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Existence of Multiple Positive Solutions for Semipositone Fractional Boundary Value Problems

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Abstract. In this paper, we study the existence and multiplicity of positive solutions to the four-point boundary value problems of nonlinear semipositone fractional differential equations. Our results extend some recent works in the literature.

1. Introduction

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields, such as physics, mechanics, chemistry, engineering, etc. For details we refer to [3, 5, 6] and references therein. Such investigations will provide an important platform for gaining a deeper understanding of our environment.

Motivated by the wide application of fractional differential equation, in the past few years, the study of positive solutions on fractional boundary value problems had aroused extensive interest (see [1, 2, 9, 11, 12, 14-17] and the references therein). In [1, 9, 12, 14, 15], the authors considered the boundary value problems of Riemann-Liouville differential equations and zero boundary values (Riemann-Liouville derivatives is not suitable for non-zero boundary values). In [2, 11, 16, 17], the authors considered the boundary value problems of Caputo fractional differential equations. However, to the best knowledge of the authors, there is less literature available on paper concerned with the fractional four-point boundary value problems [18].

The purpose of this paper is to study of existence of positive solutions for the following four-point boundary value problem for fractional differential equation

$$D^q\left(p(t)D^r u(t)\right) + f\left(t, u(t)\right) = 0, \quad t \in (0,1), \tag{1}$$

$$\begin{aligned} &\alpha_1 u(0) - \beta_1 u'(0) = -\gamma_1 u(\xi_1),\\ &\alpha_2 u(1) + \beta_2 u'(1) = -\gamma_2 u(\xi_2),\\ &D^r u(0) = 0, \end{aligned} \tag{2}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are real constants with $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, $\beta_1 > \gamma_1$, $\beta_2 > \gamma_2$, $0 < \xi_1 \leq \xi_2 < 1$, $f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}\right)$ is semipositone, i.e., $f(t,u)$ needn't be positive for all $t \in [0,1]$ and all $u \geq 0$, $p \in C\left([0,1], \mathbb{R}^+\right)$ with $p(t) \neq 0$ for all $t \in [0,1]$ and D^r and D^q are the standard Caputo fractional derivatives of fractional order r and q with $1 < r \leq 2, 0 < q \leq 1$.

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The remainder of the paper is organized as follows. In Section 2 we state some preliminary facts needed in the proofs of the main results. We also state the Krasnosel'skiĭ's and Leggett-Williams fixed point theorems in this section. In Section 3, we give the main results of the paper, that establish existence of at least one or multiple positive solutions for the problem (1)-(2). Finally, in this section we discuss an example that illustrates the main results of the paper.

2. Preliminaries

In this section we collect some preliminary definitions and results that will be used in subsequent section. Firstly, for convenience of the reader, we give some definitions and fundamental results of fractional calculus theory.

Definition 2.1 For a function f given on the interval $[a,b]$, the Caputo derivative of fractional order r is defined as

$$D^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} f^{(n)}(s)ds, \quad n = [r]+1, \tag{3}$$

where $[r]$ denotes the integer part of r.

Definition 2.2 The Riemann-Liouville fractional integral of order r for a function f is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)ds, \quad r > 0, \tag{4}$$

where $[r]$ denotes the integer part of r.

Lemma 2.1 Let $r > 0$. Then the differential equation $D^r x(t) = 0$ has solutions

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{5}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,...,n$, $n = [r]+1$.

Lemma 2.2 Let $r > 0$. Then

$$I^r(D^r x)(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{6}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,...,n$, $n = [r]+1$.

For finding a solution of the problem (1)-(2), we first consider the following fractional differential equation

$$-D^r u(t) = v(t) \tag{7}$$

$$\begin{aligned} \alpha_1 u(0) - \beta_1 u'(0) &= -\gamma_1 u(\xi_1) \\ \alpha_2 u(1) + \beta_2 u'(1) &= -\gamma_2 u(\xi_2) \end{aligned} \tag{8}$$

where $v \in C([0,1])$.

Let we define $d := \alpha_1(\alpha_2 + \beta_2 + \gamma_2\xi_2) + \gamma_1(\beta_2 + \alpha_2(1-\xi_1) + \gamma_2(\xi_2 - \xi_1)) + \beta_1(\alpha_2 + \gamma_2)$.

