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Solutions of Some Types of Soliton Equations in $\mathbb{R}^3$

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Abstract. Solutions of ome types of soliton equations in the 3-dimensional Euclidean space are given and some examples are provided.

1. Introduction

Let g be a Riemannian metric on the n-dimensional manifold M, Ric its Ricci curvature tensor field and r the scalar curvature of M. Fix a vector field V and a 1-form η on M. If there exist two smooth functions λ and μ on M such that

$$\frac{1}{2}\pounds_V g + Ric + \lambda g + \mu\eta \otimes \eta = 0, \tag{1}$$

where $\pounds_V$ denotes the Lie derivative in the direction of V, then the data (V, η, λ, μ) is called an *almost η-Ricci soliton* on (M, g) [2]; in particular, if λ and μ are constants, then (V, η, λ, μ) is an *η-Ricci soliton* [3], if $\mu = 0$, (V, λ) is an *almost Ricci soliton* [6], respectively a *Ricci soliton* [5] if λ is a function, respectively a constant. The soliton is called *shrinking*, *steady* or *expanding* according as λ is negative, zero or positive, respectively [4]. If the potential vector field V is of gradient type, $V = grad(f)$, for f a smooth function on M, then (V, η, λ, μ) is called a *gradient almost η-Ricci soliton*.

Similarly, if there exist two smooth functions λ and μ on M such that

$$\frac{1}{2}\pounds_V g + (\lambda - r)g + \mu\eta \otimes \eta = 0, \tag{2}$$

then the data (V, η, λ, μ) is called an *almost quasi-Yamabe soliton* on (M, g) [1].

2. Almost η-solitons in $\mathbb{R}^3$

Consider the 3-dimensional Euclidean space $(\mathbb{R}^3, g := \sum_{i=1}^{3} dx^i \otimes dx^i)$, where $\{x^1, x^2, x^3\}$ denotes the local coordinates. Then the components of the Levi-Civita connection, of the Ricci tensor field, and the scalar curvature all vanish. In this case, the equations (1) and (2) coincide and, in what follows, we shall call the data (V, η, λ, μ) an almost η-soliton.

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Let $V = \sum_{i=1}^{3} V^i \frac{\partial}{\partial x^i}$ and $\eta = \sum_{i=1}^{3} \eta^i dx^i$ with V^i and η^i smooth functions on $\mathbb{R}^3$.

Replacing $\pounds_V g(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ in (1) and computing it in $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ we obtain:

$$\frac{1}{2}\left[g\left(\nabla_{\frac{\partial}{\partial x^i}} V, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^j}} V\right)\right] + \lambda \delta^{ij} + \mu \eta^i \eta^j = 0, \ (\forall)\ i, j \in \{1,2,3\}$$

equivalent to

$$\frac{1}{2}\left(\frac{\partial V^j}{\partial x^i} + \frac{\partial V^i}{\partial x^j}\right) + \lambda \delta^{ij} + \mu \eta^i \eta^j = 0, \ (\forall)\ i, j \in \{1,2,3\}, \tag{3}$$

for δ^{ij} the Kronecker symbol.

Theorem 2.1. *The system (3) is verified by the functions V^i and η^i given by:*

$$V^i(x^1,x^2,x^3) = c' x^i + c'' f(x^1,x^2,x^3) \tag{4}$$

$$\eta^i(x^1,x^2,x^3) = c''' \sqrt{c'} \tag{5}$$

with $c' \in \mathbb{R}_+^*$, $c'' \in \mathbb{R}$, $c''' \in \mathbb{R}^*$ and $f : \mathbb{R}^3 \to \mathbb{R}$ one of the following functions:

$$f(x^1,x^2,x^3) \in \{\sin(x^1+x^2+x^3), \cos(x^1+x^2+x^3), \sinh(x^1+x^2+x^3), \cosh(x^1+x^2+x^3), e^{x^1+x^2+x^3}\}.$$

In this case,

$$(V,\eta,\lambda,\mu) = \left(\sum_{i=1}^{3}(c'\pi^i + c'' f)\frac{\partial}{\partial x^i}, c''' \sqrt{c'} \sum_{i=1}^{3} dx^i, -c' - c'' \frac{\partial f}{\partial x^{i_0}} - \mu c'(c''')^2, \mu\right) \tag{6}$$

define a gradient almost η-soliton on $(\mathbb{R}^3, g)$, where $\pi^i : \mathbb{R}^3 \to \mathbb{R}$, $\pi^i(x^1,x^2,x^3) := x^i$, $i \in \{1,2,3\}$. Moreover, $V = grad(h)$ with the potential function $h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(x^1,x^2,x^3) := \frac{c'}{2}\sum_{i=1}^{3}(x^i)^2 + c''\bar{f}(x^1,x^2,x^3) + c''''$, $c'''' \in \mathbb{R}$ and $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$ with the property $\frac{\partial \bar{f}}{\partial x^i} = f$, $i \in \{1,2,3\}$.

