



Some New Sufficient Conditions for $2p$ -Hamilton-Biconnectedness of Graphs

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Abstract. A balanced bipartite graph G is said to be $2p$ -Hamilton-biconnected if for any balanced subset W of size $2p$ of $V(G)$, the subgraph induced by $V(G) \setminus W$ is Hamilton-biconnected. In this paper, we prove that “ Let G be a balanced bipartite graph of order $2n$ with minimum degree $\delta(G) \geq k$, where $n \geq 2k - p + 2$ for two integers $k \geq p \geq 0$. If the number of edges $e(G) > n(n - k + p - 1) + (k + 2)(k - p + 1)$, then G is $2p$ -Hamilton-biconnected except some exceptions.” Furthermore, this result is used to present two new spectral conditions for a graph to be $2p$ -Hamilton-biconnected. Moreover, the similar results are also presented for nearly balanced bipartite graphs.

1. Introduction

Let G be an undirected simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. Denote by $\delta(G)$ the *minimum degree* of G . The *adjacency matrix* $A(G)$ of G is the $n \times n$ matrix (a_{ij}) , where $a_{ij} = 1$ if v_i is adjacent to v_j , and 0 otherwise. The matrix $Q(G) = D(G) + A(G)$ is known as the *signless Laplacian matrix* of G , where $D(G)$ is the degree diagonal matrix. The *spectral radius* and *signless Laplacian spectral radius* of G are the largest eigenvalues of $A(G)$ and $Q(G)$, denoted by $\rho(G)$ and $q(G)$, respectively.

For two disjoint graphs G and H , we denote by $G \cup H$ and $G \vee H$ the *union* of G and H , and the *join* of G and H which is obtained from $G \cup H$ by joining every vertex of G to every vertex of H , respectively. Moreover, kG denotes a graph consisting of k disjoint copies of G . Denote by $G[X, Y]$ the subgraph of G with all possible edges with one end vertex in X and the other in Y respectively. Denote $e(X, Y) = |E(G[X, Y])|$.

A cycle (path) in a graph G that contains every vertex of G is called a *Hamiltonian cycle (path)* of G , respectively. A graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle. A bipartite graph $G = (X, Y; E)$ is called (*nearly*) *balanced* if $(\|X| - |Y| = 1) |X| = |Y|$ respectively. A (nearly) balanced bipartite graph $G = (X, Y; E)$ with $(\|X| - |Y| = 1) |X| = |Y|$ is called *Hamilton-biconnected* if for (any two distinct vertices $u, v \in X$) any vertex $u \in X$ and any vertex $v \in Y$, G has a Hamiltonian path between u and v , respectively. A (nearly) balanced bipartite graph G is said to be $2p$ -Hamilton-biconnected if for any balanced subset W of

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size $2p$ of $V(G)$, the subgraph induced by $V(G)\setminus W$ is Hamilton-biconnected, respectively. Obviously for $p = 0$, $2p$ -Hamilton-biconnected graphs are exactly Hamilton-biconnected graphs. For graph notation and terminology undefined here, readers are referred to [6].

Denote by $M_{n,m}^{s,t}$ a bipartite graph obtained from $K_{s,m-t} \cup K_{n-s,t}$ by joining every vertex in X_2 to every vertex in Y_1 , where $K_{s,m-t} = (X_1, Y_1; E_1)$ and $K_{n-s,t} = (X_2, Y_2; E_2)$ with $|X_1| = s$, $|Y_1| = m - t$, $|X_2| = n - s$, and $|Y_2| = t$ (see Fig. 1). Moreover, denote by $N_{n,n}^{p,1}$ a balanced bipartite graph obtained from $K_{n-p-2,n-p-2} \cup K_{p+1,p+1} \cup K_2$ by joining every vertex in X_1 to every vertex in Y_2 , every vertex in X_2 to every vertex in $Y_1 \cup Y_3$, and every vertex in X_3 to every vertex in Y_2 , where $K_{n-p-2,n-p-2} = (X_1, Y_1; E_1)$, $K_{p+1,p+1} = (X_2, Y_2; E_2)$, and $K_2 = (X_3, Y_3; E_3)$ with $|X_1| = |Y_1| = n - p - 2$, $|X_2| = |Y_2| = p + 1$, and $|X_3| = |Y_3| = 1$ (see Fig. 1).

The problem of deciding whether a graph is Hamiltonian is NP-complete. So researchers focus on giving reasonable sufficient or necessary conditions for Hamiltonian cycles in graphs and bipartite graphs. Moon and Moser [13] studied balanced bipartite graphs and showed a sufficient condition for Hamiltonian cycles in balanced bipartite graphs with large minimum degree.

Theorem 1.1. [13] *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k$, where $1 \leq k \leq \frac{n}{2}$. If*

$$e(G) > \max \left\{ n(n - k) + k^2, n \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right) + \left\lfloor \frac{n}{2} \right\rfloor^2 \right\},$$

then G is Hamiltonian.

Amar et al. [2] proved a sufficient condition for $2p$ -Hamilton-biconnectedness of balanced bipartite graphs.

Theorem 1.2. [2] *Let $p \geq 0$ and G be a balanced bipartite graph of order $2n$. If*

$$e(G) > n(n - 1) + p + 1, \tag{1}$$

then G is $2p$ -Hamilton-biconnected.

Recently, Li and Ning [10] gave the spectral analogue of Moon–Moser’s theorem [13]. For more results, readers are referred to [1, 3, 8, 9, 11, 12, 14–16]. In this paper, we establish the analogues of Moon–Moser’s theorem for $2p$ -Hamilton-biconnectedness of balanced bipartite graphs and nearly balanced bipartite graphs, respectively.

Theorem 1.3. *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k$, where $n \geq 2k - p + 2$ for two nonnegative integers $k \geq p \geq 0$. If*

$$e(G) > n(n - k + p - 1) + (k + 2)(k - p + 1), \tag{2}$$

then G is $2p$ -Hamilton-biconnected, unless $G \subseteq M_{n,n}^{n-k,k-p}$ for $k \geq p + 1$, or $G \subseteq N_{n,n}^{p,1}$ for $k = p + 2$.

Remark 1. Theorem 1.2 [2] and Theorem 1.3 are not comparable. For $k \geq p + 1$ and large n , the condition (2) in Theorem 1.3 is weaker than the condition (1) in Theorem 1.2.

Theorem 1.4. *Let G be a nearly balanced bipartite graph of order $2n - 1$ with $\delta(G) \geq k$, where $n \geq 2k - p + 2$ for two nonnegative integers $k \geq p \geq 0$. If*

$$e(G) > n(n - k + p - 2) + (k + 2)(k - p + 1),$$

then G is $2p$ -Hamilton-biconnected, unless one of the following holds:

- (i) $G \subseteq M_{n,n-1}^{n-k-1,k-p}$ for $k \geq p + 1$;
- (ii) $G \subseteq M_{n,n-1}^{k-p,n-k-1}$ for $k \geq p + 1$;
- (iii) $G \subseteq M_{n,n-1}^{n-k,k-p-1}$ for $k \geq p + 2$.

Theorems 1.3 and 1.4 can be used to obtain some spectral conditions for $2p$ -Hamilton-biconnectedness of balanced bipartite graphs and nearly balanced bipartite graphs in terms of spectral radius or signless Laplacian spectral radius, respectively.

For balanced bipartite graphs, we have

Theorem 1.5. Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k$ and two integers $k \geq p + 1 \geq 1$.

- (i) If $k = p + 2$, $n \geq 2k^2 + 3$, and $\rho(G) \geq \rho(N_{n,n}^{k-2,1})$, then G is $2p$ -Hamilton-biconnected unless $G = N_{n,n}^{k-2,1}$.
- (ii) If $k \neq p + 2$, $n \geq (k + 2)(k - p + 1)$, and $\rho(G) \geq \rho(M_{n,n}^{n-k,k-p})$, then G is $2p$ -Hamilton-biconnected unless $G = M_{n,n}^{n-k,k-p}$.

Theorem 1.6. Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k$, where $n \geq (k + 2)(k - p + 1)$ for two integers $k \geq p + 1 \geq 1$. If $q(G) \geq q(M_{n,n}^{n-k,k-p})$, then G is $2p$ -Hamilton-biconnected unless $G = M_{n,n}^{n-k,k-p}$.

For nearly balanced bipartite graphs, we have

Theorem 1.7. Let G be a nearly balanced bipartite graph of order $2n - 1$ with $\delta(G) \geq k$ for two nonnegative integers k and p .

- (i) If $k = p + 1$, $n \geq 2k + 3$, and $\rho(G) \geq \rho(M_{n,n-1}^{1,n-k-1})$, then G is $2p$ -Hamilton-biconnected unless $G = M_{n,n-1}^{1,n-k-1}$.
- (ii) If $k \geq p + 2$, $n \geq \frac{(k+2)(k-p+1)}{2}$, and $\rho(G) \geq \rho(M_{n,n-1}^{n-k,k-p-1})$, then G is $2p$ -Hamilton-biconnected unless $G = M_{n,n-1}^{n-k,k-p-1}$.

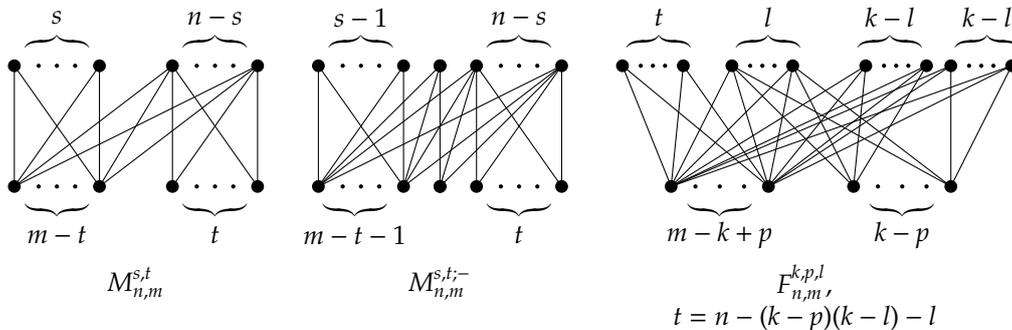
Theorem 1.8. Let G be a nearly balanced bipartite graph of order $2n - 1$ with $\delta(G) \geq k$ for two nonnegative integers k and p .

- (i) If $k = p + 1$, $n \geq 2k + 4$, and $q(G) \geq q(M_{n,n-1}^{n-k-1,1})$, then G is $2p$ -Hamilton-biconnected unless $G = M_{n,n-1}^{n-k-1,1}$.
- (ii) If $k \geq p + 2$, $n \geq \frac{(k+2)(k-p+1)}{2}$, and $q(G) \geq q(M_{n,n-1}^{n-k,k-p-1})$, then G is $2p$ -Hamilton-biconnected unless $G = M_{n,n-1}^{n-k,k-p-1}$.

The rest of this paper is organized as follows. In Section 2, we state some known and new results that will be used in the proofs of Theorems 1.3–1.8. In Section 3, we present some necessary lemmas and prove Theorems 1.3 and 1.4. In Section 4, we present some necessary lemmas and prove Theorems 1.5 and 1.6. Some corollaries are also included. In Section 5, we present some necessary lemmas and prove Theorems 1.7 and 1.8. Some corollaries are also included.

