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Conformal Semi-invariant Riemannian Maps from Almost Hermitian Manifolds

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Abstract. Conformal semi-invariant Riemannian maps from Kaehler manifolds to Riemannian manifolds are introduced. We give examples, study the geometry of leaves of certain distributions and investigate certain conditions for such maps to be horizontally homothetic. Morever, we introduce special pluriharmonic maps and obtain characterizations.

1. Introduction

Fischer introduced Riemannian maps between Riemannian manifolds in [7] as a generalization of the notions of isometric immersions and Riemannian submersions, [6], [9], [13] and [23]. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{dimM_1, dimM_2\}$. Then the tangent bundle of M_1 has the following decomposition:

$$TM_1 = kerF_* \oplus (kerF_*)^{\perp}.$$

Since $rankF < min\{dimM_1, dimM_2\}$, we always have $(rangeF_*)^{\perp}$. Thus tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^{\perp}.$$

Now, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^{\perp} \longrightarrow (rangeF_*)$ is a linear isometry. Therefore Fischer stated in [7] that a Riemannian map satisfies the equation

$$g_1(X, Y) = g_2(F_*X, F_*Y) \quad (1)$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^{\perp} = \{0\}$. There are many applications of this type maps in different research areas such geometric modelling, computer vision and medical imaging [10, 21, 22].

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Let $(\bar{M}, g)$ be a Kaehler manifold. This means [23] that $\bar{M}$ admits a tensor field J of type (1,1) on $\bar{M}$ such that, $\forall X, Y \in \Gamma(T\bar{M})$, we have

$$J^2 = -I, \quad g(X,Y) = g(JX, JY), \quad (\bar{\nabla}_X J)Y = 0,$$

where g is the Riemannian metric and $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. Certain Riemannian maps from Kaehler manifolds to arbitrary Riemannian manifolds were introduced such as anti-invariant Riemannian maps, semi-invariant Riemannian maps and slant Riemannian maps and such maps were studied widely, see:[14] and references therein. On the other hand, conformal anti-invariant Riemannian maps from Kaehler manifolds to Riemannian manifolds were recently introduced in [18].

In this paper, we introduce and investigate geometric structures for conformal semi-invariant Riemannian maps from Kaehler manifolds to Riemannian manifolds.

2. Preliminaries

We recall useful results which are related to the second fundamental form and conformal Riemannian maps from [4], [13] and [14]. Let $(M, g_{_M})$ and $(N, g_{_N})$ be Riemannian manifolds and suppose that $F : M \longrightarrow N$ is a smooth map between them. The second fundamental form of F is given by

$$(\nabla F_*)(X,Y) = \overset{N}{\nabla}{}^F{}_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y) \tag{2}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric.

Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) to a Riemannian manifold (N^n, g_N). Then we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E}\mathcal{H}F, \quad \mathcal{T}_E F = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}E}\mathcal{H}F, \tag{3}$$

for vector fields $E, F \in \Gamma(TM)$, where $\overset{M}{\nabla}$ is the Levi-Civita connection of g_M [19]. In fact, we can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field $\mathcal{T}$ is symmetric on the vertical distribution [20]. On the other hand, from (3) we have

$$\overset{M}{\nabla}_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \overset{M}{\nabla}_V X = \mathcal{H}\overset{M}{\nabla}_V X + \mathcal{T}_V X, \overset{M}{\nabla}_X V = \mathcal{A}_X V + \mathcal{V}\overset{M}{\nabla}_X V, \overset{M}{\nabla}_X Y = \mathcal{H}\overset{M}{\nabla}_X Y + \mathcal{A}_X Y, \tag{4}$$

for $X, Y \in \Gamma((ker\, F_*)^\perp)$and $V, W \in \Gamma(kerF_*)$, where $\hat{\nabla}_V W = \mathcal{V}\overset{M}{\nabla}_V W$.