Remark 2.2. *For the gradient almost η-soliton (6):*

i) If $f = c \in \mathbb{R}$ is a constant, then

$$(V,\eta,\lambda,\mu) = \left(\sum_{i=1}^{3}(c'\pi^i + c)\frac{\partial}{\partial x^i}, c''' \sqrt{c'} \sum_{i=1}^{3} dx^i, -c'[1 + \mu(c''')^2], \mu\right) \tag{7}$$

define an almost η-soliton on $(\mathbb{R}^3, g)$ with $c' \in \mathbb{R}_+^*$, $c''' \in \mathbb{R}^*$, where $\pi^i : \mathbb{R}^3 \to \mathbb{R}$, $\pi^i(x^1,x^2,x^3) := x^i$, $i \in \{1,2,3\}$.

If $\mu = -\frac{1}{(c''')^2}$, the soliton is steady, if $\mu > -\frac{1}{(c''')^2}$, the soliton is shrinking and if $\mu < -\frac{1}{(c''')^2}$, the soliton is expanding.

ii) If $\mu = 0$ we obtain the almost Ricci soliton

$$(V,\lambda) = \left(\sum_{i=1}^{3}(c'\pi^i + c'' f)\frac{\partial}{\partial x^i}, -c' - c'' \frac{\partial f}{\partial x^{i_0}}\right) \tag{8}$$

and in particular, a shrinking Ricci soliton given by $(V,\lambda) = \left(\sum_{i=1}^{3}(c'\pi^i + c)\frac{\partial}{\partial x^i}, -c'\right)$, $c' \in \mathbb{R}_+^*$, $c \in \mathbb{R}$.

Proposition 2.3.

$$(V,\eta,\lambda,\mu) = \left(\sum_{1\le i\le 3, i\ne i_0} c_i \frac{\partial}{\partial x^i} + f\frac{\partial}{\partial x^{i_0}}, \eta^{i_0} dx^{i_0}, -f' - \mu(\eta^{i_0})^2, \mu\right) \tag{9}$$

define a gradient almost η-soliton on $(\mathbb{R}^3, g)$, where η^{i_0} is a nowhere zero function and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function depending only on x^{i_0}. Moreover, $V = grad(h)$ with the potential function $h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(x^1,x^2,x^3) := \sum_{1\le i\le 3, i\ne i_0} c_i x^i + \bar{f}(x^{i_0}) + c$, $c \in \mathbb{R}$ and $\bar{f} : \mathbb{R}^3 \to \mathbb{R}$ a smooth function that depends only on x^{i_0}, with the property $\frac{\partial \bar{f}}{\partial x^{i_0}} = f$.

Remark 2.4. *For the gradient almost η-soliton given by (9):*

i) If $\mu = -\frac{f'}{(\eta^{i_0})^2}$, the soliton is steady, if $\mu > -\frac{f'}{(\eta^{i_0})^2}$, the soliton is shrinking and if $\mu < -\frac{f'}{(\eta^{i_0})^2}$, the soliton is expanding.

ii) If $\mu = 0$ we obtain the almost Ricci soliton

$$(V, \lambda) = \left(\sum_{1 \le i \le 3, i \ne i_0} c_i \frac{\partial}{\partial x^i} + f \frac{\partial}{\partial x^{i_0}}, -f' \right) \tag{10}$$

which is steady if f is a constant map, shrinking if f is strictly increasing, and expanding if f is strictly decreasing.

iii) If f is a constant, then

$$(V, \eta, \lambda, \mu) = \left(\sum_{i=1}^{3} c_i \frac{\partial}{\partial x^i}, \eta^{i_0} dx^{i_0}, -\mu(\eta^{i_0})^2, \mu \right) \tag{11}$$

define an almost η-soliton on $(\mathbb{R}^3, g)$ with $c_i \in \mathbb{R}$ at least one of them nonzero and η^{i_0} a nowhere zero function.

In this case, if $\mu = 0$, we have a steady Ricci soliton, if $\mu > 0$, the soliton is shrinking and if $\mu < 0$, the soliton is expanding.