2. Preliminaries

We now introduce some more terminologies and notations, which will be used in this section and the proofs of Theorems 1.3 and 1.4. Recall that the k -biclosure of a bipartite graph $G = (X, Y; E)$ [5] is the unique smallest bipartite graph H of order $|V(H)| := |V(G)|$ such that $G \subseteq H$ and $d_H(x) + d_H(y) < k$ for any two non-adjacent vertices $x \in X$ and $y \in Y$. The k -biclosure of G is denoted by $cl_k(G)$, and $cl_k(G)$ can be obtained from G by a recursive procedure which consists of joining non-adjacent vertices in different classes with degree sum at least k until no such pair remains. A bipartite graph is called k -closed if $G = cl_k(G)$.



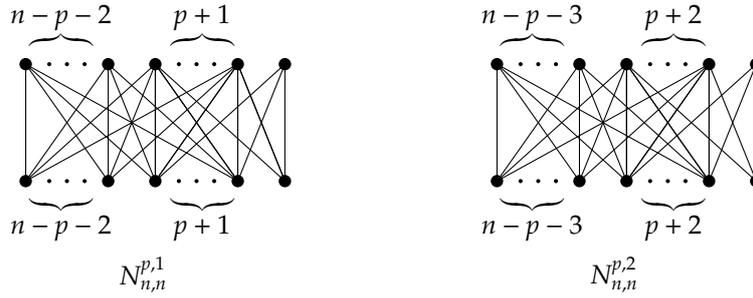


Fig. 1. Graphs $M_{n,m}^{s,t}$, $M_{n,m}^{s,t;-}$, $F_{n,m}^{k,p,l}$, $N_{n,n}^{p,1}$ and $N_{n,n}^{p,2}$.

Denote by $M_{n,m}^{s,t;-}$ a bipartite graph obtained from $K_{s-1,m-t-1} \cup K_2 \cup K_{n-s,t}$ by joining every vertex in X_2 to every vertex in Y_1 , and every vertex in X_3 to every vertex in $Y_1 \cup Y_2$, where $K_{s-1,m-t-1} = (X_1, Y_1; E_1)$, $K_2 = (X_2; Y_2; E_2)$, and $K_{n-s,t} = (X_3, Y_3; E_3)$ with $|X_1| = s - 1$, $|Y_1| = m - t - 1$, $|X_2| = |Y_2| = 1$, $|X_3| = n - s$, and $|Y_3| = t$ (see Fig. 1). Obviously $M_{n,m}^{s,t;-} \subseteq M_{n,m}^{s,t}$.

Denote by $N_{n,n}^{p,2}$ a balanced bipartite graph obtained from $K_{n-p-3,n-p-3} \cup K_{p+2,p+2} \cup \bar{K}_2$ by joining every vertex in X_1 to every vertex in Y_2 , every vertex in X_2 to every vertex in $Y_1 \cup Y_3$, and every vertex in X_3 to every vertex in Y_2 , where $K_{n-p-3,n-p-3} = (X_1, Y_1; E_1)$, $K_{p+2,p+2} = (X_2, Y_2; E_2)$, and $\bar{K}_2 = (X_3, Y_3; E_3)$ with $|X_1| = |Y_1| = n - p - 3$, $|X_2| = |Y_2| = p + 2$, and $|X_3| = |Y_3| = 1$ (see Fig. 1).

Given integers n, m, k, p, l , where $k \geq p + 2$, $0 \leq l \leq k - 1$, $n \geq (k - p)(k - l) + l$, and $n - 1 \leq m \leq n$, we denote by $F_{n,m}^{k,p,l}$ a bipartite graph obtained from $M_{n-(k-p)(k-l),m}^{n-(k-p)(k-l)-l,k-p}$ by attaching $k - l$ pendant vertices at every vertex of those $k - p$ vertices with degree l , respectively, and then joining every pendant vertex to every vertex with degree $n - (k - p)(k - l)$ in $M_{n-(k-p)(k-l),m}^{n-(k-p)(k-l)-l,k-p}$ (see Fig. 1).

The following lemma follows from the Perron–Frobenius theorem.

Lemma 2.1. *Let G be a connected graph. If H is a (proper) subgraph of G , then $\rho(H)(<) \leq \rho(G)$ and $q(H)(<) \leq q(G)$, respectively.*

Lemma 2.2. [4] *Let G be a bipartite graph. Then*

$$\rho(G) \leq \sqrt{e(G)},$$

with equality if and only if G is a disjoint union of a complete bipartite graph and isolated vertices.

Lemma 2.3. [10] *Let G be a balanced bipartite graph of order $2n$. Then*

$$q(G) \leq \frac{e(G)}{n} + n.$$

with equality if and only if $G = K_{n,n}$.

Remark 2: The extremal graph in Lemma 2.3 is not characterized in [10]. But it is easy to obtain the extremal graph by combining the proof of Lemma 2.3 and Das’s bound [7, Theorem 4.5].

Note that $G \subseteq cl_{n+p+2}(G)$. If G is $2p$ -Hamilton-biconnected then so is $cl_{n+p+2}(G)$. Combining this with [2, Theorem 3.3.1], we have the following lemma.

Lemma 2.4. *Let $p \geq 0$ and G be a balanced bipartite graph of order $2n$. Then G is $2p$ -Hamilton-biconnected if and only if $cl_{n+p+2}(G)$ is $2p$ -Hamilton-biconnected.*

Lemma 2.5. *Let $p \geq 0$ and G be a nearly balanced bipartite graph of order $2n - 1$. Then G is $2p$ -Hamilton-biconnected if and only if $cl_{n+p+1}(G)$ is $2p$ -Hamilton-biconnected.*

Proof. Since $G \subseteq cl_{n+p+1}(G)$, if G is $2p$ -Hamilton-biconnected then so is $cl_{n+p+1}(G)$. Conversely, suppose that $cl_{n+p+1}(G)$ is $2p$ -Hamilton-biconnected. Denote $G = (X, Y; E)$ with $|X| = n$ and $|Y| = n - 1$. We show that if $G + xy$ is $2p$ -Hamilton-biconnected for two non-adjacent vertices $x \in X$ and $y \in Y$ with $d_G(x) + d_G(y) \geq n + p + 1$, then G is $2p$ -Hamilton-biconnected. Indeed, if G is not $2p$ -Hamilton-biconnected, then there exists a balanced subset W of size $2p$ of $V(G)$ and two vertices $x_1, x_2 \in X \setminus W$ such that the subgraph F induced by $V(G) \setminus W$ has no Hamiltonian path between x_1 and x_2 . On the other hand, since $G + xy$ is $2p$ -Hamilton-biconnected, the graph $F + xy$ has a Hamiltonian path between x_1 and x_2 and thus $x \in X \setminus W$ and $y \in Y \setminus W$. Let H be a graph obtained from F by adding a new vertex v in Y and two edges vx_1 and vx_2 . Then H is not Hamiltonian, but $H + xy$ is Hamiltonian. Note that

$$\begin{aligned} d_H(x) + d_H(y) &\geq d_F(x) + d_F(y) \geq (d_G(x) - p) + (d_G(y) - p) \\ &= d_G(x) + d_G(y) - 2p \geq n - p + 1 = \frac{1}{2}|V(H)| + 1. \end{aligned}$$

It follows from [5, Theorem 6.2] that H is Hamiltonian, a contradiction. Note that $cl_{n+p+1}(G)$ is a graph obtained from G by a recursive procedure joining non-adjacent vertices in different classes with degree sum at least $n + p + 1$ until no such pair remains. Since $cl_{n+p+1}(G)$ is $2p$ -Hamilton-biconnected, G is also $2p$ -Hamilton-biconnected. \square

The proofs of Lemmas 2.6–2.8 are put in the appendix, since they are technical and complicated.

Lemma 2.6. $F_{n,m}^{k,p,l}$ is $2p$ -Hamilton-biconnected.

Lemma 2.7. (i) For integers $p \geq 0, s \geq 2, t \geq 1$, and $n = s + t + p + 1$, $M_{n,n}^{s,t,-}$ is $2p$ -Hamilton-biconnected.

(ii) For two integers $p \geq 0$ and $n \geq p + 6$, $N_{n,n}^{p,2}$ is $2p$ -Hamilton-biconnected.

Lemma 2.8. (i) For integers $p \geq 0, s, t \geq 1$, and $\max\{s + p + 1, t + p + 2\} \leq n \leq s + t + p + 1$, $M_{n,n-1}^{s,t}$ is not $2p$ -Hamilton-biconnected.

(ii) For integers $p \geq 0, s, t \geq 1$, and $n = s + t + p$, $M_{n,n}^{s,t}$ is not $2p$ -Hamilton-biconnected.

(iii) For two integers $p \geq 0$ and $n \geq p + 6$, $N_{n,n}^{p,1}$ is not $2p$ -Hamilton-biconnected.

3. Proofs of Theorems 1.3 and 1.4

In order to prove Theorems 1.3 and 1.4, we first prove the following lemma, in which the techniques are from [10, Lemma 4].

Lemma 3.1. *Let G be an $(n + p + 2)$ -closed balanced bipartite graph of order $2n$, where $n \geq 2k - p + 2$ for two integers $k \geq p \geq 0$. If*

$$e(G) > n(n - k + p - 1) + (k + 2)(k - p + 1),$$

then G contains a complete bipartite graph of order $2n - k + p$. Furthermore, if $\delta(G) \geq k$, then $K_{n,n-k+p} \subseteq G$, or $G \in \{N_{n,n}^{p,1}, N_{n,n}^{p,2}\}$ for $k = p + 2$.

Proof. Denote $G = (X, Y; E)$ with $|X| = |Y| = n$. Let $U = \{x \in X : d_G(x) \geq \frac{n+p+2}{2}\}$ and $W = \{y \in Y : d_G(y) \geq \frac{n+p+2}{2}\}$. Then

$$n(n - k + p - 1) + (k + 2)(k - p + 1) < e(G) \leq n|U| + \frac{(n - |U|)(n + p + 1)}{2}.$$

Since $k \geq p$ and $n \geq 2k - p + 2$, we have

$$|U| \geq \frac{n^2 - (2k - p + 3)n + 2(k + 2)(k - p + 1) + 2}{n - p - 1} > k + 1,$$

which implies that $|U| \geq k + 2$. By symmetry, $|W| \geq k + 2$. Since G is an $(n + p + 2)$ -closed balanced bipartite graph, every vertex in U is adjacent to every vertex in W and thus $K_{k+2,k+2} \subseteq G$. Let t be the largest integer such that $K_{t,t} \subseteq G$.

Claim 1. $t \geq n - k + p$.

