Perturbation of (*m*, *C*)-isometric operators by algebraic operators

In this section, we will give some properties of perturbation of (m, C)-isometric operators by algebraic operators, we will consider the decomposability of $T + A$ and TA where T is (m,C)-isometric and A is an algebraic operator.

Proposition 17. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (m, C)-isometric operator and $A \in \mathcal{L}(\mathcal{H})$ be a algebraic operator of order k, if $R = T + A$ or TA where T commutes with A, then the following properties are equivalent:*

(i) T is decomposable.

(ii) T^* has the property (β).

(iii) R^* has the property (β).

Proof. *(1) Since the proof of from Theorem 3 and Corollary 4, we get $(i) \Leftrightarrow (ii)$, we only consider the following implication $(ii) \Leftrightarrow (iii)$. Assume that T^* has the property (β), let $R = T + A$, since $A \in \mathcal{L}(\mathcal{H})$ is a algebraic operator of order k, there exists a nonconstant polynomial $P(\lambda) = (\lambda - \gamma_1)(\lambda - \gamma_2)(\lambda - \gamma_3)...(\lambda - \gamma_k)$ such that $P(A) = 0$, set $P_0(\lambda) = 1$ and $P_j(\lambda) = (\lambda - \gamma_1)(\lambda - \gamma_2)(\lambda - \gamma_3)...(\lambda - \gamma_j)$ for $j = 1, 2, 3, ..., k$. Let G be an open set in $\mathbb{C}$ and $f_n : G \to \mathcal{H}$ be a sequence of analytic functions such that*

$$\lim_{n\to\infty} \|(R^* - z)f(z)\|_K = \lim_{n\to\infty} \|(T^* + A^* - z)f(z)\|_K = 0 \tag{10}$$

for every compact set K in D, fix any compact subset K of D. Since $P_k(A)^* = (A^* - \overline{\gamma_1})(A^* - \overline{\gamma_2})(A^* - \overline{\gamma_3})...(A^* - \overline{\gamma_{k-1}})(A^* - \overline{\gamma_k}) = 0$, we obtain $P_{k-1}(A)^*(A^* - \overline{\gamma_k}) = 0$, therefore $P_{k-1}(A)^*A^* = P_{k-1}(A)^*\overline{\gamma_k}$ this gives that

$$\begin{aligned}\lim_{n\to\infty} \|P_{k-1}(A)^*(R^* - z)f(z)\|_K &= \lim_{n\to\infty} \|P_{k-1}(A)^*(T^* + A^* - z)f(z)\|_K \\ &= \lim_{n\to\infty} \|P_{k-1}(A)^*(T^* + \overline{\gamma_k} - z)f(z)\|_K \\ &= \lim_{n\to\infty} \|(T^* + \overline{\gamma_k} - z)P_{k-1}(A)^*f(z)\|_K \\ &= 0.\end{aligned}$$

Moreover, since $T^* + \overline{\gamma_k}$ has the property(β), we have

$$\lim_{n\to\infty} \|P_{k-1}(A)^*f(z)\|_K = 0. \tag{11}$$

Equations (10) and (11) imply that $\lim_{n\to\infty} \|P_{k-2}(A)^*(T^* + \overline{\gamma_{k-1}} - z)f(z)\|_K = \lim_{n\to\infty} \|P_{k-2}(A)^*(T^* + A^* - z)f(z)\|_K = 0$. *Since* $T^* + \overline{\gamma_{k-1}}$ *has the property* (β), *we get that* $\lim_{n\to\infty} \|P_{k-1}(A)^* f(z)\|_K = 0$. *Hence, by induction* $\lim_{n\to\infty} \|f(z)\|_K = 0$. *Therefore,* R^* *has the property* (β).

(2) In the case $R = TA$. *Assume that* T^* *has the property* (β). *Let* G *be an open set in* $\mathbb{C}$ *and* $f_n : G \to \mathcal{H}$ *be a sequence of analytic functions such that* $\lim_{n\to\infty} \|(R^* - z)f(z)\|_K = \lim_{n\to\infty} \|(T^*A^* - z)f(z)\|_K = 0$ *for every compact set K in D, fix any compact subset K of D. Thus, it holds that*

$$\lim_{n\to\infty} \|(A^* - \overline{\gamma_k})T^* f(z) + \overline{\gamma_k} T^* f(z) - z f(z)\|_K = 0. \tag{12}$$

Since $P_k(A)^* = (A^* - \overline{\gamma_1})(A^* - \overline{\gamma_2})(A^* - \overline{\gamma_3})...(A^* - \overline{\gamma_{k-1}})(A^* - \overline{\gamma_k}) = P_{k-1}(A)^*(A^* - \overline{\gamma_k}) = 0$ *and* $A^*T^* = T^*A^*$. *We obtain that from (12)*

$$\lim_{n\to\infty} \|P_{k-1}(A)^*(A^* - \overline{\gamma_k})T^* f(z) + P_{k-1}(A)^* \overline{\gamma_k} T^* f(z) - P_{k-1}(A)^* z f(z)\|_K = 0.$$

So we have

$$\lim_{n\to\infty} \|P_{k-1}(A)^* \overline{\gamma_k} T^* f(z) - P_{k-1}(A)^* z f(z)\|_K = 0. \tag{13}$$

Since $T^*\overline{\gamma_k}$ *has the property* (β), *we have* $\lim_{n\to\infty} \|P_{k-1}(A)^* f(z)\|_K = 0$. *Following from the proof of (1), so we have* $\lim_{n\to\infty} \|f(z)\|_K = 0$. *Hence,* R^* *has the property* (β). *The converse implication holds by the similar arguments above.* □

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