On the Sampson Laplacian

Sergey Stepanov[a], Irina Tsyganok[b], Josef Mikeš[c]

[a]All-Russian Institute for Scientific and Technical Information of the Russian Academy of Sciences, Moscow, Russia
[b]Finance University under the Government of Russian Federation, Moscow, Russia
[c]Palacky University, Olomouc, Czechia

Abstract. In the present paper we consider the little-known Sampson operator that is strongly elliptic and self-adjoint second order differential operator acting on covariant symmetric tensors on Riemannian manifolds. First of all, we review the results on this operator. Then we consider the properties of the Sampson operator acting on one-forms and symmetric two-tensors. We study this operator using the analytical method, due to Bochner, of proving vanishing theorems for the null space of a Laplace operator admitting a Weitzenböck decomposition. Further we estimate operator's lowest eigenvalue.

1. Introduction

Let M be a smooth differentiable manifold of dimension n and g be a Riemannian metric on M. One can associate a number of natural elliptic differential operators to the Riemannian manifold (M, g), which arise from the geometric structure of (M, g). Usually these operators act in the space $C^\infty(E)$ of smooth sections of some vector bundle E over M. The most famous elliptic operator on a Riemannian manifold is the Hodge-de Rham Laplacian Δ_H which acts on C^∞-sections of the vector bundle $\Lambda^q M$ of exterior differential forms (see [4, p. 54]; [36, p. 204]). Forty five years ago J. H. Sampson has defined the second order differential operator Δ_S acting on C^∞-sections of the vector bundle $S^p M$ of covariant symmetric tensors defined on (M, g) (see [38, p. 147]). The operator Δ_S was defined as an analogue of Δ_H. These operators, Δ_S and Δ_H, are self-adjoint and strongly elliptic. Therefore their kernels are finite-dimensional vector spaces on a compact (without boundary) Riemannian manifold. In addition, the Sampson operator Δ_S admits the Weitzenböck decomposition formula as well as the Hodge-de Rham Laplacian (see [38, p. 147]). Therefore we can study this operator Δ_S using the analytical method, due Bochner, of proving vanishing theorems for the null space of a Laplace operator admitting a Weitzenböck decomposition, and further, we can estimate its lowest eigenvalue (see, for example, [4, p. 53]; [36, p. 211]; [2, 5, 10, 14, 28, 29, 48]).

The Sampson operator and its Weitzenböck decomposition formula can be found in the monograph [4, p. 356] and in papers from the following list [6, p. 237]; [47, p. 660]; [7, p. 456]; [15, p. 33]; [23, p. 21]. The authors of these papers and monograph have determined this operator and obtained its Weitzenböck

decomposition but did not quote the source [38]. On the other hand, we were the first and only who began to study the properties of this operator in details (see [3, 31, 33, 39, 41, 42, 44]). To these lists, we can add two papers [24] and [25] in which there are terms "*Sampson Laplacian*" and "*Sampson operator*" but there are no new results on the geometry of the Sampson operator.

The present paper is organized as follows. In the next paragraph, we give a brief review of the Riemannian geometry of the Sampson Laplace operator Δ_S. In the third and fourth paragraphs of the paper we consider the properties of the Sampson operator acting on one-forms and symmetric two-tensors. Theorems and corollaries of the present paper complement our results from the papers [3, 31, 33, 39, 41, 42, 44].

A part of these results was announced in our reports on the International Conference "*XX Geometrical Seminar*" (May 20-23, 2018, Vrnjačka Banja, Serbia)

2. Preliminaries

2.1 Let us first fix some notation and conventions. Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ with its Levi-Civita connection ∇. Everywhere in what follows we denote by $\Lambda^q M = \Lambda^q(\otimes T^*M)$ and $S^p M = S^p(\otimes T^*M)$ the vector bundles of differential q-forms and covariant symmetric p-tensors for the cotangent bundle T^*M on M. Throughout this paper we will consider the vector spaces of their C^∞-sections denoted by $C^\infty \Lambda^q M$ and $C^\infty S^p M$, respectively. The Riemannian metric g induces a metric on the fibres of each of these spaces. If (M, g) is a compact (without boundary) connected manifold then all these spaces are also endowed with *global scalar product* $\langle \cdot, \cdot \rangle$. In particular, the formula

$$\langle \varphi, \psi \rangle = \int_M \frac{1}{p!} \, g(\varphi, \psi) \, dv_g, \tag{1}$$

where $\varphi, \psi \in C^\infty S^p M$ and dv_g is the volume element of (M, g) determines the *global scalar product* or, in other words, $L^2(M, g)$-scalar product on $C^\infty S^p M$. In addition, if (M, g) is not orientable, consider its orientable double covering.