Suppose that $k + 2 \leq t \leq n - k + p - 1$. Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ with $|X_1| = |Y_1| = t$ such that $G[X_1, Y_1] = K_{t,t}$. Set $X_2 = X \setminus X_1$ and $Y_2 = Y \setminus Y_1$. Since t is the largest integer such that $K_{t,t} \subseteq G$, there exists a corresponding vertex $y \in Y_1$ such that $xy \notin E(G)$ for every $x \in X_2$ (by symmetry). It follows that $d_G(x) \leq n + p - t + 1$ for every $x \in X_2$. Hence

$$\begin{aligned} e(G) &= e(X_1, Y_1) + e(X_1, Y_2) + e(X_2, Y) \\ &\leq t^2 + t(n - t) + (n + p - t + 1)(n - t) \\ &= t^2 - (n + p + 1)t + n(n + p + 1) \\ &\leq (n - k + p - 1)^2 - (n + p + 1)(n - k + p - 1) + n(n + p + 1) \\ &= n(n - k + p - 1) + (k + 2)(k - p + 1) \\ &< e(G), \end{aligned}$$

a contradiction. Thus Claim 1 holds.

Let s be the largest integer such that $K_{s,t} \subseteq G$. Obviously, $s \geq t$. Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that $G[X_1, Y_1] = K_{s,t}$, where $|X_1| = s$ and $|Y_1| = t$. Set $X_2 = X \setminus X_1$ and $Y_2 = Y \setminus Y_1$.

Claim 2. $s + t \geq 2n - k + p$.

Suppose that $s + t \leq 2n - k + p - 1$. It follows from Claim 1 that $n - k + p \leq t \leq n - \frac{k-p+1}{2}$ and $t \leq s \leq 2n - k + p - t - 1$. Since G is an $(n + p + 2)$ -closed balanced bipartite graph, $d_G(x) \leq n + p - s + 1$ for every $x \in X_2$ and $d_G(y) \leq n + p - t + 1$ for every $y \in Y_2$. Hence

$$\begin{aligned} e(G) &\leq e(X_1, Y_1) + e(X_2, Y) + e(X, Y_2) \\ &\leq st + (n + p - s + 1)(n - s) + (n + p - t + 1)(n - t) \\ &= s^2 - (2n + p - t + 1)s + (n - t)(n + p - t + 1) + n(n + p + 1) \\ &\leq (2n - k + p - t - 1)^2 - (2n + p - t + 1)(2n - k + p - t - 1) + \\ &\quad (n - t)(n + p - t + 1) + n(n + p + 1) \\ &= t^2 - (2n - k + p - 1)t + 2n^2 - 2(k - p + 1)n + (k + 2)(k - p + 1) \\ &\leq (n - k + p)^2 - (2n - k + p - 1)(n - k + p) + 2n^2 - 2(k - p + 1)n + \\ &\quad (k + 2)(k - p + 1) \\ &= n(n - k + p - 1) + (k + 1)(k - p + 1) + 1 \\ &< e(G), \end{aligned}$$

a contradiction. Thus Claim 2 holds.

It follows from Claim 2 that $K_{s,t}$ is a complete bipartite graph of order at least $2n - k + p$. Hence G contains a complete bipartite graph of order $2n - k + p$.

Claim 3. If $\delta(G) \geq k$, then $K_{n,n-k+p} \subseteq G$, or $G \in \{N_{n,n}^{p,1}, N_{n,n}^{p,2}\}$ for $k = p + 2$.

If $t = n - k + p$, then Claim 2 implies that $s = n$ and thus $K_{n,n-k+p} \subseteq G$. So we can assume that $t \geq n - k + p + 1$. Next we consider the following two cases.

Case 1. $s = n - k + p + 1$. Obviously, $k \geq p + 1$ and $t = n - k + p + 1$. If $k \geq p + 3$, then $s + t = 2(n - k + p + 1) < 2n - k + p$, which contradicts Claim 2. Hence $p + 1 \leq k \leq p + 2$. Furthermore, if $k = p + 1$, then $G = K_{n,n}$ and thus $K_{n,n-k+p} \subseteq G$. If $k = p + 2$, then $s = t = n - 1$. Note that G is an $(n + p + 2)$ -closed bipartite graph with $\delta(G) \geq k$. If there exists a vertex $v \in V(G) \setminus V(K_{n-1,n-1})$ such that $d_G(v) \geq k + 1$, then $K_{n,n-1} \subseteq G$. If $d_G(v) = k = p + 2$ for every $v \in V(G) \setminus V(K_{n-1,n-1})$, then $G = N_{n,n}^{p,1}$ or $G = N_{n,n}^{p,2}$ for $k = p + 2$.

Case 2. $s \geq n - k + p + 2$. Clearly, $d_G(y) \geq n - k + p + 2$ for every $y \in Y_1$. Then every vertex in Y_1 is adjacent to every vertex in X . This implies that $K_{n,n-k+p} \subseteq G$. \square

Corollary 3.2. Let G be an $(n + p + 1)$ -closed nearly balanced bipartite graph of order $2n - 1$, where $n \geq 2k - p + 2$ for two integers $k \geq p \geq 0$. If

$$e(G) > n(n - k + p - 2) + (k + 2)(k - p + 1),$$

then G contains a complete bipartite graph of order $2n - k + p - 1$. Furthermore, if $\delta(G) \geq k$, then $K_{n-1,n-k+p} \subseteq G$ or $K_{n,n-k+p-1} \subseteq G$.

Proof. Denote $G = (X, Y; E)$ with $|X| = n$ and $|Y| = n - 1$. Let H be a graph with vertex set $V(G) \cup \{y\}$ and edge set $E(G) \cup \{xy : x \in X\}$, where $y \notin V(G)$. Then H is an $(n + p + 2)$ -closed balanced bipartite graph of order $2n$ and $e(H) = e(G) + n > n(n - k + p - 1) + (k + 2)(k - p - 1)$. By Lemma 3.1, H contains a complete bipartite graph of order $2n - k + p$. Thus G contains a complete bipartite graph of order $2n - k + p - 1$. Note that if $\delta(G) \geq k$, then $\delta(H) \geq \delta(G) \geq k$. It follows from Lemma 3.1 that $K_{n,n-k+p} \subseteq H$, or $H \in \{N_{n,n}^{p,1}, N_{n,n}^{p,2}\}$ for $k = p + 2$. If $K_{n,n-k+p} \subseteq H$, then $K_{n,n-k+p-1} \subseteq G$ or $K_{n-1,n-k+p} \subseteq G$. If $H \in \{N_{n,n}^{p,1}, N_{n,n}^{p,2}\}$ for $k = p + 2$, then $\delta(G) = p + 1 < k$, a contradiction. This completes the proof. \square

Lemma 3.3. Let k, p be two noneagative integers and $G = (X, Y; E)$ be an $(m + p + 2)$ -closed bipartite graph with $|X| = n, |Y| = m$, and $\delta(G) \geq k$, where $n \geq 2k - p + 2$ and $n - 1 \leq m \leq n$. Suppose that $k \geq p$ and $K_{n,m-k+p} \subseteq G$.

- (i) If $m = n$, then G is $2p$ -Hamilton-biconnected, unless $G = M_{n,n}^{n-k,k-p}$ for $k \geq p + 1$.
- (ii) If $m = n - 1$, then G is $2p$ -Hamilton-biconnected, unless one of the following holds:

- (a) $G = M_{n,n-1}^{n-k,k-p}$ for $k \geq p + 1$;
- (b) $G = M_{n,n-1}^{n-k-1,k-p}$ for $k \geq p + 1$;
- (c) $G = M_{n,n-1}^{n-k,k-p-1;-}$ for $k \geq p + 2$;
- (d) $G = M_{n,n-1}^{n-k,k-p-1}$ for $k \geq p + 2$.

Proof. Suppose that G is not $2p$ -Hamilton-biconnected. Let t be the largest integer such that $K_{n,t} \subseteq G$, and $Y_1 \subseteq Y$ such that $G[X, Y_1] = K_{n,t}$. Obviously, $t \geq m - k + p$. We claim that $m - k + p \leq t \leq m - k + p + 1$. Note that G is an $(m + p + 2)$ -closed bipartite graph and $\delta(G) \geq k$. If $t > m - k + p + 1$, then every vertex in Y is adjacent to every vertex in X , and thus $G = K_{n,m}$, a contradiction. Next we consider the following two cases.

Case 1. $t = m - k + p$. Then $|Y_1| = m - k + p$ and $|Y \setminus Y_1| = k - p \geq 1$. We show that $k \leq d_G(y) \leq k + 1$ for every $y \in Y \setminus Y_1$. Indeed, if there exists a vertex $y \in Y \setminus Y_1$ such that $d_G(y) \geq k + 2$, then y is adjacent to every vertex in X and thus $t \geq m - k + p + 1$, a contradiction. Next we consider the following two subcases.

Case 1.1. For every $y \in Y \setminus Y_1, d_G(y) = k$. Set $X = \bigcup_{i=1}^3 X_i$, where $X_1 = \{x \in X : d_G(x) = m - k + p\}$, $X_2 = \{x \in X : d_G(x) = m - k + p + 1\}$, and $X_3 = \{x \in X : d_G(x) \geq m - k + p + 2\}$. Set $Y_2 = Y \setminus Y_1$ and $l = |X_3|$. Since G is an $(m + p + 2)$ -closed bipartite graph with $\delta(G) \geq k$, every vertex in Y_2 is adjacent to every vertex in X_3 . This implies that $0 \leq l \leq k$. Furthermore, every vertex in Y_2 is adjacent to $k - l$ vertices in X_2 and any two distinct vertices in Y_2 have no common neighbors in X_2 . This implies that $|X_2| = (k - p)(k - l)$. Moreover, if $k \geq p + 2$ and $0 \leq l \leq k - 1$, then $G = F_{n,m}^{k,p,l}$. By Lemma 2.6, $F_{n,m}^{k,p,l}$ is $2p$ -Hamilton-biconnected, a contradiction. If $l = k \geq p + 2$ or $k = p + 1$, then $G = M_{n,m}^{n-k,k-p}$. It follows from Lemma 2.8 (i) and (ii) that $M_{n,m}^{n-k,k-p}$ is not $2p$ -Hamilton-biconnected, as desired.

Case 1.2. There exists a vertex $y \in Y \setminus Y_1$ such that $d_G(y) = k + 1$. Set $X = \bigcup_{i=1}^2 X_i$ and $Y = \bigcup_{i=1}^3 Y_i$, where $X_1 = \{x \in X : d_G(x) = m - k + p\}$, $X_2 = \{x \in X : d_G(x) \geq m - k + p + 1\}$, $Y_2 = \{y \in Y : d_G(y) = k\}$, and $Y_3 = \{y \in Y : d_G(y) = k + 1\}$. Since G is an $(m + p + 2)$ -closed bipartite graph with $\delta(G) \geq k$, every vertex in Y_3 is adjacent to every vertex in X_2 . This implies that $|X_2| = k + 1$ and thus $|X_1| = n - k - 1$.

