Filomat 33:4 (2019), 1037–1046
https://doi.org/10.2298/FIL1904037D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Dual Quaternions and Dual Quaternionic Curves

Ali Dağdeviren[a], Salim Yüce[b]

[a]*THY Aviation Academy, Yesilköy, Bakırköy, Istanbul, Turkey*
[b]*Department of Mathematics, Yıldız Technical University, Davutpasa Campus, Istanbul, Turkey*

Abstract. After a brief review of the different types of quaternions, we develop a new perspective for dual quaternions with dividing two parts. Due to this new perspective, we will define the isotropic and non-isotropic dual quaternions. Then we will also give the basic algebraic concepts about the dual quaternions. Moreover, we define isotropic dual quaternionic curves and non-isotropic dual quaternionic curves. Via these definitions we find Serret-Frenet formulae for isotropic dual quaternionic curves. Finally, we will use these results to derive the Serret-Frenet formulae for non-isotropic dual quaternionic curves.

1. Introduction

Quaternions were discovered by William R. Hamilton in 1843. In his works, he wanted to generalize complex numbers to use geometric optics. Quaternions are the expansion of complex numbers and they are used in many fields of science, computing and physical problems [1]. The approaches at the formulation of physical laws by means of quaternions and octonions, also have a deep mathematical meaning in the generalized Frobenius theorem [2]. This theorem shows that real numbers, complex numbers, quaternions and octonions have an extraordinary position within the algebras because every real alternative algebra with division is isomorphic to one of these number systems. It is obvious that the particular field of applicability of real quaternions is Euclidean 4-space. Therefore, quaternions were used in theoretical physics for the creation of special relativity with units; space and time, forming a 4-dimensional space-time. Since a quaternion has four components, all of the components of a 4-vector can be included in it, [3].

There are some kinds of quaternions. One of them is called Minkowski quaternion. The difficulty with Minkowski quaternions is that they do not form an algebraic ring, this means that the product of two Minkowski quaternions is not always a Minkowski quaternion [4].

An important extension of Hamiltonian quaternions is the so-called binary(hyperbolic) quaternions. We can see in studies that the different types of quaternions are suitable algebraic instruments for expressing important space-time transformation as well as description of the classical and quantum fields [5]. As it has been shown in [6] the general Lorenz space-time transformation can be expressed in terms of binary quaternions.

Another important extension of Hamiltonian quaternions is the dual quaternions. Galilean transformations can be expressed in terms of dual quaternions, [7, 8]. Moreover, in terms of dual quaternions, this

2010 *Mathematics Subject Classification*. Primary 11R52 ; Secondary 53A35, 53A99
Keywords. Dual Quaternions, Dual Quaternionic Curves, Serret-Frenet Frame
Received: 17 July 2018; Accepted: 12 October 2018
Communicated by Mića S. Stanković
Research supported by Scientific and Technological Research Council of Turkey (TUBITAK)
Email addresses: adagdeviren@thy.com (Ali Dağdeviren), sayuce@yildiz.edu.tr (Salim Yüce)

transformation gets an elegant, economical and compact form, showing its underlying algebraic properties. Additionally, this formulation of Galilean transformation by means of dual quaternions shows that the linkage between space and time exists also in the Newtonian physics. In many studies, [7–9], a general dual quaternion has the form

$$q = a_0 + a_1 i + a_2 j + a_3 k, \quad a_i \in \mathbb{R}$$

where i, j and k are the dual quaternion units with $i^2 = j^2 = k^2 = ijk = 0$.

If the literature is examined, it is seen that there is another quaternion which is called dual quaternion again. This kind of quaternion was introduced by E.Study, [10]. This kind of dual quaternions form a frequently used tool for the description of Euclidean kinematics in three dimensions, see [11–13] or [14]. In these studies, dual quaternions are defined as "Quaternions with dual number entries". Furthermore, a general dual quaternion has the form

$$q = a_0 + a_1 i + a_2 j + a_3 k, \quad a_i \in \mathbb{D}$$

where i, j and k are the quaternion units and $\mathbb{D} = \{\, a + \varepsilon b \mid a, b \in \mathbb{R},\ \varepsilon^2 = 0\}$ is the set of dual numbers. Multiplication is defined with the relations for quaternion. Moreover, the dual unit ε commutes with the quaternion units $\varepsilon i = i\varepsilon$, $\varepsilon j = j\varepsilon$, $\varepsilon k = k\varepsilon$. Dual quaternions form an eight-dimensional vector space over the real numbers. The basis elements are $1, i, j, k, \varepsilon, \varepsilon i, \varepsilon j, \varepsilon k$, for more information see [15].