Lemma 2.3 Let $r \in (1,2]$ and $v \in C[0,1]$. The boundary value problem $(7)-(8)$ has a unique solution u in the form

$$u(t) = \int_0^1 G(t,s)v(s)ds, \tag{9}$$

where

$$G(t,s)=\begin{cases} -\dfrac{1}{\Gamma(r)}(t-s)^{r-1}+\dfrac{\gamma_1}{d\Gamma(r)}\left[\alpha_2+\beta_2+\gamma_1\xi_2\right. \\ \left.-t(\alpha_2+\gamma_2)\right](\xi_1-s)^{r-1}+\frac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1\right. \\ \left.+(\alpha_1+\gamma_1)t\right]\Big[\gamma_2(\xi_2-s)^{r-1}+\alpha_2(1-s)^{r-1} \\ +r\beta_2(1-s)^{r}-2\Big], & s\le\xi_1, s\le t \\ \dfrac{\gamma_1}{d\Gamma(r)}\left[\alpha_2+\beta_2+\gamma_1\xi_2-t(\alpha_2+\gamma_2)\right](\xi_1-s)^{r-1} \\ \frac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right]\Big[\gamma_2(\xi_2-s)^{r-1} \\ +\alpha_2(1-s)^{r-1}+r\beta_2(1-s)^{r-2}\Big], & s\le\xi_1, s\ge t \\ -\dfrac{1}{\Gamma(r)}(t-s)^{r-1}+\dfrac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right] \\ \left[\gamma_2(\xi_2-s)^{r-1}+\alpha_2(1-s)^{r-1}+r\beta_2(1-s)^{r-2}\right], & \xi_1\le s\le\xi_2, s\le t \\ \dfrac{1}{\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right]\Big[\gamma_2(\xi_2-s)^{r-1} \\ +r\beta_2(1-s)^{r-2}\Big], & \xi_1\le s\le\xi_2, s\ge t \\ -\dfrac{1}{\Gamma(r)}(t-s)^{r-1}+\dfrac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right] \\ \left[\alpha_2(1-s)^{r-1}+r\beta_2(1-s)^{r-2}\right], & \xi_2\le s, s\le t \\ \frac{1}{\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right]\Big[\alpha_2(1-s)^{r-1} \\ +r\beta_2(1-s)^{r-2}\Big] & \xi_2\le s, s\ge t. \end{cases}$$

Proof. The equation $D^r u(t)+v(t)=0$ has a unique solution

$$u(t)=-\frac{1}{\Gamma(r)}\int_0^1 (t-s)^{r-1}v(s)ds+c_0+c_1t \tag{10}$$

where $c_0, c_1\in\mathbb{R}$. By $\alpha_1u(0)-\beta_1u'(0)=-\gamma_1u(\xi_1)$, $\alpha_2u(1)+\beta_2u'(1)=-\gamma_2u(\xi_2)$, we have

$$\begin{aligned} c_0=&\frac{\gamma_1(\alpha_2+\beta_2+\gamma_2\xi_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}v(s)ds-\frac{1}{d}(\gamma_1\xi_1-\beta_1)\left[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}v(s)ds\right. \\ &\left.+\frac{\beta_2}{\Gamma(r)}\int_0^1(1-s)^{r-2}v(s)ds+\frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}v(s)ds\right] \end{aligned}$$

and

$$\begin{aligned} c_1=&\frac{\alpha_1+\gamma_1}{d}\left[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}v(s)ds+\frac{\beta_2}{\Gamma(r-1)}\int_0^1(1-s)^{r-2}v(s)ds\right. \\ &\left.+\frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}v(s)ds\right]-\frac{\gamma_1(\alpha_2+\gamma_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}v(s)ds. \end{aligned}$$

Substituting c_0, c_1 into equation (10) we find,

$$
\begin{aligned}
u(t) = &-\frac{1}{\Gamma(r)}\int_0^1 (t-s)^{r-1}v(s)ds + \frac{\gamma_1(\alpha_2+\beta_2+\gamma_2\xi_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}v(s)ds - \frac{1}{d}(\gamma_1\xi_1-\beta_1)\\
&\left[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}v(s)ds + \frac{\beta_2}{\Gamma(r)}\int_0^1(1-s)^{r-2}v(s)ds + \frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}v(s)ds\right]\\
&+\left[\frac{\alpha_1+\gamma_1}{d}\left[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}v(s)ds + \frac{\beta_2}{\Gamma(r-1)}\int_0^1(1-s)^{r-2}v(s)ds\right.\right.\\
&\left.\left.+\frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}v(s)ds\right] - \frac{\gamma_1(\alpha_2+\gamma_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}v(s)ds\right]t\\
= &\int_0^1 G(t,s)v(s)ds.
\end{aligned}
$$

The proof is complete.