We first assume that $Y_2 = \emptyset$. It is easy to see that $G = M_{n,m}^{n-k-1,k-p}$ for $k \geq p + 1$. Suppose that $m = n$. Since $M_{n,n}^{n-k-1,k-p;-}$ is a spanning subgraph of $M_{n,n}^{n-k-1,k-p}$, it follows from Lemma 2.7 (i) that $M_{n,n}^{n-k-1,k-p}$ is

$2p$ -Hamilton-biconnected, a contradiction. Next suppose that $m = n - 1$. By Lemma 2.8 (i), $M_{n,n-1}^{n-k-1,k-p}$ is not $2p$ -Hamilton-biconnected, as desired.

We next assume that $Y_2 \neq \emptyset$. We show that $|Y_3| = 1$. Indeed, if $|Y_3| \geq 2$, then $d_G(x) \geq m - k + p + 2$ for every $x \in X_2$ and hence every vertex in Y_2 is adjacent to every vertex in X_2 . This implies that $d_G(y) = k + 1$ for every $y \in Y_2$, a contradiction. By a similar argument to the proof of $|Y_3| = 1$, there exists a vertex $x \in X_2$ adjacent to none of vertices in Y_2 . Moreover, since $\delta(G) \geq k$, every vertex in Y_2 is adjacent to every vertex in $X_2 \setminus \{x\}$. Hence $G = M_{n,m}^{n-k,k-p-1;-}$ for $k \geq p + 2$. Suppose that $m = n$. By Lemma 2.7 (i), $M_{n,n}^{n-k,k-p-1;-}$ is $2p$ -Hamilton-biconnected, a contradiction. Next suppose that $m = n - 1$. Since $M_{n,n-1}^{n-k,k-p-1;-}$ is a spanning subgraph of $M_{n,n-1}^{n-k,k-p-1}$, it follows from Lemma 2.8 (i) that $M_{n,n-1}^{n-k,k-p-1;-}$ is not $2p$ -Hamilton-biconnected, as desired.

Case 2. $t = m - k + p + 1$. Then $|Y_1| = m - k + p + 1$ and $k \geq p + 2$. Set $X = \bigcup_{i=1}^2 X_i$, where $X_1 = \{x \in X : d_G(x) = m - k + p + 1\}$, $X_2 = \{x \in X : d_G(x) \geq m - k + p + 2\}$. Set $Y_2 = Y \setminus Y_1$. Obviously, $|Y_2| = k - p - 1$. Since G is an $(m + p + 2)$ -closed bipartite graph with $\delta(G) \geq k$, every vertex in Y_2 is adjacent to every vertex in X_2 . We claim that $Y_2 = \{y \in Y : d_G(y) = k\}$. Otherwise, there exists a vertex in Y_2 adjacent to every vertex in X , and thus $t \geq m - k + p + 2$, a contradiction. It follows that $|X_2| = k$ and $|X_1| = n - k$. Hence $G = M_{n,m}^{n-k,k-p-1}$ for $k \geq p + 2$. Suppose that $m = n$. Since $M_{n,n}^{n-k,k-p-1;-}$ is a spanning subgraph of $M_{n,n}^{n-k,k-p-1}$, it follows from Lemma 2.7 (i) that $M_{n,n}^{n-k,k-p-1}$ is $2p$ -Hamilton-biconnected, a contradiction. Next suppose that $m = n - 1$. By Lemma 2.8 (i), $M_{n,n-1}^{n-k,k-p-1}$ is not $2p$ -Hamilton-biconnected, as desired. \square

Now we are ready to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Suppose that $k \geq p$, $e(G) > n(n - k + p - 1) + (k + 2)(k - p + 1)$, and G is not $2p$ Hamilton-biconnected. Let $H = cl_{n+p+2}(G)$. By Lemma 2.4, H is also not $2p$ -Hamilton-biconnected. Furthermore, $\delta(H) \geq \delta(G) \geq k$ and $e(H) \geq e(G) > n(n - k + p - 1) + (k + 2)(k - p + 1)$. By Lemma 3.1, $K_{n,n-k+p} \subseteq H$, or $H \in \{N_{n,n}^{p,1}, N_{n,n}^{p,2}\}$ for $k = p + 2$. It follows from Lemmas 2.7 (ii), 2.8 (iii), and 3.3 (i) that $H = M_{n,n}^{n-k,k-p}$ for $k \geq p + 1$, or $H = N_{n,n}^{p,1}$ for $k = p + 2$. Hence $G \subseteq M_{n,n}^{n-k,k-p}$ for $k \geq p + 1$, or $G \subseteq N_{n,n}^{p,1}$ for $k = p + 2$. \square

Let $p = 0$ in Theorem 1.3, we partially prove the following Moon and Moser’s Theorem [13].

Corollary 3.4. Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k$, where $1 \leq k \leq \frac{n-2}{3}$. If

$$e(G) > n(n - k) + k^2,$$

then G is Hamiltonian.

Proof. Note that $e(G) > n(n - k) + k^2 \geq n(n - k - 1) + (k + 2)(k + 1)$, $e(M_{n,n}^{n-k,k}) = n(n - k) + k^2$, and $e(N_{n,n}^{0,1}) = n^2 - 2n + 4$. It follows from Theorem 1.3 that G is Hamilton-biconnected. Hence G is Hamiltonian. \square

Proof of Theorem 1.4. Denote $G = (X, Y; E)$ with $|X| = n$ and $|Y| = n - 1$. Suppose that $k \geq p$, $e(G) > n(n - k + p - 2) + (k + 1)(k - p + 1)$, and G is not $2p$ -Hamilton-biconnected. Let $H = cl_{n+p+1}(G)$. By Lemma 2.5, H is also not $2p$ -Hamilton-biconnected. In addition, $\delta(H) \geq \delta(G) \geq k$ and $e(H) \geq e(G) > n(n - k + p - 2) + (k + 2)(k - p + 1)$. By Corollary 3.2, $K_{n,n-k+p-1} \subseteq H$ or $K_{n-1,n-k+p} \subseteq H$. Since H is not $2p$ -Hamilton-biconnected, we have $H \neq K_{n,n-1}$, which implies that $k \geq p + 1$. Next we consider the following two cases.

Case 1. $K_{n,n-k+p-1} \subseteq H$. Note that $G \subseteq H$, $M_{n,n-1}^{n-k,k-p} \subseteq M_{n,n-1}^{n-k-1,k-p}$, and $M_{n,n-1}^{n-k,k-p-1;-} \subseteq M_{n,n-1}^{n-k,k-p-1}$. Combining this with Lemma 3.3 (ii), G is $2p$ -Hamilton-biconnected unless $G \subseteq M_{n,n-1}^{n-k-1,k-p}$ for $k \geq p + 1$, or $G \subseteq M_{n,n-1}^{n-k,k-p-1}$ for $k \geq p + 2$.

Case 2. $K_{n-1,n-k+p} \subseteq H$ and $K_{n,n-k+p-1} \not\subseteq H$. Let s, t with $s \geq t$ be the largest integers such that $K_{s,t} \subseteq H$. It follows that $s = n - 1$ and $n - k + p \leq t \leq n - 1$. We consider the following two subcases.

Case 2.1. Let $X_1 \subseteq X$ with $|X_1| = t$ such that $H[X_1, Y] = K_{t, n-1}$. We show that $t = n - k + p$. Indeed, if $t > n - k + p$, then $d_H(y) \geq n - k + p + 1$ for every $y \in Y$. Since H is an $(n + p + 1)$ -closed bipartite graph with $\delta(H) \geq k$, every vertex in Y is adjacent to every vertex in X and thus $H = K_{n, n-1}$, a contradiction. Then $|X_1| = n - k + p$ and $|X \setminus X_1| = k - p$. Furthermore, since H is an $(n + p + 1)$ -closed bipartite graph with $\delta(H) \geq k$, $d_H(x) = k$ for every $x \in X \setminus X_1$. Let $Y_1 = \{y \in Y : d_H(y) = n - k + p\}$. Moreover, every vertex in $X \setminus X_1$ is adjacent to every vertex in $Y \setminus Y_1$. It follows that $|Y \setminus Y_1| = k$ and $|Y_1| = n - k - 1$. Hence $H = M_{n, n-1}^{k-p, n-k-1}$ for $k \geq p + 1$. On the other hand, by Lemma 2.8 (i), $M_{n, n-1}^{k-p, n-k-1}$ is not $2p$ -Hamilton-biconnected, as desired.

Case 2.2. Let $X_1 \subseteq X$ and $Y_1 \subseteq Y$ with $|X_1| = n - 1$ and $|Y_1| = t$ such that $H[X_1, Y_1] = K_{n-1, t}$. We first show that $k = p + 1$. Since H is an $(n + p + 1)$ -closed bipartite graph, if $k > p + 1$ then every vertex in X is adjacent to every vertex in Y_1 , which implies that $K_{n, n-k+p-1} \subseteq K_{n, n-k+p} \subseteq H$, a contradiction. Since $k = p + 1$, we have $t = n - 1$. Hence $K_{n-1, n-1} \subseteq H$, which can be described to Case 2.1. \square

4. Proofs of Theorems 1.5 and 1.6

In order to prove Theorems 1.5 and 1.6, we need the following lemma.

Lemma 4.1. (i) For two integers $k \geq 2$ and $n \geq 2k^2 + 3$, $\rho(N_{n, n}^{k-2, 1}) > \rho(M_{n, n}^{n-k, 2})$.
 (ii) For two integers $k \geq 2$ and $n \geq k + 1$, $q(M_{n, n}^{n-k, 2}) > q(N_{n, n}^{k-2, 1})$.

Proof. (i) Denote $M_{n, n}^{n-k, 2} = (X, Y; E)$ with $|X| = |Y| = n$. Let \mathbf{x} be the eigenvector corresponding to $\rho(M_{n, n}^{n-k, 2})$. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$, where X_1 and X_2 are the sets of vertices in X with degree $n - 2$ and n respectively, and Y_1 and Y_2 are the sets of vertices in Y with degree n and k respectively.

