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Application of the Geometry of Curves in Euclidean Space

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Abstract. In this article the so called induced spin velocities are studied, and it is an improvement of the paper [13] using the geometry of curves in 3-dimensional Euclidean space. Some essential properties of them are given, and they are rather different than the ordinary velocities. Indeed, the induced spin velocities are non-inertial and instead of the Lorentz transformations for them the Galilean transformations should be used. The induced spin velocity is derived in terms of the curvature and torsion of the trajectory. Two applications of the induced spin velocities are studied.

1. Preliminaries for high-dimensional space-time

Henri Poincare and Albert Einstein almost one century ago thought about a 3-dimensional time, in order the space and time would be of the same dimension. At present time this idea appears again and some of the authors [1–9] propose multidimensional time from different targets. This paper is a continuation of the papers [10–14], and improvement of [13]. The gravitation in multidimensional space-time is recently published in [15].

Let us denote by x, y and z the coordinates in $\mathbb{R}^3$. The bundle of all moving orthonormal frames, can be parameterized by the following 9 coordinates $x, y, z, x_s, y_s, z_s, x_t, y_t, z_t$, where the first 6 coordinates parameterize the subbundle with the fiber $SO(3,\mathbb{R})$. So it is called 3+3+3-dimensional model [10–14]. Indeed, to each body are related 3 coordinates for the position, 3 coordinates for the spatial rotation and 3 coordinates for the velocity. This 3+3+3-model is built on three 3-dimensional sets: space (S) which is homeomorphic to S^3, spatial rotations (SR) which is also homeomorphic to S^3 and velocity (V) which is homeomorphic to $\mathbb{R}^3$.

2. Research of the 6-dimensional space $SR \times S$

It is known that the Lorentz group $O_+^\uparrow(1,3)$ is isomorphic to $SO(3,\mathbb{C})$, and both of them are homeomorphic to $SR \times V \cong SO(3,\mathbb{R}) \times \mathbb{R}^3$. Instead of these real 6×6-matrices we are interest now for the product $S \times SR$, which can be considered as a fiber and Lie group G of a principal bundle over the base V. So we consider this group for a fixed inertial coordinate system up to a translation and spatial rotation and the coefficient

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$\sqrt{1-\frac{v^2}{c^2}}$ will not have any role. This group is analogous to the group of all rotations and translations in the 3-dimensional Euclidean space. The Lie algebra of G is given by

$$\begin{bmatrix} C & B \\ B & C \end{bmatrix}. \tag{1}$$

where B and C are antisymmetric 3×3 matrices.

The group G is isomorphic to the group $Spin(4)$ [14]. While the Lorentz group reduces to the group of Galilean transformations when the velocities are small, the transformations of the group G reduce to the group of all rotations and translations in the Euclidean space in case of short translations, i.e. matrices of type $\begin{bmatrix} M & \vec{h}^T \\ 0 & 1 \end{bmatrix}$, where $M \in SO(3,\mathbb{R})$ and $\vec{h}^T$ is the vector of translation.

If a rigid body is spinning, there may appear a constraint for the spatial rotation, because there is no freedom of a chosen point to rotate according to its own trajectory. As a consequence there may appear a displacement, which is called spin displacement, because it appears in case of spinning bodies. The property of conversion from unadmitted spatial rotation into spatial displacement is basic property of the space. This displacement induces the so called *induced spin velocity* or simply *spin velocity* ([13]) and will be denoted by large letter V. In section 3 a precise introduction of this motion will be given, and it will be applied in two examples in section 4.

The spin motion (displacement) has the following properties.

i) The spin velocity is non-inertial, because it can be conceived just like a displacement in the space.

ii) Instead the Lorentz transformations for these velocities we may use only the Galilean transformations and the coefficient $\sqrt{1-\frac{V^2}{c^2}}$ does not appear.

iii) If the spin velocity of any point is constrained completely or partially, then the constrained part converts into inertial velocity with opposite sign.

