Filomat 33:4 (2019), 1047–1052
https://doi.org/10.2298/FIL1904047P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Equitorsion Holomorphically Projective Mappings of Generalized m-parabolic Kähler Manifolds

Miloš Z. Petrović[a], Patrik Peška[b]

[a] *Department of Mathematics, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia*
[b] *Department of Algebra and Geometry, Faculty of Science, 17. listopadu 12, 779 00 Olomouc, Czech Republic*

Abstract. We investigate equitorsion holomorphically projective mappings of generalized m-parabolic Kähler manifolds and provide some necessary and sufficient conditions for the existence of these mappings in form of linear PDE-systems. Also, we find an invariant geometric object with respect to a holomorphically projective mapping of generalized m-parabolic Kähler manifolds which is analogous to the Thomas projective parameter.

1. Introduction and preliminaries

Many results on holomorphically projective (HP) mappings of parabolically-Kählerian spaces were obtained by J. Mikeš, M. Shiha, P. Peška, H. Chuda [1, 5–9, 16, 22–25]. These results and many other related results are included in two excellent monographs [6, 8]. In this paper we consider manifolds endowed with a non-symmetric metric and a non-symmetric linear connection. Importance of investigation of these manifolds comes from non-symmetric gravitational theory [2–4, 15]. Geometric aspects of manifolds with non-symmetric linear connection were thoroughly studied by M. Prvanović and S.M. Minčć [10–14, 21, 26]. Further, generalized elliptic, hyperbolic and parabolic Kählerian spaces were developed in [14, 17–20, 26]. Recently, generalized m-parabolic Kähler manifolds were defined in [18]. We will transform necessary and sufficient conditions for the existence of equitorsion HP mappings of generalized m-parabolic Kähler manifolds from the paper [18] into linear PDE-systems.

Definition 1.1. [18] *A generalized Riemannian manifold* (M, g) *of even dimension* n $(n > 2)$ *is said to be a* generalized m-parabolic Kähler manifold *if there exists a tensor field* F *on* M *of type* $(1, 1)$ *such that* $\mathrm{rank}(F) = m \leq \frac{n}{2}$ *and the following conditions hold*

$$\begin{aligned} F^2 &= 0, \\ \underline{g}(X, FX) &= 0, \\ \nabla F &= 0, \end{aligned}$$

2010 *Mathematics Subject Classification*. Primary 53B05; Secondary 53B20, 53B35.
Keywords. Holomorphically projective mapping, generalized Riemannian space, generalized m-parabolic Kähler manifold, PDE-system, curvature tensor.
Received: 14 August 2018; Accepted: 16 Oktober 2018
Communicated by Ljubica S. Velimirović
Research supported by Grant No. 174012 of the Ministry of Education, Science and Technological Development, Republic of Serbia.
Email addresses: milos.petrovic.phd@gmail.com (Miloš Z. Petrović), patrik_peska@seznam.cz (Patrik Peška)

where ∇ *denotes the Levi-Civita connection corresponding to the symmetric part* $\underline{g}$ *of the metric* g *and* X *is an arbitrary tangent vector field on* M*. In the case when* $\operatorname{rank}(F) = m = \frac{n}{2}$ *the manifold* $(M, \underline{g})$ *is called a* generalized parabolic Kähler manifold.

2. Equitorsion HP mappings of generalized *m*-parabolic Kähler manifolds

A diffeomorphism $f : M \to \overline{M}$ of generalized m-parabolic Kähler manifolds (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ is said to be an *equitorsion HP mapping* if it preserves holomorphically planar curves and the torsion tensor [18, 20, 26]. In this section we give necessary and sufficient conditions for the existence of an equitorsion HP mapping in terms of the symmetric part $\overline{\underline{g}}$ of the metric g and the covariant derivatives of the first and second kind with respect to (w.r.t.) the metric $\underline{g}$.