The concept of curves theory is also described in the real quaternion space by the help of Frenet elements of pure real quaternionic curves. Baharathi and Nagaraj represented the curves by unit quaternions in E^3 and E^4 and called these curves quaternionic curves [16]. However, we could not define the Frenet elements of the dual quaternionic curves with the current dual quaternion concept. By our new definitions, we will obtain the identification capability of dual quaternionic curves using Frenet elements of pure quaternionic curves.

In this paper, we will investigate dual quaternions and dual quaternionic curves. Firstly in section 2, we will mention isomorphism between spaces and the types of quaternions. After that, in section 3, we will define isotropic dual quaternions and non-isotropic dual quaternions. Moreover, we will examine some properties of dual quaternions. Finally, in section 4 and 5, we define isotropic dual quaternionic curve and non-isotropic dual quaternionic curve. Then, we will find Serret-Frenet formulae of these curves.

2. Preliminaries

Let's start with examining $\mathbb{R}^4, \mathbb{R}^4_1$ and $\mathbb{G}^4$ which are real space, Minkowski space and Galilean space, respectively. When these spaces are examined, it is seen that $\mathbb{R}^4_1$ and $\mathbb{G}^4$ are actually the similar spaces with $\mathbb{R}^4$. These are the same point sets according to the sum of two ordered 4-tuple and a multiplication of an ordered 4-tuple with a scalar. However, in vector space $\mathbb{R}^4$, for vectors $\overrightarrow{x} = (x_0, x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_0, y_1, y_2, y_3)$;

i) If the Euclidean inner product $\left\langle \overrightarrow{x}, \overrightarrow{y} \right\rangle = \sum_{i=0}^{3} x_i y_i$ is defined in $\mathbb{R}^4$, this space is called the Euclidean space.

ii) If the Minkowski inner product $\left\langle \overrightarrow{x}, \overrightarrow{y} \right\rangle_M = \sum_{i=0}^{2} x_i y_i - x_3 y_3$ is defined in $\mathbb{R}^4$, this space is called the Minkowski space.

iii) If the Galilean inner product

$$\left\langle \overrightarrow{x}, \overrightarrow{y} \right\rangle_G = \begin{cases} x_0 y_0 & , \quad x_0 \neq 0 \ \ or \ \ y_0 \neq 0 \\ x_1 y_1 + x_2 y_2 + x_3 y_3 & , \quad x_0 = 0 \ and \ \ y_0 = 0 \end{cases}$$

is defined in $\mathbb{R}^4$, this space is called the Galilean space [17]. In Galilean space, the vectors are examined in two groups according to whether their first components are zero or not. If the first component of any vector is zero, that is $x_0 = 0$, then this vector is called isotropic vector, otherwise, if it is not zero, this vector is called non-isotropic vector. If one of the vectors is non-isotropic and the other is isotropic, the first inner product is used. However, if the two vectors are isotropic vectors in the same way, these two vectors need to use the same metric because the space $\mathbb{G}^4$ falls into the space $\mathbb{R}^3$, which is the subvector space. Thus, the second inner product is used instead of the first, [17].

When the literature is examined, quaternions which are isomorphic to space $\mathbb{R}^4$ are real quaternions. Real quaternions are shown as $q = a_0 + a_1 i + a_2 j + a_3 k$, where $1, i, j$ and k denote the elements of the standard basis of $\mathbb{R}^4$ and $a_i \in \mathbb{R}$. These quaternions are defined by the following relations:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

Furthermore, when we think the set of pure real quaternions, it will be seen that pure real quaternion space is isomorphic to $\mathbb{R}^3$ space. Hence, equations (1) also apply to the product of two pure quaternions, [1]. When space $\mathbb{R}^4_1$ is examined, it is seen that this space is isomorphic to split quaternions space. Split quaternions are defined by the following relationships:

$$i^2 = j^2 = -1,\ k^2 = 1,\ ijk = 1. \tag{2}$$

In this space, the set of pure split quaternions are isomorph to space $\mathbb{R}^3_1$. Thus, when we deal with pure split quaternions, we again apply the relations in (2), [18]. However, when we investigate quaternions that are isomorphic to space $\mathbb{G}^4$, the quaternions that appear in papers [3, 7–9] are dual quaternions. The set of dual quaternions is denoted by $\mathbb{H}_{\mathbb{D}}$ and is defined by the following relations:

$$i^2 = j^2 = k^2 = ijk = 0. \tag{3}$$

However, it can not be deduced that the set of dual quaternions is given by these relations is fully isomorphic to space $\mathbb{G}^4$. Because in this space there is no equivalent of pure dual quaternions according to (3). In order for the dual quaternion space to be isomorphic to the space $\mathbb{G}^4$, a multiplication operation must be defined for pure dual quaternions. In this article, this deficiency in the literature will be eliminated.