Throughout this study we will assume the following condition is satisfied:
(H1) $(\alpha_2+(r-1)\beta_2)(\beta_1-\gamma_1\xi_1)\geq d$.
Lemma 2.4 If (H1) holds, then there exist a constant N such that $0\leq G(t,s)\leq N(1-s)^{r-2}$, $t,s\in[0,1]$, where $N := \frac{1}{d\Gamma(r)}\left[\gamma_1(\alpha_2+\beta_2+\gamma_1\xi_2)+(\alpha_1+\beta_1)(\gamma_2+\alpha_2+(r-1)\beta_2)\right]$.
Proof. Obviously $G(t,s)\geq 0$ also we get

$$
\begin{aligned}
max_{0\leq t\leq 1}G(t,s) \;\leq\;& \frac{\gamma_1}{d\Gamma(r)}(\alpha_2+\beta_2+\gamma_1\xi_2-(\alpha_2+\gamma_2)t)(\xi_1-s)^{r-1}\\
&+\frac{\gamma_2}{d\Gamma(r)}(\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t)(\xi_2-s)^{r-1}\\
&+\frac{\alpha_2+(r-1)\beta_2}{d\Gamma(r)}(\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t)(1-s)^{r-2}\\
\leq\;& \frac{\gamma_1}{d\Gamma(r)}(\alpha_2+\beta_2+\gamma_1\xi_2)(1-s)^{r-2}\\
&+\frac{\gamma_2}{d\Gamma(r)}(\beta_1-\gamma_1\xi_1+\alpha_1+\gamma_1)(1-s)^{r-2}\\
&+\frac{\alpha_2+(r-1)\beta_2}{d\Gamma(r)}(\beta_1-\gamma_1\xi_1+\alpha_1+\gamma_1)(1-s)^{r-2}\\
\leq\;& \frac{1}{d\Gamma(r)}\left[\gamma_1(\alpha_2+\beta_2+\gamma_1\xi_2)+(\alpha_1+\beta_1)(\gamma_2+\alpha_2+(r-1)\beta_2)\right](1-s)^{r-2}\\
\leq\;& N(1-s)^{r-2}.
\end{aligned}
$$

The proof is complete.
Lemma 2.5 If $0<s<1$, $\theta\in(0,\frac{1}{2})$, then there exists a constant μ such that

$$\min_{t\in[\theta,1-\theta]} G(t,s)\geq \mu N(1-s)^{r-2}, \tag{11}$$

where

$$\mu := \frac{-d+(\alpha_2+(r-1)\beta_2)(\beta_1-\gamma_1\xi_1+min\{\theta,1-\theta\}(\alpha_1+\gamma_1))}{\gamma_1(\alpha_2+\beta_2+\gamma_1\xi_2)+(\alpha_1+\beta_1)(\gamma_2+\alpha_2+(r-1)\beta_2)}. \tag{12}$$

Proof: We have two cases:

Case 1. For $0 \leq s \leq t \leq 1-\theta$, we get

$$G(t,s) \geq -\frac{1}{\Gamma(r)}(1-s)^{r-2} + \frac{\alpha_2 + (r-1)\beta_2}{d\Gamma(r)}\left[\beta_1 - \gamma_1\xi_1 + (\alpha_1+\gamma_1)t\right](1-s)^{r-2}. \tag{13}$$

Case 2. For $\theta \leq t \leq s \leq 1$, we get

$$G(t,s) \geq \frac{\alpha_2 + (r-1)\beta_2}{d\Gamma(r)}\left[\beta_1 - \gamma_1\xi_1 + (\alpha_1+\gamma_1)t\right](1-s)^{r-2}. \tag{14}$$

Hence we have

$$G(t,s) \geq \frac{-d + (\alpha_2 + (r-1)\beta_2)\left(\beta_1 - \gamma_1\xi_1 + (\alpha_1+\gamma_1)min\{\theta, 1-\theta\}\right)}{d\Gamma(r)}(1-s)^{r-2}. \tag{15}$$