Proposition 2.1. *A necessary and sufficient condition for the existence of an equitorsion HP mapping* $f : M \to \overline{M}$ *of generalized m-parabolic Kähler manifolds* M *and* $\overline{M}$ *is given by*

$$\underset{1}{P}(X, Y) = \psi(X)Y + \psi(Y)X + \varphi(X)FY + \varphi(Y)FX,$$

where φ *is a linear form,* ψ *is a gradient-like form such that* $\psi(X) = \varphi(FX)$*.*

Corollary 2.1. *A generalized m-parabolic Kähler manifold* M *with a metric* g *admits an equitorsion HP mapping onto a generalized m-parabolic Kähler manifold* $\overline{M}$ *with a metric* $\overline{g}$ *if and only if*

$$\begin{aligned}(\underset{\mu}{\nabla}_Z \overline{\underline{g}})(X, Y) =& 2\psi(Z)\overline{\underline{g}}(X, Y) + \psi(X)\overline{\underline{g}}(Y, Z) + \varphi(X)\overline{\underline{g}}(Y, FZ) \\ &+ \psi(Y)\overline{\underline{g}}(X, Z) + \varphi(Y)\overline{\underline{g}}(X, FZ),\end{aligned} \tag{1}$$

where $\mu \in \{1, 2\}$*,* $\overline{\underline{g}}$ *denotes the symmetric part of the metric* $\overline{g}$*,* φ *is a linear form,* ψ *is a gradient-like form such that* $\psi(X) = \varphi(FX)$*.*

Remark 2.1. *The condition* (1) *is equivalent with the condition* [8]

$$\begin{aligned}(\nabla_Z \overline{\underline{g}})(X, Y) =& 2\psi(Z)\overline{\underline{g}}(X, Y) + \psi(X)\overline{\underline{g}}(Y, Z) + \varphi(X)\overline{\underline{g}}(Y, FZ) \\ &+ \psi(Y)\overline{\underline{g}}(X, Z) + \varphi(Y)\overline{\underline{g}}(X, FZ),\end{aligned}$$

where ∇ *is the symmetric part of the non-symmetric linear connections* $\underset{\mu}{\nabla}$*,* $\mu \in \{1, 2\}$*.*

2.1. Linear PDE-systems for the existence of an equitorsion HP mapping of generalized m-parabolic Kähler spaces

Following the idea of M. Shiha and J. Mikeš [23] we transform the non-linear systems (1) into linear PDE-systems in covariant derivatives of the first and second kind.

Theorem 2.1. *A necessary and sufficient condition for the existence of an equitorsion HP mapping* $f : M \to \overline{M}$ *of generalized m-parabolic Kähler manifolds* M *and* $\overline{M}$ *is given by*

$$\begin{aligned}(\underset{\mu}{\nabla}_Z a)(X, Y) =& \lambda(X)\underline{g}(Y, Z) + \lambda(Y)\underline{g}(X, Z) \\ &+ \theta(X)\underline{g}(FY, Z) + \theta(Y)\underline{g}(FX, Z), \quad \mu \in \{1, 2\},\end{aligned} \tag{2}$$

or in local form

$$\underset{\mu}{\nabla}_k a_{ij} = \lambda_i g_{\underline{jk}} + \lambda_j g_{\underline{ik}} + \theta_i g_{\underline{pk}} F^p_j + \theta_j g_{\underline{pk}} F^p_i, \quad \mu \in \{1, 2\},$$

where

$$a_{ij} = e^{2\psi} \overline{g}^{\underline{pq}} g_{\underline{pi}} g_{\underline{qj}}, \quad \lambda_i = \theta_p F^p_i, \quad \theta_i = -e^{2\psi} \overline{g}^{\underline{pq}} g_{\underline{qi}} \varphi_p.$$

Remark 2.2. *The condition* (2) *is equivalent with the condition* [8]

$$\begin{aligned}(\nabla_Z a)(X,Y) =& 2\psi(Z)\underline{\overline{g}}(X,Y) + \psi(X)\underline{\overline{g}}(Y,Z) + \varphi(X)\underline{\overline{g}}(Y,FZ)\\ &+ \psi(Y)\underline{\overline{g}}(X,Z) + \varphi(Y)\underline{\overline{g}}(X,FZ),\end{aligned}$$

which in local coordinates reads

$$\nabla_k a_{ij} = \lambda_i g_{\underline{jk}} + \lambda_j g_{\underline{ik}} + \theta_i g_{\underline{pk}} F^p_j + \theta_j g_{\underline{pk}} F^p_i,$$

where

$$a_{ij} = e^{2\psi}\overline{g}^{\underline{pq}} g_{\underline{pi}} g_{\underline{qj}}, \quad \lambda_i = \theta_p F^p_i, \quad \theta_i = -e^{2\psi}\overline{g}^{\underline{pq}} g_{\underline{qi}} \varphi_p.$$

Here ∇ *denotes the symmetric part of the non-symmetric linear connections* $\underset{\mu}{\nabla}$, $\mu \in \{1,2\}$.