By symmetry, the coordinate of \mathbf{x} corresponding to any vertex in X_i , denoted by x_i , is a positive constant for $1 \leq i \leq 2$. Similarly, the coordinate of \mathbf{x} corresponding to any vertex in Y_i , denoted by y_i , is also a positive constant for $1 \leq i \leq 2$. By eigenequation $A(M_{n, n}^{n-k, 2})\mathbf{x} = \rho(M_{n, n}^{n-k, 2})\mathbf{x}$, we have

$$\begin{aligned} \rho x_1 &= (n - 2)y_1, \\ \rho x_2 &= (n - 2)y_1 + 2y_2, \\ \rho y_1 &= (n - k)x_1 + kx_2, \\ \rho y_2 &= kx_2. \end{aligned}$$

By a simple calculation, $\rho(M_{n, n}^{n-k, 2})$ is the largest root of $f(x) = 0$, where

$$f(x) = x^4 - (n^2 - 2n + 2k)x^2 + 2k(n - k)(n - 2).$$

Since

$$f(n - 1) = n(n - 2k^2 - 2) + 4k^2 - 2k + 1 > 0$$

and for $x \geq n - 1$,

$$f'(x) = 2x(2x^2 - n^2 + 2n - 2k) \geq 2(n - 1)(n^2 - 2n - 2k + 2) > 0,$$

we have $\rho(M_{n, n}^{n-k, 2}) < n - 1$. On the other hand, since $K_{n-1, n-1}$ is a subgraph of $N_{n, n}^{k-2, 1}$, it follows from Lemma 2.1 that

$$\rho(N_{n, n}^{k-2, 1}) \geq \rho(K_{n-1, n-1}) = n - 1 > \rho(M_{n, n}^{n-k, 2}).$$

(ii) Let $f(x) = x(x - n)f_1(x)$ and $g(x) = x(x - n)^2(x - k + 1)g_1(x)$, where

$$f_1(x) = x^2 - (2n + k - 2)x + 2kn - 4k,$$

$$g_1(x) = x^2 - (2n + k - 1)x + 2kn + 2n - 4k.$$

By a similar argument to the proof of (i), $q(M_{n,n}^{n-k,2})$ and $q(N_{n,n}^{k-2,1})$ are the largest roots of $f(x) = 0$ and $g(x) = 0$, respectively. Furthermore, since $K_{n,n-2}$ and $K_{n-1,n-1}$ are proper subgraphs of $M_{n,n}^{n-k,2}$ and $N_{n,n}^{k-2,1}$, respectively, it follows from Lemma 2.1 that

$$q(M_{n,n}^{n-k,2}) > 2n - 2, \quad q(N_{n,n}^{k-2,1}) > 2n - 2.$$

Hence $q(M_{n,n}^{n-k,2})$ and $q(N_{n,n}^{k-2,1})$ are the largest roots of $f_1(x) = 0$ and $g_1(x) = 0$, respectively. On the other hand, since both $M_{n,n}^{n-k,2}$ and $N_{n,n}^{k-2,1}$ are proper subgraphs of $K_{n,n}$, it follows from Lemma 2.1 that

$$q(M_{n,n}^{n-k,2}) < 2n, \quad q(N_{n,n}^{k-2,1}) < 2n.$$

Since for $x < 2n$

$$g_1(x) - f_1(x) = 2n - x > 0,$$

we have $q(M_{n,n}^{n-k,2}) > q(N_{n,n}^{k-2,1})$. □

Proof of Theorem 1.5. (i) Suppose that $\rho(G) \geq \rho(N_{n,n}^{k-2,1})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n-1,n-1}$ is a proper subgraph of $N_{n,n}^{k-2,1}$, Lemma 2.1 implies that

$$\rho(G) \geq \rho(N_{n,n}^{k-2,1}) > \rho(K_{n-1,n-1}) = n - 1.$$

By Lemma 2.2, $\sqrt{e(G)} \geq \rho(G) > n - 1$, which implies that

$$e(G) > (n - 1)^2 \geq n(n - 3) + 3(k + 2).$$

It follows from Theorem 1.3 that $G \subseteq M_{n,n}^{n-k,2}$ or $G \subseteq N_{n,n}^{k-2,1}$. If $G \subseteq M_{n,n}^{n-k,2}$, then Lemmas 2.1 and 4.1 (i) imply that $\rho(G) \leq \rho(M_{n,n}^{n-k,2}) < \rho(N_{n,n}^{k-2,1})$, a contradiction. If G is a proper subgraph of $N_{n,n}^{k-2,1}$, then Lemma 2.1 implies that $\rho(G) < \rho(N_{n,n}^{k-2,1})$, a contradiction. Hence $G = N_{n,n}^{k-2,1}$.

(ii) Suppose that $\rho(G) \geq \rho(M_{n,n}^{n-k,k-p})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n,n-k+p}$ is a proper subgraph of $M_{n,n}^{n-k,k-p}$, Lemma 2.1 implies that

$$\rho(G) \geq \rho(M_{n,n}^{n-k,k-p}) > \rho(K_{n,n-k+p}) = \sqrt{n(n - k + p)}.$$

By Lemma 2.2, $\sqrt{e(G)} \geq \rho(G) > \sqrt{n(n - k + p)}$, which implies that

$$e(G) > n(n - k + p) \geq n(n - k + p - 1) + (k + 2)(k - p + 1).$$

It follows from Theorem 1.3 that $G \subseteq M_{n,n}^{n-k,k-p}$. If G is a proper subgraph of $M_{n,n}^{n-k,k-p}$, then Lemma 2.1 implies that $\rho(G) < \rho(M_{n,n}^{n-k,k-p})$, a contradiction. Hence $G = M_{n,n}^{n-k,k-p}$. □

Corollary 4.2. Let k, p be two noneagative integers and G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k \geq p + 1$, where $n \geq n_0(k, p)$ and

$$n_0(k, p) = \begin{cases} 2k^2 + 3, & \text{if } k = p + 2 \\ (k + 2)(k - p + 1), & \text{otherwise.} \end{cases}$$

If $\rho(G) \geq \sqrt{n(n - k + p) + k(k - p)}$, then G is $2p$ -Hamilton-biconnected.

Proof. Suppose that $k = p + 2$. Note that $e(N_{n,n}^{k-2,1}) = n^2 - 2n + 2k$. By Lemma 2.2 and Theorem 1.5 (i), the result follows. Next suppose that $k \neq p + 2$. Note that $e(M_{n,n}^{n-k,k-p}) = n(n - k + p) + k(k - p)$. By Lemma 2.2 and Theorem 1.5 (ii), the result follows. □

Proof of Theorem 1.6. Suppose that $q(G) \geq q(M_{n,n}^{n-k,k-p})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n,n-k+p}$ is a proper subgraph of $M_{n,n}^{n-k,k-p}$, Lemma 2.1 implies that

$$q(G) \geq q(M_{n,n}^{n-k,k-p}) > q(K_{n,n-k+p}) = 2n - k + p.$$

By Lemma 2.3, $\frac{e(G)}{n} + n \geq q(G) > 2n - k + p$, which implies that

$$e(G) > n(n - k + p) \geq n(n - k + p - 1) + (k + 2)(k - p + 1).$$

It follows from Theorem 1.3 that $G \subseteq M_{n,n}^{n-k,k-p}$ for $k \geq p + 1$, or $G \subseteq N_{n,n}^{p,1}$ for $k = p + 2$. If G is a proper subgraph of $M_{n,n}^{n-k,k-p}$, then Lemma 2.1 implies that $q(G) < q(M_{n,n}^{n-k,k-p})$, a contradiction. If G is a subgraph of $N_{n,n}^{p,1}$ for $k = p + 2$, then Lemmas 2.1 and 4.1 (ii) imply that $q(G) \leq q(N_{n,n}^{p,1}) < q(M_{n,n}^{n-k,k-p})$, a contradiction. Hence $G = M_{n,n}^{n-k,k-p}$. \square

Corollary 4.3. Let k, p be two noneagative integers and G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq k \geq p + 1$, where $n \geq (k + 2)(k - p + 1)$. If $q(G) \geq 2n - k + p + \frac{k(k-p)}{n}$, then G is $2p$ -Hamilton-biconnected.

Proof. Note that $n + \frac{e(M_{n,n}^{n-k,k-p})}{n} = 2n - k + p + \frac{k(k-p)}{n}$. By Lemma 2.3 and Theorem 1.6, the result follows. \square

5. Proofs of Theorems 1.7 and 1.8

The proofs of Lemmas 5.1 and 5.2 are similar to that of Lemma 4.1, so we put them in the appendix.

Lemma 5.1. (i) For integers $p \geq 0, k \geq p + 1$, and $n \geq 2k - p + 2$,

$$\rho(M_{n,n-1}^{k-p,n-k-1}) > \rho(M_{n,n-1}^{n-k-1,k-p}).$$

(ii) For integers $p \geq 0, k \geq p + 2$, and $n \geq 2k - p + 2$,

$$\rho(M_{n,n-1}^{n-k,k-p-1}) > \rho(M_{n,n-1}^{k-p,n-k-1}).$$

Lemma 5.2. (i) For integers $p \geq 0, k \geq p + 1$, and $n \geq 2k - p + 2$,

$$q(M_{n,n-1}^{n-k-1,k-p}) > q(M_{n,n-1}^{k-p,n-k-1}).$$

(ii) For integers $p \geq 0, k \geq p + 2$, and $n \geq 2k - p + 2$,

$$q(M_{n,n-1}^{n-k,k-p-1}) > q(M_{n,n-1}^{n-k-1,k-p}).$$

Proof of Theorem 1.7. (i) Suppose that $\rho(G) \geq \rho(M_{n,n-1}^{1,n-k-1})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n-1,n-1}$ is a proper subgraph of $M_{n,n-1}^{1,n-k-1}$, it follows from Lemma 2.1 that

$$\rho(G) \geq \rho(M_{n,n-1}^{1,n-k-1}) > \rho(K_{n-1,n-1}) = n - 1.$$

By Lemma 2.2 $\sqrt{e(G)} \geq \rho(G) > n - 1$, which implies that

$$e(G) > n^2 - 2n + 1 \geq n(n - 3) + 2k + 4.$$

Then it follows from Theorem 1.4 that $G \subseteq M_{n,n-1}^{n-k-1,1}$ or $G \subseteq M_{n,n-1}^{1,n-k-1}$. By Lemmas 2.1 and 5.1 (i), $G = M_{n,n-1}^{1,n-k-1}$.

(ii) Suppose that $\rho(G) \geq \rho(M_{n,n-1}^{n-k,k-p-1})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n,n-k+p}$ is a proper subgraph of $M_{n,n-1}^{n-k,k-p-1}$, it follows from Lemma 2.1 that

$$\rho(G) \geq \rho(M_{n,n-1}^{n-k,k-p-1}) > \rho(K_{n,n-k+p}) = \sqrt{n(n-k+p)}.$$

By Lemma 2.2, $\sqrt{e(G)} \geq \rho(G) > \sqrt{n(n-k+p)}$, which implies that

$$e(G) > n(n-k+p) \geq n(n-k+p-2) + (k+2)(k-p+1).$$

It follows from Theorem 1.4 that $G \subseteq M_{n,n-1}^{n-k-1,k-p}$, $G \subseteq M_{n,n-1}^{k-p,n-k-1}$, or $G \subseteq M_{n,n-1}^{n-k,k-p-1}$. By Lemmas 2.1 and 5.1, $G = M_{n,n-1}^{n-k,k-p-1}$. □

Corollary 5.3. *Let k, p be two nonnegative integers and G be a nearly balanced bipartite graph of order $2n - 1$ with $\delta(G) \geq k$.*

(i) *If $k = p + 1$, $n \geq 2k + 3$, and $\rho(G) \geq \sqrt{(n-1)^2 + k}$, then G is $2p$ -Hamilton-biconnected.*

(ii) *If $k \geq p + 2$, $n \geq \frac{(k+2)(k-p+1)}{2}$, and $\rho(G) \geq \sqrt{n(n-k+p) + k(k-p-1)}$, then G is $2p$ -Hamilton-biconnected.*

Proof. (i) Note that $e(M_{n,n-1}^{1,n-k-1}) = (n-1)^2 + k$. By Lemma 2.2 and Theorem 1.7 (i), the result follows.