In examining dual quaternions in our paper, we will use the $e_0 = 1, e_1, e_2, e_3$ symbols instead of the $1, i, j, k$ symbols. Furthermore, when pure dual quaternions are examined, it is seen that pure dual quaternions correspond to space $\mathbb{R}^3$ which is the subspace of $\mathbb{G}^4$. Hence, metric in space $\mathbb{R}^3$ should be used when processing in pure dual quaternions. Also, the elements of pure dual quaternions should provide the following relations:

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1 \tag{4}$$

Therefore, by expanding the concept of dual quaternions in the literature, with the equation (4) we will examine dual quaternions in two parts as isotropic and non-isotropic dual quaternions according to whether the first components are zero or not. So, after our definition of dual quaternions, it will be discussed that dual quaternionic space will be isomorphic to $\mathbb{G}^4$.

We can show the spaces, corresponding quaternions and their multiplication rules in the current literature with the following table:

Spaces	Corresponding Quaternions	Multiplication Rules	References
Euclidean Space, $\mathbb{R}^4$	Real Quaternions	$i^2 = j^2 = k^2 = ijk = -1$	1, 2
Minkowski Space, $\mathbb{R}^4_1$	Split Quaternions	$i^2 = j^2 = -1,\ k^2 = 1,\ ijk = 1$	4, 6, 18
Galilean Space, $\mathbb{G}^4$	Dual Quaternions	$i^2 = j^2 = k^2 = ijk = 0$	3, 7, 8

3. Dual Quaternions

In the literature, there are many articles about Dual quaternions. Some basic information and properties can be found in [3, 7–9]. But, in this section, we will define dual quaternions with a new perspective inspired by Galilean geometry.

A dual quaternion is written in the form $q = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ where a_0, a_1, a_2, a_3 are real numbers and $e_0 = 1, e_1, e_2, e_3$ are dual quaternion units. Also, dual quaternions can be written in form $q = S_q + V_q$, where $S_q = a_0$ denotes the scalar part of q and $V_q = a_1e_1 + a_2e_2 + a_3e_3$ denotes vector part of q. If $a_0 = 0$, then q is called isotropic dual quaternion. If $a_0 \neq 0$, then q is called non-isotropic dual quaternion. Dual quaternion units e_1, e_2, e_3 satisfy the following multiplication rules:

(*i*) *if q is a non-isotropic dual quaternion, that is $a_0 \neq 0$, then*

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = 0$$

where $e_1^2 = e_1 \times e_1 = e_1e_1$, here " $\times$ " denotes that the product of two non-isotropic dual quaternion.

(*ii*) *if q is an isotropic dual quaternion, that is $a_0 = 0$, then*

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$$

where $e_1^2 = e_1 \times_\delta e_1 = e_1e_1$, here " $\times_\delta$ " denotes that the product of two isotropic dual quaternion.

The set of non-isotropic dual quaternions and isotropic dual quaternions are shown with $\mathbb{H}_{\mathbb{D}}$ and $\mathbb{H}_{\mathbb{D}}^p$ respectively. Moreover, these spaces are called non-isotropic dual quaternionic space and isotropic dual quaternionic space. Additionally, the set of isotropic dual quaternionic space is the subspace of non-isotropic dual quaternionic space and it is clear that isotropic dual quaternionic space $\mathbb{H}_{\mathbb{D}}^p$ is isomorphic to $\mathbb{R}^3$.

The sum of two dual quaternions is also a dual quaternion. This sum is the usual componentwise sum. The sum of two dual quaternions can be defined as follows,
$\oplus : \mathbb{H}_{\mathbb{D}} \times \mathbb{H}_{\mathbb{D}} \to \mathbb{H}_{\mathbb{D}}$
$(q, p) \to q \oplus p = \left(S_q + V_q\right) \oplus \left(S_p + V_p\right) = S_q + S_p + V_q + V_p = S_{q+p} + V_{q+p}$
It is easy to show $(\mathbb{H}_{\mathbb{D}}, \oplus)$ is an Abelian group with identity element.