3. Spinning bodies in gravitational field

Let us consider a trajectory over a spinning sphere, which rests in our coordinate system, but it is under the gravitational acceleration or any mechanical force. We assume that the barycentre is at the coordinate origin, that at the initial moment the spin axis is determined by $\vec{b}^* = (0,0,1)$ and at the initial moment the considered point has coordinates $(r\cos\alpha, r\sin\alpha, h)$.

In order to calculate the spin velocity we will use the group of affine transformations in 3-dimensional Euclidean space. Its Lie algebra has the following form

$$A = \begin{bmatrix} 0 & -\varphi_z & \varphi_y & s_x \\ \varphi_z & 0 & -\varphi_x & s_y \\ -\varphi_y & \varphi_x & 0 & s_z \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

where $\vec{\varphi} = (w_x, w_y, w_z)t$, $\vec{w}$ is the angular velocity of the sphere, t is short time and $\vec{s} = (g_x, g_y, g_z)t^2/2$ is small translation as a consequence of the acceleration $\vec{g}$. The quantities $\vec{\varphi}$ and $\vec{s}$ may dependent on time, so we use the Taylor series. Since $\vec{\varphi}(0) = 0$, $\vec{s}(0) = 0$, and $\vec{s}'(0) = 0$ we obtain

$$\vec{\varphi}(t) = \vec{\varphi}(0) + \vec{\varphi}'(0)\frac{t}{1!} + \vec{\varphi}''(0)\frac{t^2}{2!} + \cdots = \vec{w}t + \vec{w}'\frac{t^2}{2!} + \vec{w}''\frac{t^3}{3!} + \cdots$$

and

$$\vec{s}(t) = \vec{s}(0) + \vec{s}'(0)\frac{t}{1!} + \vec{s}''(0)\frac{t^2}{2!} + \cdots = \vec{g}\frac{t^2}{2} + \vec{g}'\frac{t^3}{6} + \cdots.$$

After these replacements into (2) the required trajectory is determined by the matrix $I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$. Then the image $(x(t), y(t), z(t))$ of the starting vector $(r\cos\alpha, r\sin\alpha, h)$, where α is an arbitrary parameter of the circle, is given by the equality

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ 1 \end{bmatrix} = (I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots) \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \\ h \\ 1 \end{bmatrix}. \tag{3}$$

Hence $\vec{r} = (x(t), y(t), z(t))$ is well defined, and the first three derivatives are

$$\vec{r}' = (-w\sin\alpha, w\cos\alpha, 0)r, \tag{4}$$

$$\vec{r}'' = (-rw^2\cos\alpha - rw_z'\sin\alpha + hw_y' + g_x, -rw^2\sin\alpha + rw_z'\cos\alpha - hw_x' + g_y, -rw_y'\cos\alpha + rw_x'\sin\alpha + g_z), \tag{5}$$

$$\begin{aligned} \vec{r}''' &= (-3rww_z'\cos\alpha + rw^3\sin\alpha - w_z''r\sin\alpha + hw_y'' + \frac{3}{2}hww_x', \\ &\quad -3rww_z'\sin\alpha - rw^3\cos\alpha + w_z''r\cos\alpha - hw_x'' + \frac{3}{2}hww_y', \\ &(-w_y'' + \frac{3}{2}ww_x')r\cos\alpha + (w_x'' + \frac{3}{2}ww_y')r\sin\alpha) - \frac{3}{2}(\vec{g}\times\vec{w}) + \vec{g}'. \end{aligned} \tag{6}$$

Any point of the spinning sphere intends to move in its own osculating plane, orthogonal to the binormal vector $\vec{b}$, but as a part of the sphere at the chosen moment all points will move in the plane which is orthogonal to the vector $\vec{b}^*$. Note that in general case $\vec{b} \neq \vec{b}^*$. We will assume further that $\frac{d\vec{b}^*}{dt} << w$.