Lemma 2.6 Let $f \in C([0,1]\times\mathbb{R}^+)$, then the problem (1)-(2) has a unique solution

$$u(t) = \int_0^1 G(t,s)\frac{1}{p(s)}I^q f(s,u(s))ds. \tag{16}$$

Proof. Let $p(t)D^r x(t) = h(t)$, we have the following problem

$$\begin{array}{l} D^q h(t) + f(t,x(t)) = 0 \\ h(0) = 0. \end{array} \tag{17}$$

By Lemma 2.2, we have

$$h(t) = c_1 t^{q-1} - I^q\left(f(t,x(t))\right). \tag{18}$$

Since $h(0) = 0$ we get

$$h(t) = -I^q\left(f(t,x(t))\right), \quad 0 < t < 1. \tag{19}$$

So, using Lemma 2.3, the problem

$$\begin{array}{l} D^r u(t) = \frac{1}{p(t)}I^q\left(-f(t,u(t))\right) = -\frac{1}{p(t)}I^q f(t,u(t)) \\ \alpha_1 u(0) - \beta_1 u'(0) = -\gamma_1 u(\xi_1) \\ \alpha_2 u(1) - \beta_2 u'(1) = -\gamma_2 u(\xi_2) \end{array} \tag{20}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)\frac{1}{p(s)}I^q f(s,u(s))ds. \tag{21}$$

Lemma 2.7 Let ω be a solution of

$$D^q\left(p(t)D^r u(t)\right) + 1 = 0 \tag{22}$$

with the boundary condition (2), then $\omega(t) \leq \frac{N}{\Gamma(q+1)}P$ where $P = \int_0^1 \frac{1}{p(s)}ds$.

Proof. Using Lemma 2.6, we obtain the solution of the problem (22)-(2) is

$$\omega(t) = \int_0^1 G(t,s)\frac{1}{p(s)}I^q(1)ds.$$

Since

$$I^q(1) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds = \frac{t^q}{\Gamma(q+1)},$$

we get, for $t \in [0,1]$,

$$\begin{aligned} \omega(t) &\leq \frac{1}{\Gamma(q+1)} \int_0^1 G(t,s) s \frac{1}{p(s)} ds \leq \frac{1}{\Gamma(q+1)} \int_0^1 N(1-s)^{r-2} \frac{1}{p(s)} ds \\ &\leq \frac{N}{\Gamma(q+1)} \int_0^1 \frac{1}{p(s)} ds = \frac{N}{\Gamma(q+1)} P. \end{aligned}$$

This completes the proof.

The following fixed point theorems are fundamental and important to the proof of our main results.

Theorem 2.1. [7] *Let $E = (E, \|\,.\,\|)$ be a Banach space, $P \subset E$ be a cone in E. Suppose that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*
(1) $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$, or
(2) $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$ holds.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Define the sets $P_c := \{u \in P : \| u \| < c\}$ and $P(\alpha, a, b) := \{u \in P : a \leq \alpha(u), \| u \| \leq b\}$ where $a, b, c > 0$ and α on P is a nonnegative functional.

Theorem 2.2. [8] *Let $E = (E, \|\,.\,\|)$ be a Banach space, $P \subset E$ a cone of E and $c > 0$ a constant. Suppose that there exists a nonnegative continuous concave functional α on P with $\alpha(u) \leq \| u \|$ for $u \in \overline{P}_c$ and let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous map. Assume that there exist a, b, c, d with $0 < a < b < d \leq c$ such that*
(S1) $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Tu) > b$ for all $u \in P(\alpha, b, d)$;
(S2) $\| Tu \| < a$ for all $u \in \overline{P}_a$;
(S3) $\alpha(Tu) > b$ for all $u \in P(\alpha, b, c)$ with $\| Tu \| > d$.

Then T has at least three fixed points $u_1, u_2, u_3 \in P$ such that $\| u_1 \| < a, \alpha(u_2) > b, \| u_3 \| > a$ and $\alpha(u_3) < b$.

We consider the Banach space $E = C([0,1], \mathbb{R})$ endowed with the norm defined by $\|u\| = \sup_{0 \leq t \leq 1} |u(t)|$. Let $P = \{u \in E : \mu \| u \| \leq min_{t \in [\theta, 1-\theta]} u(t)\}$, then P is a cone in E.