We say that $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rankF_{*p} \leq min\{m, n\}$ and F_{*p} maps the horizontal space $\mathcal{H}(p) = ((ker(F_{*p})^\perp)$ conformally onto $range(F_{*p})$, i.e., there exist a number$\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p}X, F_{*p}Y) = \lambda^2(p) g_M(X,Y)$$

for $X, Y \in \mathcal{H}(p)$. Also F is called conformal Riemannian if F is conformal Riemannian at each $p \in M$ [15]. On the other hand, let F be a conformally Riemannian map between Riemannian manifolds $(M^m, g_{_M})$ and $(N^n, g_{_N})$. Then, we have

$$(\nabla F_*)(X,Y)\mid_{rangeF_*} = X(ln\lambda)F_*(Y) + Y(ln\lambda)F_*(X) - g_M(X,Y)F_*(grad(ln\lambda)), \tag{5}$$

where $X, Y \in \Gamma((kerF_*)^\perp)$ [15].

Therefore from (5), we can write $\overset{N}{\nabla}{}^F{}_X F_*(Y)$ as

$$\overset{N}{\nabla}{}^F{}_X F_*(Y) = F_*(h\overset{M}{\nabla}_X Y) + X(ln\lambda)F_*(Y) + Y(ln\lambda)F_*(X) - g_M(X,Y)F_*(grad(ln\lambda)) + (\nabla F_*)^\perp(X,Y) \tag{6}$$

where $(\nabla F_*)^\perp(X,Y)$ is the component of $(\nabla F_*)(X,Y)$ on $(rangeF_*)^\perp$ for $X, Y \in \Gamma((kerF_*)^\perp)$[18].

3. Conformal Semi-invariant Riemannian maps

Firstly, we give definition of conformal semi-invariant Riemannian maps.

Definition 3.1. *Let $F : (M, g_M, J_M) \longrightarrow (N, g_N)$ be a conformal Riemannian map from a Kaehlerian manifold (M, g_M, J_M) to a Riemannian manifold (N, g_N) . Then we say that F is a conformal semi-invariant Riemannian map if the following conditions are satisfied;*

i- There exist a subbundle of $kerF_*$ such that $J(D_1) = D_1$.

ii- There exist a complementary subbundle D_2 to D_1 in $kerF_*$ such that $J(D_2) \subset (kerF_*)^\perp$.

From definition, we have

$$kerF_* = D_1 \oplus D_2. \tag{7}$$

We provide some examples of conformal semi-invariant Riemannian maps.

Example 3.2. *Every conformal anti-invariant Riemannian submersion [3] from an almost Hermitian manifold to a Riemannian manifold is a conformal semi-invariant Riemannian map with $D_2 = kerF_*$.*

We say that a conformal semi-invariant Riemannian map is proper if $D_1 \neq 0$, $D_2 \neq 0$ and $\mu \neq 0$. Here, there is an example of a proper conformal semi-invariant Riemannian map, where μ is the complementary subbundle to D_2 in $\mathcal{H}$.

Example 3.3. *Let $F : (R^8, g_8, J) \longrightarrow (R^4, g_4)$ be a map from a Kaehlerian manifold (R^8, g_8, J) to a Riemannian manifold (R^4, g_4) defined by*

$$(e^{x_1} cosx_3, -e^{x_1} cosx_3, e^{x_1} cosx_6, -e^{x_1} cosx_6).$$

Then, we obtain horizontal distribution and vertical distribution,

$$H = (kerF_*)^\perp = \{X_1 = (e^{x_1} cosx_3 \frac{\partial}{\partial x_1} - e^{x_1} sinx_3 \frac{\partial}{\partial x_3}), X_2 = (e^{x_1} cosx_6 \frac{\partial}{\partial x_1} - e^{x_1} sinx_6 \frac{\partial}{\partial x_6})\},$$

and

$$V = (kerF_*) = \{V_1 = \frac{\partial}{\partial x_2}, V_2 = \frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_7}, V_5 = \frac{\partial}{\partial x_8}, V_6 = (k\frac{\partial}{\partial x_1} + k\cot x_3 \frac{\partial}{\partial x_3} + k\cot x_6 \frac{\partial}{\partial x_6})\},$$

respectively, $k \in \mathbb{R}$. Hence, we get with $J = (-a_8, -a_7, -a_6, -a_5, a_4, a_3, a_2, a_1)$