Proposition 2.5.

$$(V, \eta, \lambda, \mu) = \left(\sum_{1 \le i \le 3, i \ne i_0, i \ne i_1} c_i \frac{\partial}{\partial x^i} + f_0 \frac{\partial}{\partial x^{i_0}} + f_1 \frac{\partial}{\partial x^{i_1}}, \eta^{i_0} dx^{i_0}, -f_0' + \frac{\eta^{i_0}}{2\eta^{i_1}} f_1', -\frac{1}{2\eta^{i_0}\eta^{i_1}} f_1' \right) \tag{12}$$

define a gradient almost η-soliton on $(\mathbb{R}^3, g)$, where η^{i_0} is a nowhere zero function and f_0, $f_1 : \mathbb{R} \to \mathbb{R}$ are two smooth functions depending only on x^{i_0} and x^{i_1}, respectively. Moreover, $V = grad(h)$ with the potential function $h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(x^1, x^2, x^3) := \sum_{1 \le i \le 3, i \ne i_0, i \ne i_1} c_i x^i + \bar{f}_0(x^{i_0}) + \bar{f}_1(x^{i_1}) + c$, $c \in \mathbb{R}$ and $\bar{f}_0$, $\bar{f}_1 : \mathbb{R}^3 \to \mathbb{R}$ two smooth functions that depend only on x^{i_0} and x^{i_1}, respectively, with the property $\frac{\partial \bar{f}_0}{\partial x^{i_0}} = f_0$ and $\frac{\partial \bar{f}_1}{\partial x^{i_1}} = f_1$.

Remark 2.6. *For the gradient almost η-soliton (12):*

i) If $f_0' = \frac{\eta^{i_0}}{2\eta^{i_1}} f_1'$, the soliton is steady, if $f_0' > \frac{\eta^{i_0}}{2\eta^{i_1}} f_1'$, the soliton is shrinking and if $f_0' < \frac{\eta^{i_0}}{2\eta^{i_1}} f_1'$, the soliton is expanding.

ii) If f_1 is a constant, then

$$(V, \lambda) = \left(\sum_{1 \le i \le 3, i \ne i_0} c_i \frac{\partial}{\partial x^i} + f_0 \frac{\partial}{\partial x^{i_0}}, -f_0' \right) \tag{13}$$

define an almost Ricci soliton on $(\mathbb{R}^3, g)$ with f_0 a nonzero function or at least one of $c_i \in \mathbb{R}$ nonzero. In this case, the soliton is steady if f_0 is a constant map, shrinking if f_0 is strictly increasing, and expanding if f_0 is strictly decreasing.

Remark 2.7. *Assume that $\sum_{i=1}^{3} (\eta^i)^2 \ne 0$ everywhere. Then the compatibility conditions of the system (3) in λ and μ are:*

$$\begin{cases} (i)\ (\eta^1)^2 \left(\frac{\partial V^2}{\partial x^2} - \frac{\partial V^3}{\partial x^3} \right) + (\eta^2)^2 \left(\frac{\partial V^3}{\partial x^3} - \frac{\partial V^1}{\partial x^1} \right) + (\eta^3)^2 \left(\frac{\partial V^1}{\partial x^1} - \frac{\partial V^2}{\partial x^2} \right) = 0 \\ (ii)\ ((\eta^i)^2 - (\eta^j)^2) \left(\frac{\partial V^k}{\partial x^i} + \frac{\partial V^i}{\partial x^k} \right) - 2\eta^k \eta^i \left(\frac{\partial V^i}{\partial x^i} - \frac{\partial V^j}{\partial x^j} \right) = 0 \\ (iii)\ \eta^l \eta^j \left(\frac{\partial V^j}{\partial x^t} + \frac{\partial V^t}{\partial x^j} \right) - \eta^j \eta^t \left(\frac{\partial V^l}{\partial x^j} + \frac{\partial V^j}{\partial x^l} \right) = 0 \end{cases} \tag{14}$$

for any $i, j, k, l, t \in \{1, 2, 3\}$ with $i \ne k$, $l \ne j$ and $t \ne j$.

If there exists $i_1 \in \{1, 2, 3\}$ such that $\eta^{i_1} \ne 0$ everywhere and $\eta^i = 0$, for any $i \in \{1, 2, 3\}$, $i \ne i_1$, then:

$$\frac{\partial V^i}{\partial x^i} = \frac{\partial V^j}{\partial x^j}, \text{ for any } i \ne i_1, j \ne i_1,$$

$$\frac{\partial V^{i_1}}{\partial x^j}=-\frac{\partial V^j}{\partial x^{i_1}},\ \textit{for any}\ \ j\neq i_1$$

and

$$\lambda+\mu(\eta^{i_1})^2=-\frac{\partial V^{i_1}}{\partial x^{i_1}}.$$

If there exists $i_2\in\{1,2,3\}$, $i_2\neq i_1$ such that $\eta^{i_2}\neq 0$ everywhere and $\eta^i=0$, for $i\neq i_1$, $i\neq i_2$, then:

$$\frac{\partial V^{i_1}}{\partial x^i}=-\frac{\partial V^i}{\partial x^{i_1}},\quad \frac{\partial V^{i_2}}{\partial x^i}=-\frac{\partial V^i}{\partial x^{i_2}},\quad (\eta^{i_1})^2\left(\frac{\partial V^{i_2}}{\partial x^{i_2}}-\frac{\partial V^i}{\partial x^i}\right)=(\eta^{i_2})^2\left(\frac{\partial V^{i_1}}{\partial x^{i_1}}-\frac{\partial V^i}{\partial x^i}\right),$$

$$2\left(\frac{\partial V^{i_1}}{\partial x^{i_1}}-\frac{\partial V^{i_2}}{\partial x^{i_2}}\right)=\left(\frac{\eta^{i_1}}{\eta^{i_2}}-\frac{\eta^{i_2}}{\eta^{i_1}}\right)\left(\frac{\partial V^{i_1}}{\partial x^{i_2}}+\frac{\partial V^{i_2}}{\partial x^{i_1}}\right),\quad 2\left(\frac{\partial V^{i_1}}{\partial x^{i_1}}-\frac{\partial V^i}{\partial x^i}\right)=\frac{\eta^{i_1}}{\eta^{i_2}}\left(\frac{\partial V^{i_1}}{\partial x^{i_2}}+\frac{\partial V^{i_2}}{\partial x^{i_1}}\right)$$

and

$$\begin{cases}\lambda=-\frac{\partial V^{i_1}}{\partial x^{i_1}}+\frac{\eta^{i_1}}{2\eta^{i_2}}\left(\frac{\partial V^{i_1}}{\partial x^{i_2}}+\frac{\partial V^{i_2}}{\partial x^{i_1}}\right)\\ \mu=-\frac{\frac{\partial V^{i_1}}{\partial x^{i_2}}+\frac{\partial V^{i_2}}{\partial x^{i_1}}}{2\eta^{i_1}\eta^{i_2}}\end{cases}.$$

We end these considerations by giving two families of examples of almost η-Ricci solitons in $\mathbb{R}^3$.

Example 2.8. *Let $\tilde{g}:=k_1dx^1\otimes dx^1+k_2dx^2\otimes dx^2+dx^3\otimes dx^3$ with $k_1,k_2\in\mathbb{R}$ be a Riemannian metric on $\mathbb{R}^3$. Then:*
i) there exists no almost η-Ricci soliton (with η nontrivial) having the potential vector field $V:=\frac{\partial}{\partial x^3}$;
ii) if $k_1=k_2=:\frac{1}{c^2}$, $c\in\mathbb{R}_+^*$, for $\eta:=dx^3$ and

$$\begin{cases}V^1(x^1,x^2,x^3)=c_1x^1+c_2x^2+h_1(x^3)\\ V^2(x^1,x^2,x^3)=-c_2x^1+c_1x^2+h_2(x^3)\\ V^3(x^1,x^2,x^3)=-\frac{x^1}{c}h_1'(x^3)-\frac{x^2}{c}h_2'(x^3)+h_3(x^3)\end{cases}$$

with $h_i:\mathbb{R}\to\mathbb{R}$, $i\in\{1,2,3\}$, smooth functions depending only on x^i, respectively, and c_1, $c_2\in\mathbb{R}$, then the data:

$$(V,\eta,\lambda,\mu)=\left(\sum_{i=1}^{3}V^i\frac{\partial}{\partial x^i},dx^3,-cc_1,cc_1+\frac{x^1}{c}h_1''+\frac{x^2}{c}h_2''-h_3'\right)\tag{15}$$

define an almost η-Ricci soliton which is shrinking if $c_1>0$, steady if $c_1=0$ and expanding if $c_1<0$.

Example 2.9. *For $f:I\subset\mathbb{R}^3\to\mathbb{R}$ one of the following functions*

$$f(x^1,x^2,x^3)\in\{\sin(x^1+x^2+x^3),\cos(x^1+x^2+x^3),\sinh(x^1+x^2+x^3),\cosh(x^1+x^2+x^3),e^{x^1+x^2+x^3}\}$$

with f nowhere zero on the domain I, the almost η-Ricci soliton (V,η,λ,μ) on I with the induced metric by $\tilde{g}:=\frac{1}{f^2}dx^1\otimes dx^1+\frac{1}{f^2}dx^2\otimes dx^2+dx^3\otimes dx^3$, for $\eta:=dx^3$ and $V:=\frac{\partial}{\partial x^3}$, is given by

$$\lambda=-\mu=\frac{1}{f}\left[\frac{\partial^2 f}{\partial(x^3)^2}-\frac{\partial f}{\partial x^3}-2\frac{1}{f}\cdot\left(\frac{\partial f}{\partial x^3}\right)^2\right].$$

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