Next, if D is a differential operator between some tensor bundles over M, its formal adjoint D^* is uniquely defined by the formula $\langle D \cdot, \cdot \rangle = \langle \cdot, D^* \cdot \rangle$ (see [4, p. 460]). For example, the covariant derivative $\nabla \colon C^\infty(\otimes^{p+1} T^*M) \to C^\infty(\otimes^p T^*M)$ has the formal adjoint operator ∇^* such that $\nabla^* \colon C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^{p+1} T^*M)$ (see [4, p. 54]).

We recall well-known facts of the *Hodge-de Rham theory*. Firstly, we write $d \colon C^\infty \Lambda^q M \to C^\infty \Lambda^{q+1} M$ for the *familiar exterior differential operator*. Then the codifferentiation operator $\delta \colon C^\infty \Lambda^{q+1} M \to C^\infty \Lambda^q M$ is defined as the formal adjoint to d with respect to the global scalar product (1) by the formula $\langle d\omega, \omega' \rangle = \langle \delta \omega', \omega \rangle$ for arbitrary $\omega \in C^\infty \Lambda^q M$ and $\omega' \in C^\infty \Lambda^{q+1} M$. Secondary, one can construct the well-known *Hodge-de Rham Laplacian* $\Delta_H = \delta d + d\delta$, using the operators d and δ, which is a non-negative self-adjoint elliptic second-order differential operator $\Delta_H \colon C^\infty \Lambda^q M \to C^\infty \Lambda^q M$. In turn, we have an orthogonal (with respect to the global scalar product (1)) Hodge decomposition on compact (without boundary) manifold (M, g)

$$C^\infty \Lambda^q M = \operatorname{Im} \Delta_H \oplus \operatorname{Ker} \Delta_H = \left(\operatorname{Im} d|_{C^\infty \Lambda^{q-1} M} \right) \oplus \left(\operatorname{Im} \delta|_{C^\infty \Lambda^{q+1} M} \right) \oplus \operatorname{Ker} \Delta_H. \tag{2}$$

The space $\operatorname{Ker} \Delta_H$ consists of *harmonic q-forms* on (M, g) (see [36, p. 205]). It is a finite-dimensional vector space $\mathbf{H}^q(M, \mathbb{R})$ with its dimension equal to the Betti number $b_q(M)$ of (M, g) for $q = 1, \ldots, n-1$. In addition, the Hodge-de Rham Laplacian Δ_H admits the *Weitzenböck decomposition* (see [4, pp. 57]) of the form $\Delta_H = \bar{\Delta} + H_q$, where $\bar{\Delta} = \nabla^* \nabla$ is the *rough Laplacian*, or in other words, the *Bochner Laplacian* (see [4, p. 54]; [36, p. 210]; [2, p. 377, 379]), and $H_q \colon \Lambda^q M \to \Lambda^q M$ is an algebraic symmetric operator that depends linearly in a known way on the curvature tensor R and the Ricci tensor Ric of the metric g. In particular, for special case of 1-forms, we have $\Delta_H = \bar{\Delta} + Ric$ (see [4, p. 57]).

2.2 We will apply the above to the operator $\delta^* \colon C^\infty S^p M \to C^\infty S^{p+1} M$ of degree 1 such that $\delta^* = (p+1) \operatorname{Sym} \circ \nabla$ where $\operatorname{Sym} \colon \otimes^p T^*M \to S^p M$ is the linear operator of symmetrization. This means that δ^* is a symmetrized covariant derivative defined by equation (see [4, p. 356])

$$(\delta^* \varphi)(X_1, X_2, \ldots, X_p, X_{p+1}) = (\nabla_{X_1} \varphi)(X_2, \ldots, X_p, X_{p+1}) + \cdots + (\nabla_{X_{p+1}} \varphi)(X_1, X_2, \ldots, X_p, X_p) \tag{3}$$

for any $\varphi \in C^\infty S^p M$ and $X_1, X_2, \ldots, X_p, X_{p+1} \in TM$. Then there exists its formal adjoint operator $\delta \colon C^\infty S^{p+1} M \to C^\infty S^p M$ with respect to the $L^2(M, g)$-product which is called the *divergence operator* (see [4, p. 356]). Notice that δ is nothing but the $\otimes^{p+1} T^* M$ restriction of ∇^* to $S^{p+1} M$. The operators δ^* and δ play the role somewhat analogous to d and δ of the Hodge-de Rham theory.