3. Main Result

In this section, we prove the existence of multiple positive solutions of the problem (1) – (2) by using Theorem 2.1 and Theorem 2.2.

First we shall show that the following boundary value problem

$$D^q \left(p(t) \left(D^r y(t)\right)\right) + F\left(t, y^*(t)\right) = 0 \tag{23}$$

$$\begin{aligned} &\alpha_1 y(0) - \beta_1 y'(0) = -\gamma_1 y(\xi_1) \\ &\alpha_2 y(1) + \beta_2 y'(1) = -\gamma_2 y(\xi_2) \\ &D^r y(0) = 0, \end{aligned} \tag{24}$$

has at least one and three positive solutions where $F : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$,

$$F(t,z) = \begin{cases} f(t,z) + M, & z \geq 0, \\ f(t,0) + M, & z \leq 0, \end{cases} \tag{25}$$

$y^*(t) = max\{(y-x)(t), 0\}$ and $x(t) = M\omega(t)$ such that ω is the unique solution of the problem (22)-(2). Thereafter we shall obtain the existence of multiple positive solutions of the problem $(1)-(2)$

We give the following assumptions:

(H2) $f(t, u(t)) \not\equiv 0$ for $(t, u) \in [0, 1] \times (0, \infty)$,

(H3) There exists a constant $M > 0$ such that $f(t, u) \geq -M$ for all $(t, u) \in [0, 1] \times [0, \infty]$.

Theorem 3.1. *Assume that (H1)-(H3) are satisfied. Let F satisfies the following conditions:*

(A1) *There exist $t_1, t_2 \in (0, 1)$ such that $lim_{y\to\infty} s\dfrac{F(t, y)}{y} = \infty$ uniformly on $[t_1, t_2]$,*

(A2) *R_1 is a positive real number such that $R_1 \geq M_1 \dfrac{NP}{\Gamma(q+1)}$ where $M_1 = max\{F(t, y) : (t, y) \in [0, 1] \times [0, R_1]\}$, then the problem (23)-(24) has at least one positive solution.*

Proof: It is well known that the existence of positive solution to the boundary value problem (23)-(24) is equivalent to the existence of fixed point of the operator T. So we shall seek a fixed point of T in our cone P where the operator $T : E \to E$ is defined by

$$Ty(t) = \int_0^1 G(t, s)\frac{1}{p(s)} I^q F(s, y^*(s))\, ds, \quad t \in [0, 1]. \tag{26}$$

First it is obvious that T is completely continuous. Now we will prove that $T(P) \subset P$.

$$\begin{aligned} Ty(t) &= \int_0^1 G(t, s)\frac{1}{p(s)} I^q F(s, y^*(s))\, ds \\ &\leq \int_0^1 N(1-s)^{r-2}\frac{1}{p(s)} I^q F(s, y^*(s))\, ds \\ &\leq min_{t\in[\theta,1-\theta]} \int_0^1 \frac{1}{\mu} G(t, s)\frac{1}{p(s)} I^q F(s, y^*(s))\, ds. \end{aligned}$$

and so

$$\| Ty \| = \frac{1}{\mu}\int_0^1 min_{t\in[\theta,1-\theta]} G(t, s)\frac{1}{p(s)} I^q F(s, y^*(s))\, ds, \quad t \in [0, 1]. \tag{27}$$

thus, we get

$$\mu \| Ty \| \leq min_{t\in[\theta,1-\theta]} Ty(t). \tag{28}$$

This shows that $T(P) \subset P$. Let $\Omega_{R_1} = \{y \in E : \| y \| < R_1\}$. We shall prove that $\| Ty \| \leq \| y \|$, for $y \in P \bigcap \partial\Omega_{R_1}$. Then $\| y \| = R_1$. It is clear that $y^*(t) \leq y(t) \leq R_1$, for all $t \in [0, 1]$. Then, using the definition of F, we find for $t \in [0, 1]$,

$$\begin{aligned} Ty(t) &= \int_0^1 G(t, s)\frac{1}{p(s)} I^q F(s, y^*(s))\, ds \\ &\leq \int_0^1 G(t, s)\frac{1}{p(s)} I^q (M_1)\, ds \\ &= M_1\int_0^1 G(t, s)\frac{1}{p(s)} I^q (1)\, ds \\ &= M_1\omega(t) \leq M_1\frac{NP}{\Gamma(q+1)} \leq R_1. \end{aligned}$$

Therefore $\| Ty \| \leq R_1 = \| y \|$ for $y \in P \bigcap \partial\Omega R_1$.