Definition 3.1. *Multiplication with a scalar of a dual quaternion is defined as follows,*
$\odot : \mathbb{R} \times \mathbb{H}_{\mathbb{D}} \to \mathbb{H}_{\mathbb{D}}$
$(\lambda, q) \to \lambda \odot q = \lambda \odot \left(S_q + V_q\right) = \lambda S_q + \lambda V_q$
this operation implies the following statements

(i) $\lambda \odot (q + p) = (\lambda \odot q) \oplus (\lambda \odot p)$,

(ii) $(\lambda_1 + \lambda_2) \odot q = (\lambda_1 \odot q) \oplus (\lambda_2 \odot q)$,

(iii) $(\lambda_1.\lambda_2) \odot q = \lambda_1 \odot (\lambda_2 \odot q)$,

(iv) $1 \odot q = q$.

Thus, $(\mathbb{H}_{\mathbb{D}}, \oplus, \odot)$ is a vector space over the real number field. Moreover, we get $\dim \mathbb{H}_{\mathbb{D}} = 4$ and $\mathbb{H}_{\mathbb{D}} = Sp\{1, e_1, e_2, e_3\}$.

Definition 3.2. *Quaternion product over dual quaternions is defined as follows; Let $q = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $p = b_0 + b_1e_1 + b_2e_2 + b_3e_3$ be two dual quaternions. Then*
if $a_0 \neq 0$ or $b_0 \neq 0$, that is q or p is non-isotropic dual quaternion, then the quaternion product of them is

$$\begin{aligned} q \times p &= (a_0 + a_1e_1 + a_2e_2 + a_3e_3) \times (b_0 + b_1e_1 + b_2e_2 + b_3e_3) \\ &= S_qS_p + S_qV_p + S_pV_q. \end{aligned}$$

if $a_0 = 0$ and $b_0 = 0$, that is q and p are isotropic dual quaternion, then the special quaternion product of them is

$$\begin{aligned} q \times_\delta p &= (a_1e_1 + a_2e_2 + a_3e_3) \times_\delta (b_1e_1 + b_2e_2 + b_3e_3) \\ &= -< V_q, V_p > +V_q \wedge V_p. \end{aligned}$$

where $<,>$ and $\wedge$ denote the inner product and vector product in Euclidean 3-space, respectively. Additionally note that, if q and p are isotropic dual quaternion, then their quaternion product is $q \times p = 0$.

Definition 3.3. *Let $q = S_q + V_q \in \mathbb{H}_{\mathbb{D}}$. Conjugate of a dual quaternion is defined as $\overline{q} = S_q - V_q \in \mathbb{H}_{\mathbb{D}}$.*

Other properties of dual quaternions can be shown as follows:

(i) Let $q = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ be any non-isotropic dual quaternion. The following equation implies

$$\begin{aligned} q \times \overline{q} = \overline{q} \times q &= S_q^2 - S_qV_q + S_qV_q \\ &= S_q^2 = a_0^2 \, . \end{aligned}$$

Beside, if $q = a_1e_1 + a_2e_2 + a_3e_3$ is any isotropic dual quaternion, then the equation

$$\begin{aligned} q \times_\delta \overline{q} = \overline{q} \times_\delta q &= -< V_q, -V_q > + V_q \wedge -V_q \\ &= < V_q, V_q > = a_1^2 + a_2^2 + a_3^2. \end{aligned}$$

implies. From the last equation, we obtain $q \times_\delta \overline{q} \geq 0$ and if $q \times_\delta \overline{q} = 0$ then $q = 0$.

(ii) Let $q = S_q + V_q$ and $p = S_p + V_p$ be two dual quaternion. Then, even if q and p are isotropic or non-isotropic

$$\overline{q+p} = \overline{q} + \overline{p} \ \ , \ \ \overline{\overline{q}} = q \ \ , \ \ \overline{\lambda q} = \lambda \overline{q}$$

equalities are always hold, where $\lambda \in \mathbb{R}$.

(iii) Let $q = S_q + V_q$ and $p = S_p + V_p$ be two dual quaternions. If q and p are non-isotropic dual quaternion, then $\overline{q \times p} = \overline{q} \times \overline{p}$ and $S_q = \frac{q+\overline{q}}{2}, V_q = \frac{q-\overline{q}}{2}$. In addition to this, if $q = S_q$ then $\overline{q} = q$ and if $q = V_q$ then $\overline{q} = -q$. On the other hand, if they are isotropic dual quaternion, then $\overline{q \times_\delta p} \neq \overline{q} \times_\delta \overline{p}$.