If there are no constraints, the Frenet antisymmetric matrix

$$\begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} ds \tag{7}$$

corresponds to angular rotation of the trihedron $(\vec{t}, \vec{n}, \vec{b})$ by ([16], sec.28)

$$\tau\vec{t}ds + k\vec{b}ds. \tag{8}$$

One can explain why arbitrary point of the considered trajectories over the sphere $r^2 + h^2 = const.$ tends to rotate with accordance to the rotation of the trihedron $(\vec{t}, \vec{n}, \vec{b})$, but we omit this discussion.

Assume that the considered point, which moves on the considered trajectory, may be displaced without constraint. There may exist different approaches in determining the spin velocity, but all of them have the same approximation when $\tau << k$. In [13] is given one such procedure. In this article we deduce the spin velocity which is free of any intuition and it is based on the following two invariants ([13]). Analogous to the invariant $dx^2 + dy^2 + dz^2 - c^2dt^2$ in the Special Relativity, in the space $SR \times S$ there exist two invariants ([13])

$$I_1 = (d\vec{\eta})^2 + (d\vec{\xi})^2, \qquad I_2 = d\vec{\eta} \cdot d\vec{\xi}, \tag{9}$$

where $d\vec{\eta}$ is vector of spatial displacement caused by the space (i.e. translation), while $d\vec{\xi}$ is displacement caused by the rotation given by (8). The property that they are invariant means that they are unchanged independently whether there is a constraint, or there doesn't exist a constraint. Now we have the following theorem.

Theorem. *The induced spin velocity of arbitrary point on a spinning sphere, whose center rests in our coordinate system, is given by*

$$\vec{V} = -\frac{\tau k}{k^2 + \tau^2} rw\vec{b} - \frac{\tau^2}{k^2 + \tau^2} rw\vec{t}, \tag{10}$$

where w is the angular velocity and r is distance to the axis of the sphere.

Proof. The vector of displacement caused by the rotation for angle (8) is orthogonal with the unit vector $-\vec{n}$ and the vector (8) and hence

$$d\vec{\xi} = \mu[(\tau\vec{t} + k\vec{b}) \times (-\vec{n})ds] = \mu(-\tau\vec{b} + k\vec{t})ds.$$

The vector of displacement caused by rotation (8) where $\tau \approx 0$ since $d\vec{b^*}/dt << w$ is $d\vec{\xi}_{con.} = \vec{t}ds$, while $d\vec{\eta}_{con.} = 0$ because the center of the spinning sphere rests. According to (9) we have the following system

$$(d\vec{\eta})^2 + (\mu(-\tau\vec{b} + k\vec{t})ds)^2 = (\vec{t}ds)^2,$$

$$(d\vec{\eta}) \cdot (\mu(-\tau\vec{b} + k\vec{t})ds) = 0.$$

Using also that

$$(\mu(-\tau\vec{b} + k\vec{t})ds) + (\vec{\eta}ds) = \vec{t}ds,$$

one can easily obtain that $\mu = k/(k^2 + \tau^2)$. The spin velocity $\vec{V}$ is indeed the vector $\frac{d\vec{\xi}}{dt} - \frac{d\vec{\xi}_{con.}}{dt} = \frac{d\vec{\eta}_{con.}}{dt} - \frac{d\vec{\eta}}{dt}$ and now using that $ds = rwdt$, it is given by (10). □

According to (10) we notice that *i)* $|\vec{V}| = |\frac{\tau}{\sqrt{k^2+\tau^2}} rw| \leq |rw|$, *ii)* $|\vec{b} \cdot \vec{V}| \leq |rw|/2$, and *iii)* $\vec{V}$ is collinear with the vector of rotation (8). If $\lambda = \tau/k$, then the spin velocity becomes

$$\vec{V} = -\frac{\lambda}{1+\lambda^2} rw\vec{b} - \frac{\lambda^2}{1+\lambda^2} rw\vec{t}. \tag{11}$$