$$F_*(X_1) = (e^{2x_1}, -e^{2x_1}, e^{2x_1} cosx_3 cosx_6, -e^{2x_1} cosx_3 cosx_6), \quad F_*(X_2) = (-e^{2x_1}, e^{2x_1}, -e^{2x_1} cosx_3 cosx_6, e^{2x_1} cosx_3 cosx_6)$$

which show that F is a conformal Riemannian map with $\lambda = e^{x_1}\sqrt{2(1 + cos^2x_3 + cos^2x_6)}$ and $rankF = 2$. By some calculations, we get

$$\begin{aligned}
JV_1 &= V_4, \quad JV_2 = V_3, \\
JX_1 &= e^{x_1}\cos x_3 V_5 - \frac{e^{x_1}\cot x_6 \sin x_3}{k(1 + \cot^2 x_3 + \cot^2 x_6)} V_6 + \frac{\sin 2x_3 \sin 2x_6}{4(1 - \cos^2 x_3 \cos^2 x_6)} X_1 \\
&+ \frac{\sin x_3 \sin x_6}{1 - \cos^2 x_3 \cos^2 x_6} X_2, \\
JX_2 &= e^{x_1}\cos x_6 V_5 - \frac{e^{x_1}\cot x_3 \sin x_6}{k(1 + \cot^2 x_3 + \cot^2 x_6)} V_6 + \frac{\cos x_3 \sin x_6}{1 - \cos^2 x_3 \cos^2 x_6} X_1 \\
&+ \frac{\cos^2 x_3 \sin 2x_6}{2(\cos^2 x_3 \cos^2 x_6 - 1)} X_2.
\end{aligned}$$

One can easily see that F is a proper conformal semi-invariant Riemannian map with $D_1 = span\{V_1, V_2, V_3, V_4\}$, $D_2 \neq 0$, $\mu \neq 0$.

We say that a conformal semi-invariant Riemannian map is anti-holomorphic if $J(D_2) = (kerF_*)^\perp$. Here, there is an example of an anti-holomorphic conformal semi-invariant Riemannian map.

Example 3.4. *Let $F : (R^6, g_6, J) \longrightarrow (R^4, g_4)$ be a map from a Kaehlerian manifold (R^6, g_6, J) to a Riemannian manifold (R^4, g_4) defined by*

$$(e^{x_1}cosx_3, e^{x_1}sinx_3, -e^{x_1}cosx_3, -e^{x_1}sinx_3).$$

Then, we obtain horizontal distribution and vertical distribution,

$$H = (kerF_*)^\perp = \{X_1 = (e^{x_1}cosx_3\frac{\partial}{\partial x_1} - e^{x_1}sinx_3\frac{\partial}{\partial x_3}), X_2 = (e^{x_1}sinx_3\frac{\partial}{\partial x_1} + e^{x_1}cosx_3\frac{\partial}{\partial x_3})\},$$

and

$$V = (kerF_*) = \{V_1 = \frac{\partial}{\partial x_2}, V_2 = \frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_6}\},$$

respectively. Hence, we get with $J = (-a_2, a_1, -a_4, a_3, -a_6, a_5)$

$$F_*(X_1) = e^{2x_1}\frac{\partial}{\partial x_1} - e^{2x_1}\frac{\partial}{\partial x_3}, \quad F_*(X_2) = e^{2x_1}\frac{\partial}{\partial x_2} - e^{2x_1}\frac{\partial}{\partial x_4}$$

which show that F is a conformal Riemannian map with $\lambda = e^{x_1}\sqrt{2}$. On the other hand, by direct computations we have

$$\begin{aligned} JV_1 &= -\frac{\partial}{\partial x_1} = -e^{-x_1}sinx_3X_1 - e^{-x_1}cosx_3X_2, \ JV_2 = -\frac{\partial}{\partial x_3} = -e^{-x_1}cosx_3X_1 + e^{-x_1}sinx_3X_2, \\ JV_3 &= \frac{\partial}{\partial x_6} = V_4, \ JV_4 = -\frac{\partial}{\partial x_5} = -V_3. \end{aligned}$$

Thus, F is an anti-holomorphic conformal semi-invariant Riemannian map with $D_1 = span\{V_3, V_4\}$, $D_2 = span\{V_1, V_2\}$ and $J(D_2) = (kerF_*)^\perp = span\{X_1, X_2\}$.