Definition 3.4. *The norm of a non-isotropic dual quaternion is defined by*

$$\left\| q \right\| = \sqrt{q \times \overline{q}} = \sqrt{\overline{q} \times q} = \sqrt{a_0^2} = |a_0| \, .$$

On the other hand, if q is isotropic dual quaternion, the norm of q is defined by

$$\left\| q \right\|_\delta = \sqrt{q \times_\delta \overline{q}} = \sqrt{\overline{q} \times_\delta q} = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

If $\left\| q \right\| = 1$, then q is called unit non-isotropic dual quaternion. For any non-isotropic dual quaternion or isotropic dual quaternion, the following properties of norm operations hold:

(i) $\left\| q \times p \right\| = \left\| q \right\| \left\| p \right\| = \left\| p \times q \right\|$,

(ii) $\left\| q + p \right\| \leq \left\| q \right\| + \left\| p \right\|$,

(iii) $\left\| q \right\|^2 + \left\| p \right\|^2 = \frac{1}{2} \left(\left\| q + p \right\|^2 + \left\| q - p \right\|^2 \right)$,

(iv) $\left\| q \right\| = \left\| \overline{q} \right\|$.

In addition to this equalities, if q is an isotropic dual quaternion then $\left\| q \right\| = 0$ holds only $q = 0$.

The set of non-isotropic dual quaternion is a commutative ring under the non-isotropic dual quaternion multiplication and also it is 4-dimensional vector space on $\mathbb{R}$ and its basis is $\{1, e_1, e_2, e_3\}$. The interesting property of dual quaternions is that by their means one can express the Galilean transformation in one dual quaternion equation. Because of the multiplication and ratio of two non-isotropic dual quaternions are also a non-isotropic dual quaternion, the set of non-isotropic dual quaternions form a division algebra under addition and multiplication. On the other hand, the set of isotropic dual quaternions is a 3-dimensional vector space over $\mathbb{R}$ and its basis is the set $\{e_1, e_2, e_3\}$. Moreover, the set of isotropic dual quaternions are isomorphic with $\mathbb{R}^3$.

Definition 3.5. *Let $q = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ be a non-isotropic dual quaternion. The inverse of q is defined as*

$$q^{-1} = \frac{\overline{q}}{\left\|q\right\|^2} = \frac{a_0 - a_1e_1 - a_2e_2 - a_3e_3}{a_0^2}.$$

On the other hand, let $q = a_1e_1 + a_2e_2 + a_3e_3$ be isotropic dual quaternion, the inverse of q is defined as

$$q^{-1} = \frac{\overline{q}}{\left\|q\right\|^2} = \frac{-a_1e_1 - a_2e_2 - a_3e_3}{a_1^2 + a_2^2 + a_3^2}.$$

For any dual quaternion q, inverse operation holds following properties,

(i) $\left\|q^{-1}\right\| = \left\|q\right\|^{-1}$,

(ii) $q \times q^{-1} = q \times \left(\left\|q\right\|^{-2} \overline{q}\right) = \left\|q\right\|^{-2} (q \times \overline{q}) = \left\|q\right\|^{-2} \left\|q\right\|^2 = 1 = q^{-1} \times q.$

Definition 3.6. *The inner product of non-isotropic dual quaternions is defined as follows*

$$\begin{aligned} <,>: \quad \mathbb{H}_{\mathbb{D}} \times \mathbb{H}_{\mathbb{D}} &\to \mathbb{R} \\ < q, p > &= \frac{1}{2}\left(q \times \overline{p} + p \times \overline{q}\right). \end{aligned}$$

4. Isotropic Dual Quaternionic Curves

In this section, firstly Serret-Frenet formulae will be given for isotropic dual quaternionic curves. Here, it will be considered that the isotropic dual quaternion space is isomorphic with the 3-dimensional Euclidean space. Then we will obtain the Serret-Frenet formulae for non-isotropic dual quaternionic curves using the equations we have for isotropic dual quaternions.

Definition 4.1. *The set of isotropic dual quaternionic curves is defined with the space of* $\left\{\gamma \in \mathbb{H}_{\mathbb{D}}^{p}, \ \gamma + \overline{\gamma} = 0\right\}$. *Let $I = [0, 1]$ denote the unit interval of the real line and $s \in I$ be the parameter along the smooth curve*

$$\begin{aligned} \gamma : I \ \subset \ \mathbb{R} &\to \mathbb{H}_{\mathbb{D}}^{p} \\ s \ &\to \gamma(s) = \sum_{i=1}^{3} \gamma_i(s)\, e_i \end{aligned}$$

is called an isotropic dual quaternionic curve.