4. Applications of the spin velocities

The spin velocity in case of a homogeneous spinning circle will be calculated. The components $-\frac{\lambda}{1+\lambda^2} rw\vec{b}$ and $-\frac{\lambda^2}{1+\lambda^2} rw\vec{t}$ should be calculated and then they should be averaged for $\alpha \in [-\pi, \pi]$. These two components will be denoted respectively by $\vec{V}_b$ and $\vec{V}_t$. Since $\vec{V}_t$ is collinear to $\vec{t}$, the averaging

$$\langle \vec{V}_t/(r\vec{t}) \rangle = (-w)\langle \frac{\lambda^2}{1+\lambda^2} \rangle = -\frac{w}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda^2 d\alpha}{1+\lambda^2}$$

leads to the change of the angular velocity of the spinning body. On the other hand, since $\vec{V}_b$ is orthogonal to $\vec{t}$, the averaging

$$\langle \vec{V}_b \rangle = \frac{-rw}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda\vec{b}}{1+\lambda^2} d\alpha$$

leads to the global spin velocity of the spinning body.

Example 1. Let us consider a spinning circle where $\vec{b^*}$ is a constant vector, and only $w = |\vec{w}|$ is a function of t. We use the formulas (4), (5) and (6) where $\vec{b^*} = (0, 0, 1)$. Using that $w'_x = w'_y = w''_x = w''_y = 0$, $w'_z = w'$ and $w''_z = w''$, we obtain

$$\vec{r}' = wr(-\sin\alpha, \cos\alpha, 0),$$

$$\vec{r}'' = -rw^2(\cos\alpha, \sin\alpha, 0) + rw'(-\sin\alpha, \cos\alpha, 0) + \vec{g},$$

$$\vec{r}''' = (rw^3 - rw'')(\sin\alpha, -\cos\alpha, 0) - 3rww'(\cos\alpha, \sin\alpha, 0) - \frac{3}{2}w(\vec{g} \times \vec{b^*}) + \frac{d\vec{g}}{dt}.$$

Further we obtain

$$\vec{r}' \times \vec{r}''' = 3r^2w^2w'\vec{b^*} + \frac{3}{2}rw^2(-g_x \sin\alpha + g_y \cos\alpha)\vec{b^*} + wr(-\sin\alpha, \cos\alpha, 0) \times \frac{d\vec{g}}{dt},$$

$$(\vec{r}',\vec{r}'',\vec{r}''') = -\frac{3}{2}rw^2(2rw' - g_x\sin\alpha + g_y\cos\alpha)(\vec{g}\cdot\vec{b}^*)+$$

$$+rw\Big[rw^2\Big(\vec{b}^*\cdot\frac{d\vec{g}}{dt}\Big) + (g_z\cos\alpha, g_z\sin\alpha, -g_y\sin\alpha - g_x\cos\alpha)\cdot\frac{d\vec{g}}{dt}\Big],$$

$$\vec{r}'\times\vec{r}'' = r^2w^3\vec{b}^* + wr(g_z\cos\alpha, g_z\sin\alpha, -g_y\sin\alpha - g_x\cos\alpha).$$

Let us put $g_z = g\cos\varphi$, $g_x = g\sin\varphi$ and assume that $g_y = 0$. Then for

$$-\lambda wr\vec{b} = -wr\frac{\tau}{k}\vec{b} = -\frac{|\vec{r}'|^4(\vec{r}',\vec{r}'',\vec{r}''')}{|\vec{r}'\times\vec{r}''|^4}(\vec{r}'\times\vec{r}''),$$

we obtain

$$-\lambda wr\vec{b} = \Big\{\frac{\frac{3}{2}rw^2(2rw' - g\sin\varphi\sin\alpha)g\cos\varphi}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2} -$$

$$\frac{rw\Big[rw^2\Big(\vec{b}^*\cdot\frac{d\vec{g}}{dt}\Big) + g(\cos\varphi\cos\alpha, \cos\varphi\sin\alpha, -\sin\varphi\cos\alpha)\cdot\frac{d\vec{g}}{dt}\Big]}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2}\Big\}\cdot[r^2w^3\vec{b}^* + wr(g_z\cos\alpha, g_z\sin\alpha, -g_x\cos\alpha)].$$