Let F be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then for $V \in \Gamma(kerF_*)$, we write

$$JV = \phi V + \omega V, \tag{8}$$

where $\phi V \in \Gamma(D_1)$ and $\omega V \in \Gamma(JD_2)$. Also for $X \in \Gamma((kerF_*)^\perp)$, we write

$$JX = BX + CX, \tag{9}$$

where $BX \in \Gamma(D_2)$ and $CX \in \Gamma(\mu)$. Hence, we write from (8) and (9)

$$g_M(X, U) = 0, \tag{10}$$

for $X \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(D_2)$. Thus we get the orthogonal complementary subbundle of $(kerF_*)^\perp$ to $J(D_2)$ by μ

$$(kerF_*)^\perp = \mu \oplus J(D_2).$$

Then it is easy to see that μ is invariant.

Theorem 3.5. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the invariant distribution D_1 is integrable if and only if*

$$(\nabla F_*)(U, JV) - (\nabla F_*)(V, JU) = 0,$$

for $U, V \in \Gamma(D_1)$.

Proof. Since M is Kaehlerian manifold for $U, V \in \Gamma(D_1)$, we have

$$T_U JV + v\overset{M}{\nabla}_U JV = BT_U V + CT_U V + \phi v\overset{M}{\nabla}_U V + \omega v\overset{M}{\nabla}_U V. \tag{11}$$

If we change roles of U and V in (11), we have

$$T_V JU + v\overset{M}{\nabla}_V JU = BT_V U + CT_V U + \phi v\overset{M}{\nabla}_V U + \omega v\overset{M}{\nabla}_V U. \tag{12}$$

Thus, if we take horizontal parts of (11), (12) and from (2) , we get

$$\omega v[U, V] = (\nabla F_*)(U, JV) - (\nabla F_*)(V, JU).$$

Hence, if $\omega v[U, V] = 0$, we obtain $v[U, V] \in \Gamma(D_1)$. The proof is complete. □

For the distribution D_2, we have the following result.

Theorem 3.6. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the distribution D_2 is always integrable.*

Proof. Since M is Kaehlerian manifold, fundamental 2-form Ω is closed, we obtain

$$3d\omega(U, V, W) = -g_M(JU, [V, W]) = 0,$$

for $U \in \Gamma(D_1)$ and $V, W \in \Gamma(D_2)$. Because of the distribution D_1 is invariant, we have $[V, W] \in \Gamma(D_2)$. □

We now obtain a new condition for the horizontal distributions.

Theorem 3.7. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then the distribution $(kerF_*)^\perp$ is integrable if*

$$BA_Y BX = -Bh\overset{M}{\nabla}_Y CX,$$

$$\phi A_Y CX = -\phi v\overset{M}{\nabla}_Y BX,$$

are satisfied for $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. Since M is Kaehlerian manifold for $X, Y \in \Gamma((kerF_*)^\perp)$, we have

$$\begin{aligned} \overset{M}{\nabla}_X Y = & - \{BA_X BY + CA_X BY + \phi v\overset{M}{\nabla}_X BY - \omega v\overset{M}{\nabla}_X BY \\ & + \phi A_X CY - \omega A_X CY + Bh\overset{M}{\nabla}_X CY + Ch\overset{M}{\nabla}_X CY\}. \end{aligned} \tag{13}$$

If we change roles of X and Y in (13), we have

$$\begin{aligned} \overset{M}{\nabla}_Y X = & - \{BA_Y BX + CA_Y BX + \phi v\overset{M}{\nabla}_Y BX - \omega v\overset{M}{\nabla}_Y BX \\ & + \phi A_Y CX - \omega A_Y CX + Bh\overset{M}{\nabla}_Y CX + Ch\overset{M}{\nabla}_Y CX\}. \end{aligned} \tag{14}$$

Thus, if we take vertical parts of (13), (14) and from (4) , we get

$$\begin{aligned} [X, Y] = & B\{A_Y BX - A_X BY + h\overset{M}{\nabla}_Y CX - h\overset{M}{\nabla}_X CY\} \\ & + \phi\{A_Y CX - A_X CY + v\overset{M}{\nabla}_Y BX - v\overset{M}{\nabla}_X BY\}. \end{aligned}$$

Hence, the proof is complete. □

Now, we recall pluriharmonic map from [12].