Let K be a positive real number such that

$$KL\mu N(1-t_2)^{r-2}\frac{t_1^q}{\Gamma(q+1)}P^* R_2^{-1} \geq 1, \tag{29}$$

where $P^* := \int_{t_1}^{t_2} \frac{1}{p(s)}ds$. In the view of (A2), there is a constant $L > 0$ such that $F(t,y) \geq Ky$, $\forall y \geq L$ and $t \in [t_1, t_2]$. Now set, $R_2 := R_1 + max\{\frac{2MNP}{\Gamma(q+1)}, 2L\}$ and define $\Omega_{R_2} = \{y \in E : \| y \| < R_2\}$. Since $x(t) = M\omega(t) \leq \frac{MNP}{\Gamma(q+1)} \leq \frac{R_2}{2}$, we get $y^*(t) = y(t) - x(t) \geq R_2 - \frac{R_2}{2} = \frac{R_2}{2} > L$ for $t \in [t_1, t_2]$. Therefore for $y \in P \bigcap \partial\Omega_{R_2}$, we have

$$F(t, y^*(t)) \geq Ky^*(t) \geq KL, \quad t \in [t_1, t_2], \tag{30}$$

which implies that

$$\begin{aligned}
Ty(t) &= \int_0^1 G(t,s)\frac{1}{p(s)} I^q F(s, y^*(s))ds \\
&\geq \int_{t_1}^{t_2} \mu N(1-s)^{r-2}\frac{1}{p(s)} I^q\,(KL)\,ds \\
&\geq KL\mu N \int_{t_1}^{t_2} (1-s)^{r-2}\frac{1}{p(s)} I^q\,(1)\,ds \\
&\geq KL\mu N(1-t_2)^{r-2}\int_{t_1}^{t_2} \frac{1}{p(s)}\frac{t^q}{\Gamma(q+1)}ds \\
&\geq KL\mu N(1-t_2)^{r-2}\frac{t_1^q}{\Gamma(q+1)}\int_{t_1}^{t_2} \frac{1}{p(s)}ds \\
&= KL\mu N(1-t_2)^{r-2}\frac{t_1^q}{\Gamma(q+1)}P^* \geq R_2 = \| y \|,
\end{aligned}$$

so we get $\| Ty \| \geq R_2 = \| y \|$.

Then it follows from Theorem 2.1 that T has a fixed y with $R_1 \leq \| y \| \leq R_2$.

Theorem 3.2. *Assume that (H1)-(H3) are satisfied. Let $0 < a < b < c \leq e$ and suppose that F satisfies the following conditions:*

(B1) $F(t,y) \leq a\frac{\Gamma(q+1)}{NP}$, *for all* $(t,y) \in [0,1] \times [0,a]$,

(B2) $F(t,y) \geq b\frac{\Gamma(q+1)}{\mu N \theta^{q+r-2} P_\theta}$, *for all* $(t,y) \in [\theta, 1-\theta] \times [b - \frac{MNP}{\Gamma(q+1)}, c]$,

(B3) $F(t,y) \leq e\frac{\Gamma(q+1)}{NP}$, *for all* $(t,y) \in [0,1] \times [0,e]$,

where $P_\theta := \int_\theta^{1-\theta} \frac{1}{p(s)}ds$, then the problem (3.23)-(3.24) has at least three positive solutions y_1, y_2, y_3 such that $\| y_1 \| < a, \alpha(y_2) > b, \| y_3 \| > a$ and $\alpha(y_3) < b$.

Proof: We shall seek fixed points of T in our cone P where the operator $T : E \to E$ is defined by (26). In what follows, we show that the all conditions of Theorem 2.2 are satisfied. We first define the nonnegative, continuous concave functional $\alpha : P \to [0, \infty)$ by $\alpha(y) = \min_{t\in[\theta,1-\theta]} |y(t)|$. For each $y \in P$, it is easy to see $\alpha(y) \leq \| y \|$. Let $e > 0$ be a constant. We prove that $T(\overline{P_e}) \subseteq \overline{P_e}$. Let $y \in \overline{P_e}$. Then

(i) if $y(t) \geq x(t)$, we have $0 \leq y(t) - x(t) \leq y(t) \leq c$ and $F(t, y^*(t)) = f(t, y(t) - x(t)) + M \geq 0$. By (B3) we have $F(t, y(t) - x(t)) \leq e\dfrac{\Gamma(q+1)}{NP}$.