2.2. Relations between curvature tensors with respect to an equitorsion HP mapping

On generalized parabolic Kähler manifolds one can define five linearly independent curvature tensors [13]:

$$\begin{aligned}
\underset{\mu}{R}(X,Y)Z =& \underset{\mu}{\nabla}_X \underset{\mu}{\nabla}_Y Z - \underset{\mu}{\nabla}_Y \underset{\mu}{\nabla}_X Z - \underset{\mu}{\nabla}_{[X,Y]} Z, \quad \mu = 1,2,\\
\underset{3}{R}(X,Y)Z =& \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{1}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{2}{\nabla}_X Y} Z,\\
\underset{4}{R}(X,Y)Z =& \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{2}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{1}{\nabla}_X Y} Z,\\
\underset{5}{R}(X,Y)Z =& \frac{1}{2}\Big(\underset{1}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{2}{\nabla}_Y \underset{1}{\nabla}_X Z + \underset{2}{\nabla}_X \underset{2}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z\\
&+ \underset{1}{\nabla}_{[Y,X]} Z + \underset{2}{\nabla}_{[Y,X]} Z\Big).
\end{aligned}$$

Corollary 2.2. *Let* $f : M \to \overline{M}$ *be an equitorsion HP mapping and let* $\underset{\nu}{R}$ *and* $\underset{\nu}{\overline{R}}$ *are* ν*-kind* ($\nu = 1,\dots,5$) *curvature tensors of the generalized m-parabolic Kähler manifolds* M *and* $\overline{M}$*, respectively. Then the following relations are valid*

$$\begin{aligned}
\underset{1}{\overline{R}}(X,Y)Z =& \underset{1}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ,\\
\underset{2}{\overline{R}}(X,Y)Z =& \underset{2}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ,\\
\underset{3}{\overline{R}}(X,Y)Z =& \underset{3}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ,\\
\underset{4}{\overline{R}}(X,Y)Z =& \underset{4}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ,\\
\underset{5}{\overline{R}}(X,Y)Z =& \underset{5}{R}(X,Y)Z - \psi(Z,Y)X + \psi(Z,X)Y - \varphi(Z,Y)FX\\
&+ \varphi(Z,X)FY + \big(\varphi(Y,X) - \varphi(X,Y)\big)FZ,
\end{aligned}$$

where $\varphi(X,Y)$ *is defined by*

$$\varphi(X,Y) = \nabla_Y \varphi(X) - \psi(X)\varphi(Y) - \varphi(X)\psi(Y),$$

and $\psi(X,Y)$ *is defined by*

$$\psi(X,Y) = \varphi(FX,Y) = \nabla_Y \psi(X) - \psi(X)\psi(Y).$$

3. An invariant geometric object of HP mappings of generalized *m*-parabolic Kähler manifolds

A necessary and sufficient condition for the existence of a HP mapping $f : M \to \overline{M}$ between generalized m-parabolic Kähler manifolds M and $\overline{M}$ is given by [18]

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta^h_{j)} + \varphi_{(i}F^h_{j)} + \xi^h_{ij}, \tag{3}$$

where φ_i is a covector, $\psi_i = \varphi_p F^p_i$, and ψ_i is a gradient, i.e., there exists a function ψ such that $\psi_i = \frac{\partial \psi}{\partial x^i}$, and ξ^h_{ij} is an anti-symmetric tensor field of type $(1,2)$ determined by

$$\xi^h_{ij} = \frac{1}{2}\Big(\underset{1}{\overline{T}}{}^h_{ij} - \underset{1}{T}{}^h_{ij}\Big).$$

It is not difficult to prove $\Big($see for instance equation (3.16) in [17]$\Big)$

$$\xi^p_{ip} = \frac{1}{2}\Big(\underset{1}{\overline{T}}{}^p_{ip} - \underset{1}{T}{}^p_{ip}\Big) = 0.$$

Now, by contracting relation (3) on the indices h and j one obtains

$$\psi_i = \frac{1}{n+2}\Big(\overline{\Gamma}^p_{ip} - \Gamma^p_{ip}\Big), \tag{4}$$

and by applying the Voss-Weyl formula in the last relation we get

$$\psi_i = \frac{\partial \psi}{\partial x^i},$$

where the function ψ is defined by

$$\psi := \frac{1}{2(n+2)} \ln\left(\frac{\det \overline{g}}{\det \underline{g}}\right).$$

Substituting (4) into (3) we get

$$\overline{G}^h_{ij} = G^h_{ij} + \varphi_{(i}F^h_{j)} + \xi^h_{ij}, \tag{5}$$

where

$$G^h_{ij} = \Gamma^h_{ij} - \frac{1}{n+2}\Gamma^p_{jp}\delta^h_i - \frac{1}{n+2}\Gamma^p_{ip}\delta^h_j,$$

and $\overline{G}^h_{ij}$ is defined in the same manner in the space $(\overline{M}, \overline{g}, \overline{F})$.