The spin velocity can be decomposed into two components: radial component $\vec{V}_b\cdot(\cos\alpha,\sin\alpha,0)$ and axial component $\vec{V}_b\cdot\vec{b}^*$, while the tangential component $\vec{V}_b\cdot(-\sin\alpha,\cos\alpha,0)$ is equal to 0. It is of interest to calculate the averaging of $\vec{V}_b\cdot(\cos\alpha,\sin\alpha,0)$ and $\vec{V}_b\cdot\vec{b}^*$. Since the expressions are very large, we will replace approximately $\vec{V}_b$ by $-\lambda rw\vec{b}$, and however if we know the value of λ, the exact value of the required spin velocity can easily be calculated.

We will consider two cases.

a) Assume that beside the constant gravitational acceleration there appears also radial acceleration $\vec{g}_r$ and tangent acceleration $\vec{g}_t$, such that $\vec{g}_r = -g_r(\cos\alpha,\sin\alpha,0)$ and $\vec{g}_t = g_t(\sin\alpha,-\cos\alpha,0)$. Then

$$\vec{g}_r' + \vec{g}_t' = -(g_r' - g_tw)(\cos\alpha,\sin\alpha,0) - (g_rw + g_t')(-\sin\alpha,\cos\alpha,0).$$

It is easy to check that the component $(g_rw+g_t')(-\sin\alpha,\cos\alpha,0)$ has no influence to $\vec{V}$, while the component $-(g_r'-g_tw)(\cos\alpha,\sin\alpha,0)$ has influence to V. Now the required influence to the spin velocity is the following

$$\langle\vec{V}_b\cdot(\cos\alpha,\sin\alpha,0)\rangle \approx \frac{-1}{2\pi}\int_{-\pi}^{\pi}\frac{3rw^2g\cos\varphi(2rw' - g\sin\varphi\sin\alpha)(wrg\cos\varphi)d\alpha}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2}+$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r^2w^2(g\cos\varphi)(g(g_r' - g_tw)\cos\varphi)d\alpha}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r^2w^2g\cos^2\varphi(g(g_r' - g_tw) + 3rww'g)d\alpha}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) + 2grw^2\cos\alpha\sin\varphi]^2} =$$

$$= \frac{r}{2\pi}\theta\cos^2\varphi\frac{(g_r + \frac{3}{2}rw^2)' - g_tw}{g}\int_{-\pi}^{\pi}\frac{d\alpha}{[\theta^2 - 2\theta\cos\alpha\sin\varphi + \cos^2\varphi + \sin^2\varphi\cos^2\alpha]^2}$$

and

$$\langle\vec{V}_b\cdot\vec{b}^*\rangle \approx \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\frac{3}{2}rw^2g\cos\varphi(2rw' - g\sin\varphi\sin\alpha)(r^2w^3 - wrg\cos\alpha\sin\varphi)d\alpha}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2}+$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{rw(r^2w^3 - wrg\cos\alpha\sin\varphi)(g(g_r' - g_tw)\cos\varphi)d\alpha}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(r^2w^3 - wrg\cos\alpha\sin\varphi)[3r^2w^2w'g + rwg(g_r' - g_tw)]\cos\varphi d\alpha}{[r^2w^4 + g^2(\cos^2\varphi + \sin^2\varphi\cos^2\alpha) - 2grw^2\cos\alpha\sin\varphi]^2} =$$

$$= \frac{r}{2\pi}\frac{(g_r+\frac{3}{2}rw^2)'-g_tw}{g}\cos\varphi\int_{-\pi}^{\pi}\frac{(\theta^2-\theta\cos\alpha\sin\varphi)d\alpha}{[\theta^2-2\theta\cos\alpha\sin\varphi+\cos^2\varphi+\sin^2\varphi\cos^2\alpha]^2}.$$