(ii) if $y(t) < x(t)$, we have $y(t) - x(t) < 0$ and $F(t, y^*(t)) = f(t, 0) + M \geq 0$. By (B3) we have $F(t, y(t) - x(t)) \leq e\dfrac{\Gamma(q+1)}{NP}$.

So, we proved that, if $y \in \overline{P_e}$, then $F(t, y(t) - x(t)) \leq e\dfrac{\Gamma(q+1)}{NP}$ for $t \in [0, 1]$. Then,

$$\begin{aligned} Ty(t) &= \int_0^1 G(t,s)\frac{1}{p(s)} I^q F(s, y^*(s))\, ds \\ &\leq \int_0^1 G(t,s)\frac{1}{p(s)} I^q\left(\frac{e\Gamma(q+1)}{NP}\right) ds \\ &= \frac{e\Gamma(q+1)}{NP}\int_0^1 G(t,s)\frac{1}{p(s)} I^q(1)\, ds \\ &= \frac{e\Gamma(q+1)}{NP}\omega(t) \leq \frac{e\Gamma(q+1)}{NP}\frac{NP}{\Gamma(q+1)} = e. \end{aligned}$$

Therefore, we have $T(\overline{P_e}) \subseteq \overline{P_e}$.

Especially, if $y \in \overline{P_a}$, then (B1) yields $F(t, y(t) - z(t)) \leq a\dfrac{\Gamma(q+1)}{NP}$ for $t \in [0, 1]$. Hence condition (S2) of Theorem 2.2 is satisfied.

Next we show that condition (S1) of Theorem 2.2 holds. To check that, we choose $y_0(t) = \dfrac{b+c}{2}$ for $t \in [0, 1]$. It is easy to see that $y_0 \in P$, $\| y_0 \| = \dfrac{b+c}{2} \leq c$ and $\alpha(y_0) = \dfrac{b+c}{2} > b$. That is $y_0 \in \{y \in P(\alpha, b, c) : \alpha(y) > b\}$, in other words $\{y \in P(\alpha, b, c) : \alpha(y) > b\} \neq \emptyset$. Moreover, if $y \in P(\alpha, b, c)$ we have $b < y(t) \leq c$ for $t \in [\theta, 1-\theta]$ and so $b - \dfrac{MNP}{\Gamma(q+1)} < y^\star = y - x < y < c$. By (B2) and Lemma 2.5, we have

$$\begin{aligned} \alpha(Ty) = \min_{t\in[\theta,1-\theta]}(Ty(t)) &\geq \int_\theta^{1-\theta} \mu N(1-s)^{r-2}\frac{1}{p(s)} I^q F(s, y^*(s))\, ds \\ &\geq \mu N\int_\theta^{1-\theta}(1-s)^{r-2}\frac{1}{p(s)} I^q\left(b\frac{\Gamma(q+1)}{\mu N\theta^{q+r-2}P_\theta}\right) ds \\ &\geq \mu N b\frac{\Gamma(q+1)}{\mu N\theta^{q+r-2}P_\theta}\theta^{r-2}\int_\theta^{1-\theta}\frac{1}{p(s)} I^q(1)\, ds \\ &= \mu N b\frac{\Gamma(q+1)}{\mu N\theta^{q+r-2}P_\theta}\theta^{r-2}\int_\theta^{1-\theta}\frac{1}{p(s)}\frac{t^q}{\Gamma(q+1)} ds \\ &\geq \mu N b\frac{\Gamma(q+1)}{\mu N\theta^{q+r-2}P_\theta}\theta^{r-2}\frac{\theta^q}{\Gamma(q+1)}\int_\theta^{1-\theta}\frac{1}{p(s)} ds \\ &= \frac{\mu N\theta^{q+r-2}}{\Gamma(q+1)} b\frac{\Gamma(q+1)}{\mu N\theta^{q+r-2}P_\theta} P_\theta = b. \end{aligned}$$

Hence condition (S1) of Theorem 2.3 is satisfied. If $c = e$, then condition (S1) of Theorem 2.3 implies the condition (S3) of this theorem, so condition (S3) of Theorem 2.3 is satisfied.