Let us suppose that there exists the bivector $\varepsilon^i\eta_h$ which satisfies

$$F^p_q \varepsilon^q \eta_p = 1, \tag{6}$$

and

$$\xi^p_{qj}\varepsilon^q\eta_p = 0. \tag{7}$$

This bivector is independent of HP mappings of generalized m-parabolic Kähler spaces (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$, see [9].

By contracting (5) with $\epsilon^i\epsilon^j\eta_h$ we get

$$\varphi_p\epsilon^p = \frac{1}{2}\Big(\overline{G}^p_{qr}\epsilon^q\epsilon^r\eta_p - G^p_{qr}\epsilon^q\epsilon^r\eta_p\Big)$$

and contracting (5) with $\epsilon^j\eta_h$ we obtain

$$\varphi_i = \overline{G}^p_{iq}\epsilon^q\eta_p - \frac{1}{2}\overline{G}^p_{qr}\epsilon^q\epsilon^r\eta_p F^s_i\eta_s - \Big(G^p_{iq}\epsilon^q\eta_p - \frac{1}{2}G^p_{qr}\epsilon^q\epsilon^r\eta_p F^s_i\eta_s\Big). \tag{8}$$

Plugging (8) into (5) we get

$$\overline{\mathcal{T}}^h_{ij} = \mathcal{T}^h_{ij},$$

where the geometric object $\mathcal{T}^h_{ij}$ is defined by

$$\begin{aligned}\mathcal{T}^h_{ij} = \Gamma^h_{ij} &- \frac{1}{n+2}\Gamma^p_{jp}\delta^h_i - \frac{1}{n+2}\Gamma^p_{ip}\delta^h_j\\ &-\Big(G^p_{jq}\epsilon^q\eta_p - \frac{1}{2}G^p_{qr}\epsilon^q\epsilon^r\eta_p F^s_j\eta_s\Big)F^h_i\\ &-\Big(G^p_{jq}\epsilon^q\eta_p - \frac{1}{2}G^p_{qr}\epsilon^q\epsilon^r\eta_p F^s_j\eta_s\Big)F^h_j\end{aligned} \tag{9}$$

and the geometric object $\overline{\mathcal{T}}^h_{ij}$ is defined in the same manner in the space $(\overline{M}, \overline{g}, \overline{F})$. Thus we can state the result which is similar to the main result from [9].

Theorem 3.1. *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be generalized m-parabolic Kähler spaces of dimension $n > 2$ and $f : M \to \overline{M}$ be a HP mapping. If there exists a bivector $\varepsilon^i\eta_h$ which satisfies* (6) *and* (7)*, then the geometric object $\mathcal{T}^h_{ij}$ determined by* (9) *is invariant w.r.t. the mapping f.*

As a direct consequence of Theorem 3.1 we have obtained the same result for an equitorsion HP mapping between two generalized *m*-parabolic Kähler spaces.

Corollary 3.1. *Let (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ be generalized m-parabolic Kähler spaces of dimension $n > 2$ and $f : M \to \overline{M}$ be an equitorsion HP mapping. If there exists a bivector $\varepsilon^i\eta_h$ which satisfies condition* (6)*, then the geometric object $\mathcal{T}^h_{ij}$ determined by* (9) *is invariant w.r.t. the mapping f.*

Proof. In the case of an equitorsion HP mapping $f : M \to \overline{M}$ of generalized *m*-parabolic Kähler spaces (M, g, F) and $(\overline{M}, \overline{g}, \overline{F})$ the anti-symmetric tensor ξ^h_{ij} from the basic equations (3) identically vanishes, so the condition (7) is fulfilled and consequently the proof directly follows from Theorem 3.1. □

Remark 3.1. *We should note that in* (9) *generalized Christoffel symbols appeared, so the geometric object $\mathcal{T}^h_{ij}$ given by* (9) *is more general than the corresponding geometric object that was found in* [9].

4. Acknowledgements

The research by Miloš Z. Petrović leading to these results has received funding from the Ministry of Education, Science and Technological Development of the Republic of Serbia, project No. 174012 and the research by Patrik Peška leading to these results has received funding from project IGA PrF_2019_015 Palacký University, Olomouc, Czech Republic.

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