In a special case, when the vectors $\vec{g}$ and $\vec{b}^*$ are collinear, such that $\cos\varphi=-1$, then if we put $\theta=\frac{rw^2}{g}$, we obtain

$$\langle\vec{V}_b\rangle = \frac{-\theta^2 r}{(1+\theta^2)^2}\frac{(g_r+\frac{3}{2}rw^2)'-g_tw}{g}\vec{b}^*.$$

Notice that in both cases the averaging is proportional with $\frac{(g_r+\frac{3}{2}rw^2)'-g_tw}{g}$. Having in mind that $g_t=rw'$, this term becomes simpler $\frac{(g_r+rw^2)'}{g}$. For example, the last formula for $\cos\varphi=-1$ becomes

$$\langle\vec{V}_b\rangle = \frac{-\theta^2 r}{(1+\theta^2)^2}\frac{(g_r+rw^2)'}{g}\vec{b}^*, \tag{12}$$

where g_r+rw^2 is the total radial acceleration toward the center, i.e. the centripetal acceleration and additional acceleration g_r toward the center.

Finally, note that the previous results can be experimentally verified, because when we measure the weight or acceleration of a spinning body, we measure the sum of the gravitational acceleration and $d\vec{V}_b/dt$.

b) Assume that $\frac{d\vec{g}}{dt}$ is collinear with $\vec{g}$. If $w'\neq 0$, then $g_t=rw'\neq 0$ has an influence to the spin velocity. According to case a) the influence of g_t is equivalent to $-\frac{1}{3}w'$. So, we neglect g_t and replace w' by $\frac{2}{3}w'$. If we replace $\frac{rw^2}{g}$ by θ, then for the averagings we obtain

$$\langle\vec{V}_b\cdot(\cos\alpha,\sin\alpha,0)\rangle \approx \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\frac{3}{2}rw^2g\cos\varphi(\frac{4}{3}rw'-g\sin\varphi\sin\alpha)(wrg\cos\varphi)d\alpha}{[r^2w^4+g^2(\cos^2\varphi+\sin^2\varphi\cos^2\alpha)-2grw^2\cos\alpha\sin\varphi]^2}+$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r^2w^2(-g\cos\varphi)(rw^2g'\cos\varphi)d\alpha}{[r^2w^4+g^2(\cos^2\varphi+\sin^2\varphi\cos^2\alpha)-2grw^2\cos\alpha\sin\varphi]^2}=$$

$$=\frac{-1}{2\pi}\int_{-\pi}^{\pi}\frac{r^2w^2g\cos^2\varphi(rw^2g'-2rww'g)d\alpha}{[r^2w^4+g^2(\cos^2\varphi+\sin^2\varphi\cos^2\alpha)-2grw^2\cos\alpha\sin\varphi]^2}=$$

$$=\frac{r}{4\pi}\frac{d\theta^2}{dt}\cos^2\varphi\int_{-\pi}^{\pi}\frac{d\alpha}{[\theta^2-2\theta\cos\alpha\sin\varphi+\cos^2\varphi+\sin^2\varphi\cos^2\alpha]^2},$$

and

$$\langle\vec{V}_b\cdot\vec{b}^*\rangle \approx \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\frac{3}{2}rw^2g\cos\varphi(\frac{4}{3}rw'-g\sin\varphi\sin\alpha)(r^2w^3-wrg\cos\alpha\sin\varphi)d\alpha}{[r^2w^4+g^2(\cos^2\varphi+\sin^2\varphi\cos^2\alpha)-2grw^2\cos\alpha\sin\varphi]^2}-$$