(ii) Note that $e(M_{n,n-1}^{n-k,k-p-1}) = n(n-k+p) + k(k-p-1)$. By Lemma 2.2 and Theorem 1.7 (ii), the result follows. □

Proof of Theorem 1.8. (i) Suppose that $q(G) \geq q(M_{n,n-1}^{n-k-1,1})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n,n-2}$ is a proper subgraph of $M_{n,n-1}^{n-k-1,1}$, Lemma 2.1 implies that

$$q(G) \geq q(M_{n,n-1}^{n-k-1,1}) > q(K_{n,n-2}) = 2n - 2.$$

By Lemma 2.3, $\frac{e(G)}{n} + n \geq q(G) > 2n - 2$. Note that here we consider G as a balanced bipartite graph with an isolated vertex. This implies that

$$e(G) > n^2 - 2n \geq n(n-3) + 2k + 4.$$

It follows from Theorem 1.4 that $G \subseteq M_{n,n-1}^{n-k-1,1}$ or $G \subseteq M_{n,n-1}^{1,n-k-1}$. By Lemmas 2.1 and 5.2 (i), $G = M_{n,n-1}^{n-k-1,1}$.

(ii) Suppose that $q(G) \geq q(M_{n,n-1}^{n-k,k-p-1})$ and G is not $2p$ -Hamilton-biconnected. Since $K_{n,n-k+p}$ is a proper subgraph of $M_{n,n-1}^{n-k,k-p-1}$, Lemma 2.1 implies that

$$q(G) \geq q(M_{n,n-1}^{n-k,k-p-1}) > q(K_{n,n-k+p}) = 2n - k + p.$$

By Lemma 2.3, $\frac{e(G)}{n} + n \geq q(G) > 2n - k + p$. Note that here we consider G as a balanced bipartite graph with an isolated vertex. This implies that

$$e(G) > n(n-k+p) \geq n(n-k+p-2) + (k+2)(k-p+1).$$

It follows from Theorem 1.4 that $G \subseteq M_{n,n-1}^{n-k-1,k-p}$, $G \subseteq M_{n,n-1}^{n-k,k-p-1}$, or $G \subseteq M_{n,n-1}^{k-p,n-k-1}$. By Lemmas 2.1 and 5.2, $G = M_{n,n-1}^{n-k,k-p-1}$. □

Corollary 5.4. *Let k, p be two noneagative integers and G be a nearly balanced bipartite graph of order $2n - 1$ with $\delta(G) \geq k$.*

(i) *If $k = p + 1$, $n \geq 2k + 4$, and $q(G) \geq \sqrt{2n - 2 + \frac{k+1}{n}}$, then G is $2p$ -Hamilton-biconnected.*

(ii) *If $k \geq p + 2$, $n \geq \frac{(k+2)(k-p+1)}{2}$, and $q(G) \geq \sqrt{2n - k + p + \frac{k(k-p-1)}{n}}$, then G is $2p$ -Hamilton-biconnected.*

Proof. (i) Note that $n + \frac{e(M_{n,n-1}^{n-k-1,1})}{n} = 2n - 2 + \frac{k+1}{n}$. By Lemma 2.3 and Theorem 1.8 (i), the result follows.

(ii) Note that $n + \frac{e(M_{n,n-1}^{n-k,k-p-1})}{n} = 2n - k + p + \frac{k(k-p-1)}{n}$. By Lemma 2.3 and Theorem 1.8 (ii), the result follows.

□

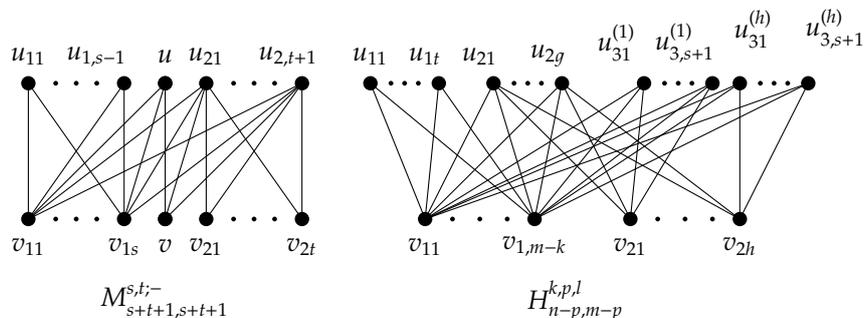
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Appendix

Denote by P_{uv} a path between u and v . Denote by $P_{uv} \sqcup P_{wz}$ a path obtained from two disjoint paths P_{uv} and P_{wz} by joining v and w .



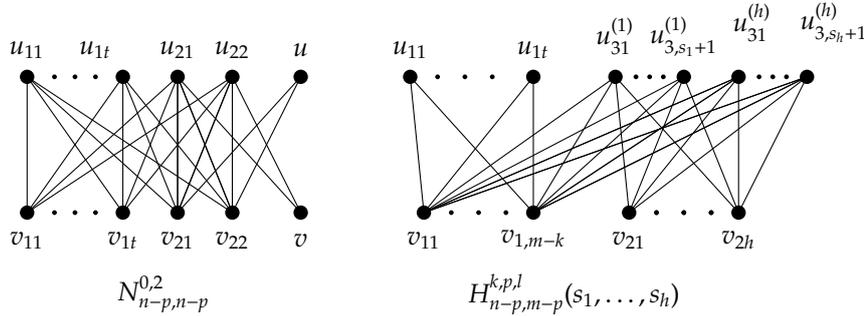


Fig. 2. Graphs $M_{s+t+1, s+t+1}^{s,t}$, $M_{s+t+1, s+t+1}^{s,t;-}$, $N_{n-p, n-p}^{0,2}$, and $H_{n-p, m-p}^{k,p,l}(s_1, \dots, s_h)$.

Proof of Lemma 2.6. We first assume that $l > p$. Let $g = l - p$, $h = k - p$, $s = k - l - 1$, and $t = n - (k - p)(k - l) - l$. Let $H_{n-p, m-p}^{k,p,l}$ be a bipartite graph obtained from $F_{n,m}^{k,p,l}$ by deleting all vertices in a balanced set of size $2p$ which consists of vertices with as large as possible degree (see Fig. 2). Note that every bipartite graph of order $m + n - 2p$ obtained from $F_{n,m}^{k,p,l}$ by deleting all vertices in a balanced set of size $2p$ contains $H_{n-p, m-p}^{k,p,l}$ as a spanning subgraph. It suffices to prove that $H_{n-p, m-p}^{k,p,l}$ is Hamiltonian-biconnected. Label the vertices of $H_{n-p, m-p}^{k,p,l}$ as $u_{11}, \dots, u_{1t}, u_{21}, \dots, u_{2g}, u_{31}^{(1)}, \dots, u_{3, s_1+1}^{(1)}, \dots, u_{31}^{(h)}, \dots, u_{3, s_h+1}^{(h)}, v_{11}, \dots, v_{1, m-k}, v_{21}, \dots, v_{2h}$ (see Fig. 2).

Let $m = n$. We assume that $l \leq k - 2$. Clearly, $g \geq 1$, $h \geq 2$, and $s \geq 1$. Denote

$$\begin{aligned}
 P_{u_{11}v_{1t}} &= \bigsqcup_{i=1}^t u_{1i}v_{1i}, & P_{u_{21}v_{1,t+g}} &= \bigsqcup_{i=1}^g u_{2i}v_{1,t+i}, \\
 P_{u_{31}^{(h)}v_{2h}} &= \bigsqcup_{i=1}^s u_{3i}^{(h)}v_{1,t+g+(h-1)s+i} \bigsqcup u_{3, s_1+1}^{(h)}v_{2h}, \\
 Q_{u_{31}^{(h)}v_{2h}} &= \bigsqcup_{i=1}^{s-1} u_{3i}^{(h)}v_{1,t+g+(h-1)s+i} \bigsqcup u_{3s}^{(h)}v_{2h}, \\
 P_{u_{3, s_1+1}^{(h)}v_{1,t+g+i}} &= u_{3, s_1+1}^{(h)}v_{2i} \bigsqcup \left(\bigsqcup_{j=1}^s u_{3j}^{(h)}v_{1,t+g+(i-1)s+j} \right) \text{ for } 1 \leq i \leq h.
 \end{aligned}$$

$H_{n-p, n-p}^{k,p,l}$ has seven kinds of Hamiltonian paths, denoted by R_1, \dots, R_7 . We present them as follows:

$$\begin{aligned}
 R_1 &= P_{u_{11}v_{1t}} \bigsqcup P_{u_{21}v_{1,t+g}} \bigsqcup \left(\bigsqcup_{i=1}^h P_{u_{3, s_1+1}^{(i)}v_{1,t+g+i}} \right), \\
 R_2 &= P_{u_{11}v_{1t}} \bigsqcup P_{u_{21}v_{1,t+g}} \bigsqcup \left(\bigsqcup_{i=1}^{h-1} P_{u_{3, s_1+1}^{(i)}v_{1,t+g+i}} \right) \bigsqcup P_{u_{31}^{(h)}v_{2h}}, \\
 R_3 &= P_{u_{21}v_{1,t+g}} \bigsqcup P_{u_{11}v_{1t}} \bigsqcup \left(\bigsqcup_{i=1}^h P_{u_{3, s_1+1}^{(i)}v_{1,t+g+i}} \right),
 \end{aligned}$$

$$\begin{aligned}
 R_4 &= P_{u_{21}v_{1,t+g}} \sqcup P_{u_{11}v_{1t}} \sqcup \left(\bigsqcup_{i=1}^{h-1} P_{u_{3,s+1}^{(i)}v_{1,t+g+is}} \right) \sqcup P_{u_{31}^{(h)}v_{2h}}, \\
 R_5 &= \bigsqcup_{i=1}^h P_{u_{3,s+1}^{(i)}v_{1,t+g+is}} \sqcup P_{u_{11}v_{1t}} \sqcup P_{u_{21}v_{1,t+g}}, \\
 R_6 &= \bigsqcup_{i=1}^{h-1} P_{u_{3,s+1}^{(i)}v_{1,t+g+is}} \sqcup P_{u_{11}v_{1t}} \sqcup P_{u_{21}v_{1,t+g}} \sqcup P_{u_{31}^{(h)}v_{2h}}, \\
 R_7 &= u_{3,s+1}^{(h)}v_{1,n-k} \sqcup \left(\bigsqcup_{i=1}^{h-1} P_{u_{3,s+1}^{(i)}v_{1,t+g+is}} \right) \sqcup P_{u_{11}v_{1t}} \sqcup P_{u_{21}v_{1,t+g}} \sqcup Q_{u_{31}^{(h)}v_{2h}}.
 \end{aligned}$$

Hence $H_{n-p,n-p}^{k,p,l}$ is Hamilton-biconnected. Thus $F_{n,n}^{k,p,l}$ is $2p$ -Hamilton-biconnected for $p < l \leq k - 2$. Similarly we can prove that $F_{n,n}^{k,p,l}$ is also $2p$ -Hamilton-biconnected for $p < l = k - 1$.