Definition 4.2. *If an isotropic dual quaternionic curve provides the following equation, it is called the unit speed isotropic dual quaternionic curve*

$$\left\|\gamma'(s)\right\|^2 = \left\langle \gamma'(s), \gamma'(s) \right\rangle = \frac{1}{2}[\gamma'(s) \times_\delta \overline{\gamma'(s)} + \gamma'(s) \times_\delta \overline{\gamma'(s)}] = \gamma'(s) \times_\delta \overline{\gamma'(s)} = 1.$$

Definition 4.3. *The tangent vector of any unit speed isotropic dual quaternionic curve is defined as*

$$t(s) = \gamma'(s) = \sum_{i=1}^{3} \gamma_i'(s)\, e_i.$$

This tangent vector has unit length, $\|t(s)\| = 1$, for all s. This unitary condition implies:

$$t' \times_\delta \overline{t} + t \times_\delta \overline{t'} = 0 \tag{5}$$

Equation (5) implies the following, that's why it is interesting:

(i) t' is orthogonal to t and

(ii) $t' \times_\delta \bar{t}$ is an isotropic dual quaternion.

Because t' is itself an isotropic dual quaternion, we define the isotropic dual quaternion n_1 by $n_1 = \frac{t'}{\|t'\|}$. Then the property above (*i*) implies: n_1 is orthogonal to t. Moreover, property above (*ii*) implies that there is a unit isotropic dual quaternion n_2 such that

$$n_2 = t \times_\delta n_1 = -\langle t, n_1 \rangle + t \wedge n_1 = -n_1 \times_\delta t. \tag{6}$$

Furthermore, it implies $t \times_\delta n_2 = -n_1 = -n_2 \times_\delta t$ and $n_2 \times_\delta n_1 = -t = -n_1 \times_\delta n_2$. Thus $t, n_1$and n_2 are mutually orthogonal unit isotropic dual quaternions. That is, $\{t, n_1, n_2\}$ is orthonormal vector system. This system is called Frenet frame of γ.

Definition 4.4. *Let γ be an isotropic dual quaternionic curve with Frenet frame $\{t, n_1, n_2\}$. The first and second curvature functions of this curve are defined as*

$$\begin{aligned} k_1(s) &= \kappa = \left\langle t', n_1 \right\rangle = \frac{1}{2}(t' \times_\delta \overline{n_1} + n_1 \times_\delta \overline{t'}), \\ k_2(s) &= \tau = \left\langle n_1', n_2 \right\rangle = \frac{1}{2}(n_1' \times_\delta \overline{n_2} + n_2 \times_\delta \overline{n_1'}). \end{aligned}$$

Here, it is easy to see that $k_1(s) = \kappa = \left\| t' \right\|$.

Theorem 4.5. *Let $(t, n_1, n_2, \kappa, \tau)$ be the Frenet apparatus for isotropic dual quaternionic curve γ. The Serret-Frenet formulae of γ are*

$$\begin{aligned} t' &= \kappa n_1, \\ n_1' &= -\kappa t + \tau n_2, \\ n_2' &= -\tau n_1. \end{aligned}$$

Proof. First of all, from the definition $n_1 = \frac{t'}{\|t'\|}$, we get $t' = \left\| t' \right\| n_1$. Then via using $\kappa = \left\| t' \right\|$, we obtain

$$t' = \kappa n_1. \tag{7}$$

We differentiate equation (6) to obtain

$$n_2' = -\kappa.e_0 + t.n_1'$$

using equation (7) and the fact $n_1 \times_\delta n_1 = -1$, where $e_0 = -1$. Since $t \times_\delta t = -1$, the equation above now becomes

$$n_2' = t \times_\delta \left(\kappa t + n_1' \right) \tag{8}$$

Again, n_2' is an isotropic dual quaternion which is orthogonal to n_2. Therefore, we obtain $n_1' + \kappa t$ is orthogonal to t and n_1, implying

$$n_1' = -\kappa t + \tau n_2 \tag{9}$$

Substituting (9) into (8), we finally obtain

$$n_2' = -\tau n_1. \tag{10}$$

So, we get Serret Frenet formulae for the isotropic dual quaternionic curve, where t is the unit tangent, n_1 is the unit principal normal and κ is the principal curvature, n_2 is the unit binormal and τ is the torsion of γ.

5. Non-Isotropic Dual Quaternionic Curves

In this section, Serret Frenet formulas will be given for non-isotropic dual quaternionic curves. We will use the result of the previous section.

Definition 5.1. *Let $I = [0,1]$ denote the unit interval of the real line and $s \in I$ be the parameter along the smooth curve, then*

$$\begin{aligned} \beta : I \subset \mathbb{R} &\to \mathbb{H}_{\mathbb{D}} \\ s &\to \beta(s) = \beta_0 e_0 + \sum_{i=1}^{3} \beta_i(s)\, e_i = (s, \beta_1(s), \beta_2(s), \beta_3(s)) \end{aligned}$$

is called isotropic dual quaternionic curves.