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{rw(r^2w^3-wrg\cos\alpha\sin\varphi)(rw^2g'\cos\varphi)d\alpha}{[r^2w^4+g^2(\cos^2\varphi+\sin^2\varphi\cos^2\alpha)-2grw^2\cos\alpha\sin\varphi]^2}=$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(r^2w^3-wrg\cos\alpha\sin\varphi)[2r^2w^2w'g-r^2w^3g']\cos\varphi d\alpha}{[r^2w^4+g^2(\cos^2\varphi+\sin^2\varphi\cos^2\alpha)-2grw^2\cos\alpha\sin\varphi]^2}=$$

$$=-\frac{r}{4\pi}\frac{d\theta^2}{dt}\cos\varphi\int_{-\pi}^{\pi}\frac{(\theta+\cos\alpha\sin\varphi)d\alpha}{[\theta^2-2\theta\cos\alpha\sin\varphi+\cos^2\varphi+\sin^2\varphi\cos^2\alpha]^2}.$$

Note that $\langle\vec{V}_b\rangle$ is a function of $\theta=\frac{rw^2}{g}$ and its derivative. For example with the mentioned approximation $\vec{V}_b\approx-\lambda rw\vec{b}$ one can obtain

$$\langle\vec{V}_b\rangle \approx \vec{b}^*\frac{r\theta}{2}\frac{d}{dt}\frac{1}{1+\theta^2}.$$

Example 2. Let us consider a spinning circle as a gyroscope, where w is a constant and the vector $\vec{b}^*$ rotates with a constant angular velocity Ω around the vertical axis, i.e. around the vector $\vec{g}$ and the angle between the vector $\vec{b}^*$ and the vertical axis is a constant angle φ. So we can write

$$\vec{b}^* = (a\cos\Omega t, a\sin\Omega t, c),$$

where $a = \sin\varphi$ and $c = \cos\varphi$. Using that $\vec{w} = w\vec{b}^* = w(a\cos\Omega t, a\sin\Omega t, c)$, where w is a constant, we obtain

$$\vec{w}' = \Omega wa(-\sin\Omega t, \cos\Omega t, 0), \quad \vec{w}'' = -\Omega^2 wa(\cos\Omega t, \sin\Omega t, 0).$$

These formulas should be replaced in the general formulas for arbitrary $\vec{b}^*$,

$$\vec{r}' = wr\vec{t}, \qquad \vec{r}'' = -rw^2(\vec{t}\times\vec{b}^*) + \vec{t}\cdot r(\vec{w}'\cdot\vec{b}^*) - \vec{b}^*(\vec{w}'\cdot\vec{t})r + \vec{g},$$

$$\vec{r}''' = -[rw^3 - r(\vec{w}''\cdot\vec{b}^*)]\vec{t} - 3rw(\vec{w}'\cdot\vec{b}^*)(\vec{t}\times\vec{b}^*) + \vec{b}^*(-(\vec{t}\cdot\vec{w}'') + \frac{3}{2}w[(\vec{t}\times\vec{b}^*)\cdot\vec{w}'])r - \frac{3}{2}w(\vec{g}\times\vec{b}^*) + \frac{d\vec{g}}{dt}.$$

Note that $\vec{w}'\cdot\vec{b}^* = 0$, because $w = const$. The unit tangent vector $\vec{t}$, which is orthogonal to $\vec{b}^*$ at the initial moment $t = 0$ can be parameterized by $\vec{t} = (-c\sin\alpha, \cos\alpha, a\sin\alpha)$. We use also that $\vec{g} = (0, 0, -g)$, where $g = const$.

In order to avoid large expressions, we will make the calculations at the moment $t = 0$, such that

$$\vec{b}^* = (a, 0, c), \quad \vec{w}' = a\Omega w(0, 1, 0), \quad \vec{w}'' = -a\Omega^2 w(1, 0, 0).$$

Hence after all these substitutions, for the derivatives of $\vec{r}$ we obtain

$$\vec{r}' = wr(-c\sin\alpha, \cos\alpha, a\sin\alpha),$$

$$\vec{r}'' = -rw^2(c\cos\alpha, \sin\alpha, -a\cos\alpha) - arw\Omega\cos\alpha(a, 0, c) - (0, 0, g),$$

$$\vec{r}''' = -(rw^3 + ra^2w\Omega^2)(-c\sin\alpha, \cos\alpha, a\sin\alpha) - (a, 0, c)ra\Omega w\sin\alpha(\Omega c - \frac{3}{2}w) + \frac{3}{2}awg(0, 1, 0).$$