Definition 3.8. **[12]** *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a map from a complex manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then F is called a pluriharmonic map if F satisfies the following equation*

$$(\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) = 0 \tag{15}$$

for $X, Y \in \Gamma(TM)$.

If F satisfies equation (15) for $X, Y \in \Gamma((kerF_*)^{\perp})$ (respectively, $kerF_*$, D_2, D_1, $\{(kerF_*)^{\perp} - (kerF_*)\}$), F is called $(kerF_*)^{\perp}-$ pluriharmonic map (respectively, $kerF_*$, D_2, D_1, $\{(kerF_*)^{\perp} - (kerF_*)\}$).

Theorem 3.9. *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal semi-invariant Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, three of the below assertions imply the fourth assertion,*

i- The distribution $kerF_*$ *defines totally geodesic foliation on* M*,*

ii- F *is a* $kerF_*$*-pluriharmonic map,*

iii- $T_{\phi U}\phi V + A_{\omega V}\phi U + A_{\omega U}\phi V = 0$*,*

iv- F *is a horizontally homothetic map and* $(\nabla F_*)^{\perp}(\omega U, \omega V) = 0$*,*

for $U, V \in \Gamma(kerF_*)$.

Proof. From definition of a pluriharmonic map, (2) and (4), we have

$$\begin{aligned}(\nabla F_*)(U, V) + (\nabla F_*)(JU, JV) &= -F_*(T_U V) - F_*(T_{\phi U}\phi V + A_{\omega V}\phi U + A_{\omega U}\phi V)\\ &+ (\nabla F_*)^{\perp}(\omega U, \omega V) - g_M(\omega U, \omega V)F_*(gradln\lambda)\\ &+ \omega U(ln\lambda)F_*(\omega V) + \omega V(ln\lambda)F_*(\omega U),\end{aligned} \tag{16}$$

for $U, V \in \Gamma(kerF_*)$. The proof is clear. □

Theorem 3.10. *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal semi-invariant Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. Then, three of the below assertions imply the fourth assertion,*

i- $(\nabla F_*)^{\perp}(X, Y) + (\nabla F_*)^{\perp}(CX, CY) = 0$*,*

ii- F *is a horizontally homothetic map,*

iii- F *is a* $(kerF_*)^{\perp}$*-pluriharmonic map,*

iv- $A_{CX}BY + A_{CY}BX + T_{BX}BY = 0$*,*

for $X, Y \in \Gamma((kerF_*)^{\perp})$.

Proof. From definition of a $(kerF_*)^{\perp}-$ pluriharmonic map, (2) and (4), we have

$$\begin{aligned}(\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) &= -F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) + (\nabla F_*)^{\perp}(X, Y) + (\nabla F_*)^{\perp}(CX, CY)\\ &+ X(ln\lambda)F_*(Y) + Y(ln\lambda)F_*(X) + CX(ln\lambda)F_*(CY) + CY(ln\lambda)F_*(CX)\\ &- F_*(gradln\lambda)\{g_M(X, Y) + g_M(CX, CY)\}.\end{aligned} \tag{17}$$

for $X, Y \in \Gamma((kerF_*)^{\perp})$. Hence one can easily obtain the assertion of theorem. □

Theorem 3.11. *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal semi-invariant Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N)*. If F is a* $(kerF_*)^{\perp}$*-pluriharmonic map, then two of the below assertions imply third assertion,*

i- F *is a horizontally homothetic map,*

ii- $A_{CY}BX + A_{CX}BY = 0$*,*

iii- $\phi T_{BX}Y - T_{BX}CY \in D_2$,

for $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. We only proof third condition. Suppose that (i) and (ii) are satisfied in (17). We get

$$g_M(\overset{M}{\nabla}_{BX}BY, U) = g_M(\overset{M}{\nabla}_{BX}JY - CY, U) = g_M(\phi T_{BX}Y - T_{BX}CY, U),$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(D_1)$. The proof is complete. □

Corollary 3.12. *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal semi-invariant Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *If F is a* $(kerF_*)^\perp$*-pluriharmonic map, we have*

$$(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(CX, CY) = 0,$$

for $X, Y \in \Gamma((kerF_*)^\perp)$.