To sum up, all the hypotheses of Theorem 2.3 are satisfied. The proof is complete.

Lemma 3.3 $y(t)$ is the solution of the boundary value problem (23)-(24) with $y(t) > x(t)$ for all $t \in [0, 1]$ if and only if $u(t) = y(t) - x(t)$ is the positive solution of the boundary value problem (1)-(2).

Proof: Let $y(t)$ is the solution of the boundary value problem (23)-(24). Then

$$\begin{aligned} y(t) &= \int_0^1 G(t,s)\frac{1}{q(s)} I^q F(s, y^\star(s)) ds \\ &= \int_0^1 G(t,s)\frac{1}{q(s)} I^q (f(s, y^\star(s)) + M) ds \\ &= \int_0^1 G(t,s)\frac{1}{q(s)} I^q f(s, (y-x)(s)) ds + M \int_0^1 G(t,s)\frac{1}{q(s)} I^q (1) ds. \end{aligned}$$

Noticing that,

$$\omega(t) = \int_0^1 G(t,s)\frac{1}{q(s)} I^q(1) ds \text{ and } x(t) = M\omega(t), \text{ we have for } t \in [0,1],$$

$$y(t) = \int_0^1 G(t,s)\frac{1}{q(s)} I^q f(s, (y-x)(s)) ds + M\omega(t),$$

or

$$y(t) - x(t) = \int_0^1 G(t,s)\frac{1}{q(s)} I^q f(s, (y-x)(s)) ds,$$

and hence

$$u(t) = \int_0^1 G(t,s)\frac{1}{q(s)} I^q f(s, u(s)) ds.$$

This completes the proof.

Example 3.3. *Consider the following fractional boundary value problem*

$$D^{\frac{1}{2}}\left(e^{t-1} D^{\frac{3}{2}} u(t)\right) + f(t, u(t)) = 0, \quad t \in (0,1), \tag{31}$$

$$\begin{aligned} &\frac{1}{2}u(0) - \frac{1}{4}u'(0) = -\frac{1}{6}u(\frac{1}{3}), \\ &\frac{1}{3}u(1) + \frac{1}{4}u'(1) = -\frac{1}{5}u(\frac{1}{2}), \\ &D^r u(0) = 0, \end{aligned} \tag{32}$$

(i) *Consider the fractional boundary value problem (31)-(32) with the function* $f(t, u(t)) = \dfrac{1}{100(t+1)}(u(t) - 1)(u(t) - 2)$. *It is easy to check that the assumptions* $(H_1) - (H_3)$ *hold and calculate* $d \cong 0.6, N \cong 1.26, P = e - 1$. *Choosing* $M = 1$ *we get* $M_1 = 1.02$ *and so* $M_1 \dfrac{NP}{\Gamma(q+1)} \cong 2.79$. *Set* $R_1 = 3$ *and* $[t_1, t_2] = [\frac{1}{6}, \frac{5}{6}]$. *Thus we can verify that conditions* $(A_1) - (A_2)$ *are satisfied. Then applying Theorem 3.1 and choosing* $R_2 = 10$, *the problem* $(31) - (32)$ *has a positive solution* $y \in P$ *with* $3 \leq \| y \| \leq 10$.

(ii) *Consider the fractional boundary value problem (31)-(32) with the function* $f(t, u(t)) = \dfrac{e^t}{10^3}\sin(u(t))$. *It is easy to check that the assumptions* $(H_1) - (H_3)$ *hold. We can choose* $M = 1$ *and easily calculate* $d \cong 0.6, N \cong 1.26, P = e - 1, \mu \cong -0.77$ *and also* $\dfrac{NP}{\Gamma(q+1)} \cong 2.74$. *Let* $\theta = \frac{1}{4}$, *we get* $P_\theta \cong 0.83$. *Set* $a = 5, \quad b = 10, \quad d = 20, \quad c = 22$. *Thus we can verify that conditions* $(B1) - (B3)$ *are satisfied. Then applying Theorem 3.2 the problem (31)-(32) has at least three positive solutions* y_1, y_2, y_3 *such that* $\| y_1 \| < 5, \alpha(y_2) > 10, \| y_3 \| > 5$ *and* $\alpha(y_3) < 10$.

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