Definition 5.2. *If a non-isotropic dual quaternionic curve provides the following equation, it is called the unit speed non-isotropic dual quaternionic curve*

$$\left\|\beta'(s)\right\|^2 = \left\langle \beta'(s), \beta'(s) \right\rangle = \frac{1}{2}[\beta'(s) \times \overline{\beta'(s)} + \beta'(s) \times \overline{\beta'(s)}] = \beta'(s) \times \overline{\beta'(s)} = 1.$$

Then the non-isotropic dual quaternionic curve β should be defined as

$$\beta(s) = s e_0 + \sum_{i=1}^{3} \beta_i(s)\, e_i.$$

Definition 5.3. *The tangent vector of any unit speed non-isotropic dual quaternionic curve β is defined as*

$$T(s) = \beta'(s) = 1 + \sum_{i=1}^{3} \beta'(s) e_i = \left(1, \beta_1'(s), \beta_2'(s), \beta_3'(s)\right).$$

This tangent vector has unit length: $\|T(s)\| = 1$ for all s. This unitary condition implies:

$$T' \times \overline{T} + T \times \overline{T'} = 0 \tag{11}$$

Equation (11) implies the following, that's why it is interesting:

(i) T' is orthogonal to T and

(ii) $T' \times \overline{T}$ is an isotropic dual quaternion.

Definition 5.4. $N_1 = \frac{T'}{\|T'\|}$, *where T' is orthogonal to T.*

Lemma 5.5. *Since $N_1 \times \overline{T}$ is isotropic dual quaternion, let γ be the isotropic dual quaternionic curve that takes vector $N_1 \times \overline{T}$ as the unit tangent vector. Thus, we can define the curve β in $\mathbb{H}_{\mathbb{D}}$ corresponding to γ in $\mathbb{H}_{\mathbb{D}}^p$. Therefore, we can write that $N_1 \times \overline{T} = t$. Moreover, it is obvious that t has unit magnitude since both N_1 and T have unit magnitudes. We can now write N_1, as follows:*

$$N_1 = t \times T. \tag{12}$$

Definition 5.6. *For non-isotropic dual quaternionic curve $\beta : I \subset \mathbb{R} \to \mathbb{H}_{\mathbb{D}}$, let us define $N_2 = n_1 \times T$ and $N_3 = n_2 \times T$. Here n_1 and n_2 are the Frenet vectors of the isotropic curves γ which correspond with β.*

Now we obtain a Frenet frame over β. The elements of this frame are T, $N_1 = \frac{T'}{\|T'\|}$, $N_2 = n_1 \times T$ and $N_3 = n_2 \times T$. Moreover, $\{T, N_1, N_2, N_3\}$ system is orthonormal.

Definition 5.7. *Let β be a non-isotropic dual quaternionic curve with Frenet frame $\{T, N_1, N_2, N_3\}$. The curvature functions of this curve are defined as follows*

$$\begin{aligned}
K_1(s) &= \left\langle T', N_1 \right\rangle = \frac{1}{2}(T' \times \overline{N_1} + N_1 \times \overline{T'}), \\
K_2(s) &= \left\langle N_1', N_2 \right\rangle = \frac{1}{2}(N_1' \times \overline{N_2} + N_2 \times \overline{N_1'}), \\
K_3(s) &= \left\langle N_2', N_3 \right\rangle = \frac{1}{2}(N_2' \times \overline{N_3} + N_3 \times \overline{N_2'}).
\end{aligned}$$

Theorem 5.8. *Let $\{T, N_1, N_2, N_3\}$ is the Frenet Frame for non-isotropic dual quaternionic curve β. Also, let's say the curvatures of this curve are $K(s)$, $\kappa(s)$, $\tau(s)$. Here $\kappa(s)$ and $\tau(s)$ are the 1st and 2nd curvature of the curve γ which implies $N_1 \times \overline{T} = t$. So, the Serret Frenet formulae of β are*

$$\begin{aligned}
T' &= KN_1, \\
N_1' &= \kappa N_2, \\
N_2' &= -\kappa N_1 + \tau N_3, \\
N_3' &= -\tau N_2,
\end{aligned}$$

where the system $\{t, n_1, n_2, \kappa, \tau\}$ is the Frenet apparatus of the isotropic dual quaternionic curve γ.