Theorem 3.13. *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal semi-invariant Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *If F is a* $(kerF_*)$*-pluriharmonic map, then two of the below assertions imply the third assertion,*

i- The distribution D_1 *defines totally geodesic foliation on M,*

ii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(\omega U, \omega V) = 0$,

iii- $C\{T_U\phi V + h\overset{M}{\nabla}_U\omega V\} + \omega\{T_U\omega V + v\overset{M}{\nabla}_U\phi V\} = A_{\omega V}\phi U + A_{\omega U}\phi V$,

for $U, V \in \Gamma(kerF_*)$.

Proof. From definition of a $kerF_*-$ pluriharmonic map, (2) and (4), we have

$$\begin{aligned} F_*(\overset{M}{\nabla}_{\phi U}\phi V) &= F_*(CT_U\phi V) + F_*(Ch\overset{M}{\nabla}_U\omega V) + F_*(\omega T_U\omega V) + F_*(\omega v\overset{M}{\nabla}_U\phi V) - F_*(A_{\omega V}\phi U) - F_*(A_{\omega U}\phi V) \\ &+ (\nabla F_*)^\perp(\omega U, \omega V) + \omega U(ln\lambda)F_*(\omega V) + \omega V(ln\lambda)F_*(\omega U) - g_M(\omega U, \omega V)F_*(gradln\lambda), \end{aligned}$$

for $U, V \in \Gamma(kerF_*)$. Thus proof is complete. □

Theorem 3.14. *Let* $F : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal semi-invariant Riemannian map from a Kaehler manifold* (M, g_M, J) *to a Riemannian manifold* (N, g_N). *If F is a* $\{(kerF_*)^\perp - (kerF_*)\}$*-pluriharmonic map, then two of the below assertions imply the third assertion,*

i- F is a horizontally homothetic map,

ii- $A_XV + T_{BX}\phi V + A_{CX}\phi V = 0$,

iii- $(\nabla F_*)(BX, \omega V) + (\nabla F_*)^\perp(CX, \omega V) = 0$,

for $X \in \Gamma((kerF_*)^\perp)$ *and* $V \in \Gamma(kerF_*)$.

Proof. From definition of a $\{(kerF_*)^\perp - (kerF_*)\}-$ pluriharmonic map, (2) and (4), we have

$$\begin{aligned} 0 &= -F_*(A_XV) + \overset{N}{\nabla}{}^F_{JX}F_*(\omega V) - F_*(\overset{M}{\nabla}_{BX}\phi V) - F_*(\overset{M}{\nabla}_{BX}\omega V) - F_*(\overset{M}{\nabla}_{CX}\phi V) - F_*(\overset{M}{\nabla}_{CX}\omega V) \\ 0 &= -F_*(A_XV) + \overset{N}{\nabla}{}^F_{JX}F_*(\omega V) - F_*(T_{BX}\phi V) - F_*(h\overset{M}{\nabla}_{BX}\omega V) - F_*(A_{CX}\phi V) - F_*(h\overset{M}{\nabla}_{CX}\omega V). \end{aligned}$$

Using (6), we get

$$\begin{aligned} 0 &= (\nabla F_*)(BX, \omega V) + (\nabla F_*)^\perp(CX, \omega V) - CX(ln\lambda)F_*(\omega V) \\ &- \omega V(ln\lambda)F_*(CX) - F_*(A_XV + T_{BX}\phi V + A_{CX}\phi V). \end{aligned} \quad (18)$$

Suppose that (ii) and (iii) are satisfied, we have

$$0 = \omega V(ln\lambda)\lambda^2 g_M(CX, CX) \tag{19}$$

for $CX \in \Gamma(\mu)$. Thus λ is a constant on $\Gamma(\mu)$. On the other hand, we derive from (18)

$$0 = CX(ln\lambda)\lambda^2 g_M(\omega V, \omega V) \tag{20}$$

for $\omega V \in (J(D_2))$. From above equation, λ is a constant on $\Gamma(J(D_2))$. The converse is clear from (18). □

We now recall $(kerF_*)^\perp$−geodesic map from [2].