Using the operators δ^* and δ, Sampson has defined in [38, p. 147] the second order differential operator $\Delta_S \colon C^\infty S^p M \to C^\infty S^p M$ by the formula $\Delta_S = \delta \delta^* - \delta^* \delta$. He has proved that this operator is related to variational problem (see [38, p. 148]). Namely, if we define the "*energy*" of symmetric tensor field φ by $E(\varphi) = 1/2 \, \langle \varphi, \Delta_S \varphi \rangle$, then the equation $\Delta_S \varphi = 0$ is the condition for a free extremum of $E(\varphi)$. At the same time, the tensor field φ was called a *harmonic symmetric tensor* as an analog of harmonic forms of the Hodge-de Rham theory (see [38, p. 148]).

It is easy to verify that if $\langle \Delta_S \varphi, \psi \rangle = \langle \varphi, \Delta_S \psi \rangle$ for any $\varphi, \psi \in C^\infty S^p M$ on a compact (without boundary) manifold (M, g), then the operator Δ_S is a self-adjoint operator with respect to the $L^2(M, g)$-product. In addition, it can be directly verified that the principal symbol σ of the Sampson operator Δ_S satisfies the following condition $\sigma(\Delta_S)(\theta, x)\varphi_x = -g(\theta, \theta)\varphi_x$ for an arbitrary $x \in M$ and $\theta \in T_x^* M - \{0\}$. This means that the Sampson operator Δ_S is the *Laplace operator* and its kernel is a finite-dimensional vector space on a compact (without boundary) manifold (M, g) (see [4, pp. 52, 461-463]). We known that Fredholm alternative guarantees the $L^2(M, g)$-orthogonal decomposition (see also [38, p. 150])

$$C^\infty S^p M = \operatorname{Im} \Delta_S \oplus \operatorname{Ker} \Delta_S, \tag{4}$$

where subspaces $\operatorname{Ker} \Delta_S$ and $\operatorname{Im} \Delta_S$ are orthogonal complements of each other with respect to the global scalar product (1). The space $\operatorname{Ker} \Delta_S$ consists of *harmonic symmetric p-tensor fields* on (M, g) (see [38, p. 148]).

Proceeding from the above, we will always call Δ_S the *Sampson Laplacian*. Compare the Sampson Laplacian Δ_S with the Bochner Laplacian $\bar{\Delta} = \nabla^* \nabla$. First, it is easy to see that these two operators coincide if (M, g) is a locally Euclidean space. Second, the operator $\Delta_S - \bar{\Delta}$ has the order zero and can be defined by symmetric endomorphisms of the bundle $S^p M$. This means that we have the Weitzenböck decomposition formula $\Delta_S = \bar{\Delta} - \Gamma_p$, and $\Gamma_p \colon S^p M \to S^p M$ is an algebraic symmetric operator that depends linearly in a known way on the curvature tensor R and the Ricci tensor $Ricc$ of the metric g (see [38, p. 147]). In particular, for special case of 1-forms, we have $\Delta_S = \bar{\Delta} - Ricc$ (see also [32, 42, 43]).

2.3 At the end of this section we give a nontrivial example of a symmetric harmonic tensor. For this we recall that a 1-dimensional immersed submanifold γ of (M, g) is called a *geodesic* if there exists a parameterization $\gamma \colon x^i = x^i(t)$ for $t \in I \subset \mathbb{R}$ satisfying $\nabla_{\dot{x}} \dot{x} = 0$. If each solution $x^i = x^i(t)$ of the equations $\nabla_{\dot{x}} \dot{x} = 0$ of the geodesics satisfies the condition $\varphi(\dot{x}, \ldots, \dot{x}) = \text{const}$ for smooth covariant symmetric p-tensor φ and $\dot{x} = \dfrac{dx^h}{dt} \cdot \dfrac{\partial}{\partial x^k}$, then the equations $\nabla_{\dot{x}} \dot{x} = 0$ admit so called a *first integral of the p-th order of differential equations of geodesics*. The equation $\delta^* \varphi = 0$ is a necessary and sufficient condition for this [18, pp. 128-129]. The tensor field $\varphi \in C^\infty S^p M$, which satisfies the equation $\delta^* \varphi = 0$, is well known in the theory of General Relativity as a symmetric *Killing tensor* (see, for example, [1, 13, 32, 35, 39], [16, pp. 164-166]). The geometry of the vector space $\mathbf{K}^p(M, \mathbb{R})$ of covariant symmetric Killing p-tensors was studied in our papers [40] and [45]. We recall here that the space $\mathbf{K}^p(M, \mathbb{R})$ is always of finite dimension. In fact, we have the following local result

$$\dim \mathbf{K}^p(M, \mathbb{R}) \le \dim \mathbf{K}^p(\mathbb{S}^n, \mathbb{R}) = \frac{1}{n} \cdot \binom{n+p}{p+1} \cdot \binom{n+p-1}{p},$$

where $\mathbb{S}^n$ is a Euclidean unit sphere of dimension $n \ge 2$ with the standard metric g_0.