Proof. From the definition of the first curvature, it can be seen that $K_1(s) = K = \left\| T' \right\|$. Also, via using $N_1 = \frac{T'}{\|T'\|}$ we get

$$T' = KN_1.$$

We now differentiate $N_1 = t \times T$ and substitute equations $t' = \kappa n_1$ and $N_2 = n_1 \times T$, we obtain

$$N_1' = \kappa N_2.$$

We differentiate $N_2 = n_1 \times T$ and substitute equations $n_1' = -\kappa t + \tau n_2$, $N_1 = t \times T$ and $N_3 = n_2 \times T$, we get

$$N_2' = -\kappa N_1 + \tau N_3.$$

Finally, we differentiate $N_3 = n_2 \times T$ and substitute equations $n_2' = -\tau n_1$ and $N_3 = n_2 \times T$, we get

$$N_3' = -\tau N_2.$$

In summary, $\{T, N_1, N_2, N_3, K, \kappa, \tau\}$ gives the Frenet apparatus for the curve β.

Note 1. We have obtained the Serret-Frenet formulae and the Frenet apparatus for the non-isotropic curve β using Serret-Frenet formulae for the isotropic curve γ. In addition, it should be noted that the torsion of β is the principal curvature of γ and the bitorsion of β is the torsion of γ.

6. Conclusion

When we review the literature, we can see that there are many studies on quaternionic curves in the literature. For example, the Serret-Frenet formulae for a Real quaternionic curves in $\mathbb{R}^3$ are introduced by K. Bharathi and M. Nagaraj. Moreover, they obtained the Serret-Frenet formulae for the Real quaternionic curves in $\mathbb{R}^4$ by the formulae in $\mathbb{R}^3$, [16]. Then, lots of studies have been published by using this studies. One of them is A. C. Coken and A. Tuna's study [19] which they gave Serret-Frenet formulas, inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi-Euclidean spaces $\mathbb{E}_1^3$ and $\mathbb{E}_2^4$.

On the other hand, no studies have been conducted on dual quatenionic curves. In our paper, we have studied dual quaternions and their properties in detail. Moreover, we have define dual quaternionic curves and also we have find Serret-Frenet formulae of dual quaternions. Thereby, this deficiency in the literature has been eliminated with this work.

References

[1] W.R. Hamilton, Elements of Quaternions, I, II and III. Chelsea, 1899.
[2] G.F. Frobenius, J. Reine and Angew. Math. 84(1) (1879).
[3] V. Majernik, Galilean transformation expressed by the dual four- component numbers. Acta Phys.Pol.A. 87 (1995) 919–923.
[4] J.L. Synge, Quaternions, Lorenz Transformation, and Conway-Dirac-Eddington Matrices. Dublin Institute for Advanced Studies, Dublin, 1972.
[5] R. Anderson, G.C. Ioshi, Physics Essays, (1993) 308.
[6] V. Majernik, Basic space-time transformations expressed by means of two-component number systems, Acta Phys. Pol. A. 86(1) (1994) 291–295.
[7] V. Majernik, Quaternion formulation of the Galilean space-time transformation. Acta Phys. 56(1) (2006) 9–14.
[8] Y. Yaylı, E.E. Tutuncu, Generalized Galilean transformations and dual quaternions. Scientia Magna 5(1)(2009) 94–100.
[9] Z. Ercan, S. Yüce, On properties of the dual quaternions. European Journal of Pure and Applied Mathematics 4(2) (2011) 142-146.
[10] E. Study, Geometrie der Dynamen, B.G. Teubner, 1903.
[11] W. Blaschke, Kinematik und Quaternionen. Deutscher Verlag der Wissenschaften, Berlin, 1960.
[12] M. Husty, A. Karger, H. Sachs, W. Steinhilper, Kinematik und Robotik., Springer, Berlin, 1997.
[13] J. M. McCarthy, An Introduction to Theoretical Kinematics. MIT Press, Cambridge and Mass, 1990.
[14] J. M. Selig, Geometric Fundamentals of Robotics. 2nd ed. Springer, New York, 2005.
[15] D. Klawitter, Clifford Algebras, Geometric Modelling and Chain Geometries with Application in Kinematics, Dresden 2014.
[16] K. Bharathi, M. Nagaraj, Quaternion valued function of a real variable Serret-Frenet formulae. Indian J. Pure Appl. Math. 18(6) (1987) 507–511
[17] I.M. Yaglom, A simple Non-Euclidean geometry and its physical basis. Sipringer-Verlag Inc. New York 1979.
[18] L. Kula, Y. Yaylı, Split quaternions and rotations in semi Euclidean Space. Journal of Korean Mathematical Society 44 (2007) 1313–1327.
[19] A. C. Çöken and A. Tuna, On the quaternionic inclined curves in the semi Euclidean space $\mathbb{E}_2^4$, Applied Mathematics and Computation, 155, (2004), 373–389.