Definition 3.15. **[2]** *Let F be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is called a $kerF_*$-geodesic map if*

$$(\nabla F_*)(X, Y) = 0,$$

for $U, V \in \Gamma(kerF_*)$.

Theorem 3.16. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, F is a $kerF_*$-geodesic map, if and only if the following conditions are satisfied,*

i- $\hat{\nabla}_U \phi V + T_U \omega V \in D_1$,

ii- $T_U \phi V + h\overset{M}{\nabla}_U \omega V \in JD_2$,

for $U, V \in \Gamma(kerF_*)$.

Proof. Using (2) for $U, V \in \Gamma(kerF_*)$, we get

$$(\nabla F_*)(U, V) = F_*(CT_U \phi V) + F_*(\omega \hat{\nabla}_U \phi V) + F_*(\omega T_U \omega V) + F_*(Ch\overset{M}{\nabla}_U \omega V). \tag{21}$$

Now, for $W \in \Gamma(D_2)$ from (21), we obtain

$$g_N((\nabla F_*)(U, V), F_*(JW)) = \lambda^2 g_M(\omega\{\hat{\nabla}_U \phi V + T_U \omega V\}, JW). \tag{22}$$

Then, for $Z \in \Gamma(\mu)$ from (21), we obtain

$$g_N((\nabla F_*)(U, V), F_*(Z)) = \lambda^2 g_M(C\{T_U \phi V + h\overset{M}{\nabla}_U \omega V\}, Z). \tag{23}$$

From (22) and (23) we have the proof. □

We now investigate the geometry of leaves of distributions on M.

Theorem 3.17. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). The distribution D_2 defines a totally geodesic foliation on M if and only if the following conditions are satisfied,*

i- $\frac{1}{\lambda^2} g_N((\nabla F_*)(X, JU), F_*(JY)) = 0$,

ii- $\frac{1}{\lambda^2} g_N((\nabla F_*)(X, CZ), F_*(JY)) = \frac{1}{\lambda^2} g_N(\overset{N}{\nabla^F}_X F_*(CZ), F_*(JY)) + g_M(T_X BZ, JY)$,

for $X, Y \in \Gamma(D_2)$, $Z \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(D_1)$.

Proof. For $X, Y \in \Gamma(D_2)$, $U \in \Gamma(D_1)$ and using (2), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= g_M(T_X JU, JY) \\ &= -\frac{1}{\lambda^2} g_N(F_*(T_X JU), F_*(JY)) \\ &= \frac{1}{\lambda^2} g_N((\nabla F_*)(X, JU), F_*(JY)). \end{aligned} \tag{24}$$

By the similar way, for $X, Y \in \Gamma(D_2)$, $Z \in \Gamma((kerF_*)^\perp)$ and using (2), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, Z) &= -g_M(\overset{M}{\nabla}_X BZ + \overset{M}{\nabla}_X CZ, JY) \\ &= -g_M(T_X BZ + h\overset{M}{\nabla}_X CZ, JY) \\ &= -\frac{1}{\lambda^2} g_N(\overset{N}{\nabla^F}_X F_*(CZ), F_*(JY)) + \frac{1}{\lambda^2} g_N((\nabla F_*)(X, CZ), F_*(JY)) - g_M(T_X BZ, JY). \end{aligned} \tag{25}$$

The proof is clear from (24) and (25). □

In a similar way, we obtain the following result.

Theorem 3.18. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). The distribution D_1 defines a totally geodesic foliation on M if and only if the following conditions are satisfied,*

i- $\hat{\nabla}_U BX + T_U CX \in D_2$,

ii- $g_N((\nabla F_*)(U, JV), F_*(JW)) = 0$,

for $U, V \in \Gamma(D_1)$, $X \in \Gamma((kerF_*)^\perp)$ and $W \in \Gamma(D_2)$.