Let $m = n - 1$. We assume that $l \leq k - 2$. $H_{n-p,n-p-1}^{k,p,l}$ has seven Hamiltonian paths, denoted by R_1^*, \dots, R_7^* , obtained from Hamiltonian paths R_1, R_3 , and R_5 in $H_{n-p,n-p}^{k,p,l}$ by some vertex and edge operations. We present them as follows:

$$\begin{aligned}
 R_1^* &= R_1 - v_{1t} - u_{1t}v_{1,t-1} + \{u_{21}v_{1,t-1}, u_{1t}v_{1,n-k}\}, \\
 R_2^* &= R_1 - v_{1,t+g} - u_{2g}v_{1,t+g-1} + \{u_{3,s+1}^{(1)}v_{1,t+g-1}, u_{2g}v_{1,n-k}\}, \\
 R_3^* &= R_1 - v_{1,n-k}, \\
 R_4^* &= R_3 - v_{1,t+g} - u_{2g}v_{1,t+g-1} + \{u_{11}v_{1,t+g-1}, u_{2g}v_{1,n-k}\}, \\
 R_5^* &= R_3 - v_{1,n-k}, \\
 R_6^* &= R_5 - v_{1,t+g+s} - u_{3s}^{(1)}v_{1,t+g+s-1} + \{u_{3,s+1}^{(2)}v_{1,t+g+s-1}, u_{3s}^{(1)}v_{1,t+g}\}, \\
 R_7^* &= R_5 - v_{1,t+g+hs} - u_{3s}^{(h)}v_{1,t+g+hs-1} + \{u_{11}v_{1,t+g+hs-1}, u_{3s}^{(h)}v_{1,t+g}\}.
 \end{aligned}$$

Hence $H_{n-p,n-p-1}^{k,p,l}$ is Hamilton-biconnected. Thus $F_{n,n-1}^{k,p,l}$ is $2p$ -Hamilton-biconnected for $p < l \leq k - 2$. Similarly we can prove that $F_{n,n}^{k,p,l}$ is also $2p$ -Hamilton-biconnected for $p < l = k - 1$.

We next assume that $l \leq p$. Let $r_i \geq 0$ with $\sum_{i=1}^h r_i = p - l$ for $1 \leq i \leq h$. Let $h = k - p, s_0 = 0, s_i = k - l - r_i - 1$ for $1 \leq i \leq h$, and $t = n - (k - p)(k - l) - l$. Let $H_{n-p,m-p}^{k,p,l}(s_1, \dots, s_h)$ be a bipartite graph obtained from $F_{n,m}^{k,p,l}$ by deleting all vertices in a balanced set of size $2p$ which consists of vertices with as large as possible degree (see Fig. 2). Let $\mathcal{G}_{n-p,m-p}^{k,p,l}$ be a set of all bipartite graphs G satisfying $G = H_{n-p,m-p}^{k,p,l}(s_1, \dots, s_h)$. Note that every graph of order $m + n - 2p$ obtained from $F_{n,m}^{k,p,l}$ by deleting all vertices in a balanced set of size $2p$ contains a bipartite graph $G \in \mathcal{G}_{n-p,m-p}^{k,p,l}$ as a spanning subgraph. It suffices to prove that any bipartite graph $G \in \mathcal{G}_{n-p,m-p}^{k,p,l}$ is Hamilton-biconnected. Let $G \in \mathcal{G}_{n-p,m-p}^{k,p,l}$ without loss of generality, say $G = H_{n-p,m-p}^{k,p,l}(s_1, \dots, s_h)$. Label the vertices of $H_{n-p,m-p}^{k,p,l}(s_1, \dots, s_h)$ as $u_{11}, \dots, u_{1t}, u_{31}^{(1)}, \dots, u_{3,s_1+1}^{(1)}, \dots, u_{31}^{(h)}, \dots, u_{3,s_h+1}^{(h)}, v_{11}, \dots, v_{1,m-k}, v_{21}, \dots, v_{2h}$ (see Fig. 2).

Let $m = n$. Since $k \geq p + 2$ and $0 \leq r_i \leq p - l$, we have $h \geq 2$ and $s_i \geq 1$ for $1 \leq i \leq h$. Denote

$$\begin{aligned}
 P_{u_{11}v_{1t}} &= \bigsqcup_{i=1}^t u_{1i}v_{1i}, \\
 P_{u_{31}^{(h)}v_{2h}} &= \bigsqcup_{i=1}^{s_h} u_{3i}^{(h)}v_{1,t+i+\sum_{j=1}^{h-1} s_j} \bigsqcup u_{3,s_h+1}^{(h)}v_{2h}, \\
 Q_{u_{31}^{(h)}v_{2h}} &= \bigsqcup_{i=1}^{s_h-1} u_{3i}^{(h)}v_{1,t+i+\sum_{j=1}^{h-1} s_j} \bigsqcup u_{3s_h}^{(h)}v_{2h}, \\
 P_{u_{3,s_i+1}^{(i)}v_{1,t+\sum_{j=1}^i s_j}} &= u_{3,s_i+1}^{(i)}v_{2i} \bigsqcup \left(\bigsqcup_{j=1}^{s_i} u_{3j}^{(i)}v_{1,t+j+\sum_{w=1}^{i-1} s_w} \right) \text{ for } 1 \leq i \leq h.
 \end{aligned}$$

$H_{n-p,n-p}^{k,p,l}(s_1, \dots, s_h)$ has five kinds of Hamiltonian paths, denoted by R_1, \dots, R_5 . We present them as follows:

$$\begin{aligned}
 R_1 &= P_{u_{11}v_{1t}} \bigsqcup \left(\bigsqcup_{i=1}^h P_{u_{3,s_i+1}^{(i)}v_{1,t+\sum_{j=1}^i s_j}} \right), \\
 R_2 &= P_{u_{11}v_{1t}} \bigsqcup \left(\bigsqcup_{i=1}^{h-1} P_{u_{3,s_i+1}^{(i)}v_{1,t+\sum_{j=1}^i s_j}} \right) \bigsqcup P_{u_{31}^{(h)}v_{2h}}, \\
 R_3 &= \bigsqcup_{i=1}^h P_{u_{3,s_i+1}^{(i)}v_{1,t+\sum_{j=1}^i s_j}} \bigsqcup P_{u_{11}v_{1t}}, \\
 R_4 &= \bigsqcup_{i=1}^{h-1} P_{u_{3,s_i+1}^{(i)}v_{1,t+\sum_{j=1}^i s_j}} \bigsqcup P_{u_{11}v_{1t}} \bigsqcup P_{u_{31}^{(h)}v_{2h}}, \\
 R_5 &= u_{3,s_h+1}^{(h)}v_{1,n-k} \bigsqcup \left(\bigsqcup_{i=1}^{h-1} P_{u_{3,s_i+1}^{(i)}v_{1,t+\sum_{j=1}^i s_j}} \right) \bigsqcup P_{u_{11}v_{1t}} \bigsqcup Q_{u_{31}^{(h)}v_{2h}}.
 \end{aligned}$$

Hence $H_{n-p,n-p}^{k,p,l}(s_1, \dots, s_h)$ is Hamilton-biconnected. Thus $F_{n,n}^{k,p,l}$ is $2p$ -Hamilton-biconnected for $l \leq p$.

Let $m = n - 1$. $H_{n-p,n-p-1}^{k,p,l}(s_1, \dots, s_h)$ has four kinds of Hamiltonian paths, denoted by R_1^*, \dots, R_4^* , obtained from Hamiltonian paths R_1 and R_3 in $H_{n-p,n-p}^{k,p,l}(s_1, \dots, s_h)$ by some vertex and edge operations. We present them as follows:

$$\begin{aligned}
 R_1^* &= R_1 - v_{1t} - u_{1t}v_{1,t-1} + \{u_{3,s_1+1}^{(1)}v_{1,t-1}, u_{1t}v_{1,n-k}\}, \\
 R_2^* &= R_1 - v_{1,n-k}, \\
 R_3^* &= R_3 - v_{1,t+s_1} - u_{3s_1}^{(1)}v_{1,t+s_1-1} + \{u_{3,s_2+1}^{(2)}v_{1,t+s_1-1}, u_{3s_1}^{(1)}v_{1t}\}, \\
 R_4^* &= R_3 - v_{1,t+\sum_{i=1}^h s_i} - u_{3s_h}^{(h)}v_{1,t-1+\sum_{i=1}^h s_i} + \{u_{11}v_{1,t-1+\sum_{i=1}^h s_i}, u_{3s_h}^{(h)}v_{1t}\}.
 \end{aligned}$$

Hence $H_{n-p,n-p-1}^{k,p,l}(s_1, \dots, s_h)$ is Hamilton-biconnected. Thus $F_{n,n-1}^{k,p,l}$ is also $2p$ -Hamilton-biconnected for $l \leq p$. This completes the proof. \square

Proof of Lemma 2.7. (i) Note that every balanced bipartite graph of order $2n - 2p$ obtained from $M_{n,n}^{s,t;-}$ by deleting all vertices in a balanced set of size $2p$ contains $M_{s+t+1,s+t+1}^{s,t;-}$ as a spanning subgraph. It suffices to prove that $M_{s+t+1,s+t+1}^{s,t;-}$ is Hamilton-biconnected. Label the vertices of $M_{s+t+1,s+t+1}^{s,t;-}$ as $u_{11}, \dots, u_{1,s-1}, u, u_{21}, \dots, u_{2,t+1}, v_{11}, \dots, v_{1s}, v_{21}, \dots, v_{2t}$ (see Fig. 2). Denote

$$P_{u_{11}v_{1,s-1}} = \bigsqcup_{i=1}^{s-1} u_{1i}v_{1i}, \quad P_{u_{21}v_{2t}} = \bigsqcup_{i=1}^t u_{2i}v_{2i}.$$

$M_{s+t+1, s+t+1}^{s, t; -}$ has nine kinds of Hamiltonian paths, denoted by R_1, \dots, R_9 . We present them as follows:

$$\begin{aligned}
 R_1 &= P_{u_{11}v_{1, s-1}} \sqcup uv \sqcup P_{u_{21}v_{2t}} \sqcup u_{2, t+1}v_{1s}, \\
 R_2 &= P_{u_{11}v_{1, s-1}} \sqcup P_{u_{21}v_{2t}} \sqcup u_{2, t+1}v_{1s}uv, \\
 R_3 &= P_{u_{11}v_{1, s-1}} \sqcup uvu_{2, t+1}v_{1s} \sqcup P_{u_{21}v_{2t}}, \\
 R_4 &= uv \sqcup P_{u_{21}v_{2t}} \sqcup u_{2, t+1}v_{1s} \sqcup P_{u_{11}v_{1, s-1}}, \\
 R_5 &= uv_{1s} \sqcup P_{u_{11}v_{1, s-1}} \sqcup P_{u_{21}v_{2t}} \sqcup u_{2, t+1}v, \\
 R_6 &= uvu_{2, t+1}v_{1s} \sqcup P_{u_{11}v_{1, s-1}} \sqcup P_{u_{21}v_{2t}}, \\
 R_7 &= P_{u_{21}v_{2t}} \sqcup u_{2, t+1}vuv_{1s} \sqcup P_{u_{11}v_{1, s-1}}, \\
 R_8 &= P_{u_{21}v_{2t}} \sqcup u_{2, t+1}v_{1s} \sqcup P_{u_{11}v_{1, s-1}} \sqcup uv, \\
 R_9 &= u_{2, t+1}v_{1s} \sqcup P_{u_{11}v_{1, s-1}} \sqcup uv \sqcup P_{u_{21}v_{2t}}.
 \end{aligned}$$

Hence $M_{n-p, n-p}^{s, t; -}$ is Hamilton-biconnected. Thus $M_{n, n}^{s, t; -}$ is $2p$ -Hamilton-biconnected.