Further we obtain

$$\vec{r}'\times\vec{r}''' = -ar^2w^2\Omega\sin\alpha(\Omega c - \frac{3}{2}w)(c\cos\alpha, \sin\alpha, -a\cos\alpha) - \frac{3}{2}rw^2ag\sin\alpha\vec{b}^*,$$

$$(\vec{r}'\times\vec{r}''')\cdot\vec{r}'' = ar^3w^4\Omega\sin\alpha(\Omega c - \frac{3}{2}w) + \frac{3}{2}a^2r^2\Omega w^3g\cos\alpha\sin\alpha -$$

$$-ga^2r^2w^2\Omega\sin\alpha\cos\alpha(\Omega c - \frac{3}{2}w) + \frac{3}{2}g^2rw^2ac\sin\alpha,$$

$$(\vec{r}', \vec{r}'', \vec{r}''') = -ar^3w^4\Omega\sin\alpha(\Omega c - \frac{3}{2}w) + ga^2cr^2w^2\Omega^2\sin\alpha\cos\alpha - \frac{3}{2}g^2rw^2ac\sin\alpha - 3a^2r^2w^3\Omega g\sin\alpha\cos\alpha,$$

$$\vec{r}'\times\vec{r}'' = r^2w^3\vec{b}^* - ar^2w^2\Omega\cos\alpha(c\cos\alpha, \sin\alpha, -a\cos\alpha) - grw(\cos\alpha, c\sin\alpha, 0),$$

$$|\vec{r}'\times\vec{r}''|^2 = r^4w^6 + a^2r^4w^4\Omega^2\cos^2\alpha + g^2r^2w^2(\cos^2\alpha + c^2\sin^2\alpha) - 2gr^3w^4a\cos\alpha + 2acgr^3w^3\Omega\cos\alpha,$$

and hence

$$\vec{V}_b \approx -\lambda rw\vec{b} = \frac{w^4r^4\sin\alpha[ar^3w^4\Omega(\Omega c - \frac{3}{2}w) - ga^2cr^2w^2\Omega^2\cos\alpha + \frac{3}{2}g^2rw^2ac + 3a^2r^2w^3\Omega g\cos\alpha]}{[r^4w^6 + a^2r^4w^4\Omega^2\cos^2\alpha + g^2r^2w^2(1 - a^2\sin^2\alpha) - 2agr^3w^3\cos\alpha(w - c\Omega)]^2}\cdot$$

$$(r^2w^3\vec{b}^* - ar^2w^2\Omega\cos\alpha(c\cos\alpha, \sin\alpha, -a\cos\alpha) - grw(\cos\alpha, c\sin\alpha, 0))d\alpha.$$

After some transformations, the approximative spin velocity over the whole circle and for arbitrary t, can be written in the from

$$\langle\vec{V}_b\rangle \approx -\frac{1}{\pi}\frac{r^2w^3}{g}(\vec{b}^*\times\vec{g})\cdot$$

$$\int_0^{\pi}\frac{\sin^2\alpha[r^2w^2\Omega(\Omega c - \frac{3}{2}w) - gacr\Omega^2\cos\alpha + \frac{3}{2}g^2c + 3arw\Omega g\cos\alpha](arw\Omega\cos\alpha + gc)d\alpha}{[r^2w^4 + a^2r^2w^2\Omega^2\cos^2\alpha + g^2(c^2 + a^2\cos^2\alpha) - 2agrw\cos\alpha(w - c\Omega)]^2}.$$

The spin motion of the spinning circle in this case is a circle with radius $R = |\langle\vec{V}_b\rangle|/\Omega$, which can be tested.

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