Theorem 3.19. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). The distribution $kerF_*$ defines a totally geodesic foliation on M if and only if the following conditions are satisfied,*

i- $g_N((\nabla F_*)(U, JX), F_*(\omega V)) = g_N((\nabla F_*)(U, \phi V), F_*(JX)) + g_N(\nabla^F{}_U F_*(JX), F_*(\omega V))$,

ii- $\frac{1}{\lambda^2} g_N((\nabla F_*)(U, Z), F_*(\omega V)) = g_M(\hat{\nabla}_U Z, \phi V)$,

for $U, V \in \Gamma(kerF_*)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma(\mu)$.

Proof. For $U, V \in \Gamma(kerF_*)$, $X \in \Gamma(\mu)$ and using (2), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, X) &= -g_M(T_U JX, \phi V) - g_M(h\overset{M}{\nabla}_U JX, \omega V) \\ &= \frac{1}{\lambda^2}\{g_N((\nabla F_*)(U, JX), F_*(\omega V)) - g_N(\overset{N}{\nabla^F}_U F_*(JX), F_*(\omega V))\} - g_M(T_U JX, \phi V). \end{aligned}$$

At last equation, because of tensor field T is anti-symmetric, we get

$$g_M(\overset{M}{\nabla}_U V, X) = \frac{1}{\lambda^2}\{g_N((\nabla F_*)(U, JX), F_*(\omega V)) - g_N(\overset{N}{\nabla^F}_U F_*(JX), F_*(\omega V)) - g_N((\nabla F_*)(U, \phi V), F_*(JX))\}. \tag{26}$$

Similarly, for $U, V \in \Gamma(kerF_*)$, $Z \in \Gamma(D_2)$ and using (2), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, JZ) &= g_M(T_U Z, \omega V) + g_M(\hat{\nabla}_U Z, \phi V) \\ &= g_M(\hat{\nabla}_U Z, \phi V) - \frac{1}{\lambda^2} g_N((\nabla F_*)(U, Z), F_*(\omega V)). \end{aligned} \tag{27}$$

From (26) and (27), we get the proof. □

For the distribution $(kerF_*)^\perp$, we have the following result.

Theorem 3.20. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal semi-invariant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). The distribution $(kerF_*)^\perp$ defines a totally geodesic foliation on M if and only if the following conditions are satisfied,*

i- $\frac{1}{\lambda^2} g_N((\nabla F_*)(X, JV), F_*(CY)) = g_M(\hat{\nabla}_X JV, BY)$,

ii- $\frac{1}{\lambda^2} g_N((\nabla F_*)(X, JW), F_*(CY)) = \frac{1}{\lambda^2} g_N(\overset{N}{\nabla^F}{}_X F_*(JW), F_*(CY)) + g_M(\mathcal{A}_X JW, BY)$,

for $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$.

Proof. For $X, Y \in \Gamma((kerF_*)^\perp)$, $V \in \Gamma(D_1)$ and using (2), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, V) &= -g_M(\mathcal{A}_X JV, CY) - g_M(\hat{\nabla}_X JV, BY) \\ &= \frac{1}{\lambda^2} g_N((\nabla F_*)(X, JV), F_*(CY)) - g_M(\hat{\nabla}_X JV, BY). \end{aligned} \tag{28}$$

Similarly, for $X, Y \in \Gamma((kerF_*)^\perp)$, $W \in \Gamma(D_2)$ and using (2), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, W) &= -g_M(\mathcal{A}_X JW, BY) - g_M(h\overset{M}{\nabla}_X JW, CY) \\ &= \frac{1}{\lambda^2} g_N((\nabla F_*)(X, JW), F_*(CY)) - g_M(\mathcal{A}_X JW, BY) - \frac{1}{\lambda^2} g_N(\overset{N}{\nabla^F}{}_X F_*(JW), F_*(CY)). \end{aligned} \tag{29}$$

From (28) and (29), we get the proof. □

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