(ii) Note that every balanced bipartite graph of order $2n - 2p$ obtained from $N_{n, n}^{p, 2}$ by deleting all vertices in a balanced set of order $2p$ contains $N_{n-p, n-p}^{0, 2}$ as a spanning subgraph. It suffices to prove that $N_{n-p, n-p}^{0, 2}$ is Hamilton-biconnected. Let $t = n - p - 3$ and label the vertices of $N_{n-p, n-p}^{0, 2}$ as $u_{11}, \dots, u_{1t}, u_{21}, u_{22}, u, v_{11}, \dots, v_{1t}, v_{21}, v_{22}, v$ (see Fig. 2). Denote

$$P_{u_{11}v_{1, t-1}} = \bigsqcup_{i=1}^{t-1} u_{1i}v_{1i}, \quad P_{u_{11}v_{1t}} = \bigsqcup_{i=1}^t u_{1i}v_{1i}.$$

$N_{n-p, n-p}^{0, 2}$ has nine kinds of Hamiltonian paths. We present them as follows:

$$\begin{aligned}
 R_1 &= P_{u_{11}v_{1, t-1}} \sqcup u_{21}vu_{22}v_{22}uv_{21}u_{1t}v_{1t}, \\
 R_2 &= P_{u_{11}v_{1t}} \sqcup u_{21}vu_{22}v_{22}uv_{21}, \\
 R_3 &= P_{u_{11}v_{1t}} \sqcup u_{21}v_{22}uv_{21}u_{22}v, \\
 R_4 &= u_{21}vu_{22}v_{22}uv_{21} \sqcup P_{u_{11}v_{1t}}, \\
 R_5 &= u_{21}vu_{22}v_{1t} \sqcup P_{u_{11}v_{1, t-1}} \sqcup u_{1t}v_{22}uv_{21}, \\
 R_6 &= u_{21}v_{22}uv_{21} \sqcup P_{u_{11}v_{1t}} \sqcup u_{22}v, \\
 R_7 &= uv_{21}u_{22}vu_{21}v_{22} \sqcup P_{u_{11}v_{1t}}, \\
 R_8 &= uv_{21}u_{22}vu_{21}v_{1t} \sqcup P_{u_{11}v_{1, t-1}} \sqcup u_{1t}v_{22}, \\
 R_9 &= uv_{21} \sqcup P_{u_{11}v_{1t}} \sqcup u_{21}v_{22}u_{22}v.
 \end{aligned}$$

Hence $N_{n-p, n-p}^{0, 2}$ is Hamilton-biconnected. Thus $N_{n, n}^{p, 2}$ is $2p$ -Hamilton-biconnected. □

Proof of Lemma 2.8. (i) Denote $M_{n-p, n-p-1}^{s, t} = (X, Y; E)$ with $|X| = n - p$ and $|Y| = n - p - 1$. If $n \geq s + p + 2$, then let $x, y \in X$ such that $d(x) = d(y) = n - p - 1$. Since $s \geq n - t - p - 1$, $M_{n-p, n-p-1}^{s, t}$ has no Hamiltonian path between x and y . If $n = s + p + 1$, then let $x, y \in X$ such that $d(x) = n - p - t - 1$ and $d(y) = n - p - 1$. Clearly, all neighbours of y are pendant vertices. So $M_{n-p, n-p-1}^{s, t}$ has no Hamiltonian path between x and y . Hence in each case, $M_{n-p, n-p-1}^{s, t}$ is not Hamilton-biconnected. Note that $M_{n-p, n-p-1}^{s, t}$ is one of graphs obtained from

$M_{n,n-1}^{s,t}$ by deleting all vertices in a balanced set of size $2p$. It follows from definition that $M_{n-p,n-p-1}^{s,t}$ is not $2p$ -Hamilton-biconnected.

(ii) Denote $M_{n-p,n-p}^{s,t} = (X, Y; E)$ with $|X| = |Y| = n - p$. Let $x \in X$ and $y \in Y$ such that $d(x) = n - p - t$ and $d(y) = n - p$. Since $s = n - p - t$, $M_{n-p,n-p}^{s,t}$ has no Hamiltonian path between x and y . Hence $M_{n-p,n-p}^{s,t}$ is not Hamilton-biconnected. Note that $M_{n-p,n-p}^{s,t}$ is one of graphs obtained from $M_{n,n}^{s,t}$ by deleting all vertices in a balanced set of size $2p$. It follows from definition that $M_{n-p,n-p}^{s,t}$ is not $2p$ -Hamilton-biconnected.

(iii) Denote $N_{n-p,n-p}^{0,1} = (X, Y; E)$ with $|X| = |Y| = n - p$. Let $x \in X$ and $y \in Y$ such that $d(x) = d(y) = n - p$. Then $N_{n-p,n-p}^{0,1}$ has no Hamiltonian path between x and y . Hence $N_{n-p,n-p}^{0,1}$ is not Hamilton-biconnected. Note that $N_{n-p,n-p}^{0,1}$ is one of graphs obtained from $N_{n,n}^{p,1}$ by deleting all vertices in a balanced set of size $2p$. It follows from definition that $N_{n,n}^{p,1}$ is not $2p$ -Hamilton-biconnected. \square

Proof of Lemma 5.1. By a similar argument to Lemma 4.1 (i), $\rho(M_{n,n-1}^{n-k-1,k-p})$, $\rho(M_{n,n-1}^{k-p,n-k-1})$, and $\rho(M_{n,n-1}^{n-k,k-p-1})$ are the largest roots of $f(x) = 0$, $g(x) = 0$, and $h(x) = 0$ respectively, where

$$f(x) = x^4 - (n^2 - (k - p + 1)n + (k + 1)(k - p))x^2 + (n - k - 1)(n - k + p - 1)(k + 1)(k - p),$$

$$g(x) = x^4 - (n^2 - (k - p + 1)n + (k - p)(k + 1))x^2 + (n - k - 1)(n - k + p)k(k - p),$$

$$h(x) = x^4 - (n^2 - (k - p)n + k(k - p - 1))x^2 + (n - k)(n - k + p)k(k - p - 1).$$

(i) Since for all real number x ,

$$f(x) - g(x) = (n - k - 1)(n - k + p - 1)(k - p) > 0,$$

we have $\rho(M_{n,n-1}^{k-p,n-k-1}) > \rho(M_{n,n-1}^{n-k-1,k-p})$.

(ii) Since for all real number x ,

$$g(x) - h(x) = (n - 2k + p)(x^2 + kn - k^2 + kp) > 0,$$

we have $\rho(M_{n,n-1}^{n-k,k-p-1}) > \rho(M_{n,n-1}^{k-p,n-k-1})$. \square

Proof of Lemma 5.2. By a similar argument to Lemma 4.1 (i), $q(M_{n,n-1}^{n-k-1,k-p})$, $q(M_{n,n-1}^{k-p,n-k-1})$, and $q(M_{n,n-1}^{n-k,k-p-1})$ are the largest roots of $f(x) = 0$, $g(x) = 0$, and $h(x) = 0$, respectively, where $f(x) = xf_1(x)$, $g(x) = xg_1(x)$, and $h(x) = xh_1(x)$,

$$f_1(x) = x^3 - (3n + p - 1)x^2 + (2n^2 + (2k + p)n - (2k + 1)(k - p + 1))x - (2n - 1)(n - k + p - 1)(k + 1),$$

$$g_1(x) = x^3 - (3n + p - 1)x^2 + (2n^2 + (2k + p - 1)n - k(2k - 2p + 1))x - (2n - 1)(n - k + p)k,$$

$$h_1(x) = x^3 - (3n + p - 1)x^2 + (2n^2 + (2k + p - 2)n - (2k - 1)(k - p))x - (2n - 1)(n - k + p)k.$$

Since signless Laplacian spectral radius of any nonempty graph is positive, $q(M_{n,n-1}^{n-k-1,k-p})$, $q(M_{n,n-1}^{k-p,n-k-1})$ and $q(M_{n,n-1}^{n-k,k-p-1})$ are the largest roots of $f_1(x) = 0$, $g_1(x) = 0$, and $h_1(x) = 0$, respectively.

(i) Since

$$f_1(2n - 1) = (2n - 1)(n - k - 1)(k - p) > 0,$$

$$g_1(2n - 1) = (2n - 1)(n - k - 1)(k - p) > 0,$$

and for $x \geq 2n - 1$,

$$\begin{aligned}
 f_1'(x) &= 3x^2 - (6n + 2p - 2)x + 2n^2 + (2k + p)n - (2k + 1)(k - p + 1) \\
 &\geq f_1'(2n - 1) \\
 &= n(n + 2k - 3p - 2) + n^2 - (2k + 3)(k - p) \\
 &> (2k - p + 2)^2 - (2k + 3)(k - p) \\
 &\geq 7k + 4 \\
 &> 0, \\
 g_1'(x) &= 3x^2 - (6n + 2p - 2)x + 2n^2 + (2k + p - 1)n - k(2k - 2p + 1) \\
 &\geq g_1'(2n - 1) \\
 &= n(n + 2k - 3p - 3) + n^2 - (k + 1)(2k - 2p - 1) \\
 &> (2k - p + 2)^2 - (k + 1)(2k - 2p - 1) \\
 &\geq 7k + 5 \\
 &> 0,
 \end{aligned}$$

we have

$$q(M_{n,n-1}^{n-k-1,k-p}) < 2n - 1, \quad q(M_{n,n-1}^{k-p,n-k-1}) < 2n - 1.$$

Together with, for $x < 2n - 1$,

$$g_1(x) - f_1(x) = (n - 2k + p - 1)(2n - 1 - x) > 0,$$

we have $q(M_{n,n-1}^{n-k-1,k-p}) > q(M_{n,n-1}^{k-p,n-k-1})$ for $k \geq p + 1$.

(ii) Note that $K_{n,n-k+p-1}$ and $K_{n,n-k+p}$ are proper subgraphs of $M_{n,n-1}^{n-k-1,k-p}$ and $M_{n,n-1}^{n-k,k-p-1}$, respectively. By Lemma 2.1,

$$q(M_{n,n-1}^{n-k-1,k-p}) > 2n - k + p - 1, \quad q(M_{n,n-1}^{n-k,k-p-1}) > 2n - k + p.$$

Since for $x > 2n - k + p - 1 > n$,

$$f_1(x) - h_1(x) = (2n - 4k + 2p - 1)x - (2n - 1)(n - 2k + p - 1) > 0,$$

we have $q(M_{n,n-1}^{n-k,k-p-1}) > q(M_{n,n-1}^{n-k-1,k-p})$. □