Next, we shall consider a divergence-free symmetric Killing p-tensor, which is determined by the two conditions $\delta^* \varphi = 0$ and $\delta \varphi = 0$. An arbitrary a divergence-free symmetric Killing p-tensor φ belongs to $\operatorname{Ker} \Delta_S$. On the other hand, if the Riemannian manifold (M, g) is compact (without boundary), then the converse is also true. Namely, if we suppose that the conditions $\varphi \in \operatorname{Ker} \Delta_S$ and $\delta \varphi = 0$ are satisfied on a compact (without boundary) Riemannian manifold (M, g), then $0 = \langle \Delta_S \varphi, \varphi \rangle = \langle \delta^* \varphi, \delta^* \varphi \rangle$. From the formula above we obtain $\delta^* \varphi = 0$. Then φ is a symmetric Killing p-tensor. Therefore, we can formulate our first theorem.

Theorem 2.1. *Let φ be a divergence-free symmetric Killing tensor on a Riemannian manifold (M, g), then it satisfies the following systems of differential equations*

$$(i) \quad \Delta_S \varphi = 0, \qquad (ii) \quad \delta\varphi = 0$$

for the Sampson Laplacian $\Delta_S \colon C^\infty S^p M \to C^\infty S^p M$. Conversely, if (M, g) is compact and $\varphi \in C^\infty S^p M$ satisfies (i) and (ii), then φ is a divergence-free Killing tensor.

In the papers [15] and [23] we consider a traceless symmetric Killing p-tensor ($p \geq 2$) as a smooth sections of a subbundle $S_0^p M$ of $S^p M$ defined by the condition $\text{trace}_g \, \varphi = \sum\limits_{i=1,\dots,n} \varphi(e_i, e_i, X_3, \dots, X_p) = 0$ for $\varphi \in C^\infty S_0^p M$ and the orthonormal basis $\{e_i\}$ of $T_x M$ at an arbitrary point $x \in M$. It is obvious that every traceless Killing tensor is a divergence-free tensor. Therefore, an arbitrary traceless Killing p-tensor ($p \geq 2$) is a harmonic symmetric tensor. Then we can formulate the following

Corollary 2.2. *Let φ be a traceless symmetric Killing p-tensor ($p \geq 2$) on a Riemannian manifold (M, g) and $\Delta_S \colon C^\infty S^p M \to C^\infty S^p M$ be the Sampson Laplacian, then φ belongs to $\text{Ker} \, \Delta_S$.*

Remark. Some other examples of a harmonic symmetric tensor can be found in [31] and [42].

3. On the kernel and spectral properties of the Sampson Laplacian acting on one-forms

3.1 The explicit expression for Δ_S is sufficiently complicated for $p \geq 2$ but for the case $p = 1$, it has the form $\Delta_S = \bar\Delta - \Gamma_1$ where $\Gamma_1 = Ric$. We obtain $\Delta_S = \Delta_H - 2\,Ric$ from the well known Weitzenböck decomposition formula $\Delta_H = \bar\Delta + Ric$ for the Hodge-de Rham Laplacian Δ_H (see also [32, 42, 43]). Using this formula, we proved that the Sampson Laplacian $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$ is dual to the *Yano Laplacian* $\square \colon C^\infty TM \to C^\infty TM$ by the metric g (see [42]). This operator was defined by Yano in [48, p. 40] for the investigation of local isometric, conformal, affine and projective transformations of compact (without boundary) Riemannian manifolds (see details in [42]).

We recall that the vector field ξ on (M, g) is called an *infinitesimal harmonic transformation* if the one-parameter group $\psi \colon (t, x) \in \mathbb{R} \times M \to \psi_t(x) \in M$ of infinitesimal point transformations of (M, g) generated by ξ consists of harmonic diffeomorphisms (see [43]). We proved in [42] that the kernel of Yano Laplacian consists of all infinitesimal harmonic transformations on (M, g). Therefore, the following theorem is true.

Theorem 3.1. *Let (M, g) be a Riemannian manifold and $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$ be the Sampson Laplacian. An arbitrary $\varphi \in C^\infty S^1 M$ belongs to $\text{Ker} \, \Delta_S$ if and only if the vector field ξ corresponding to φ under the duality defined by the metric g is an infinitesimal harmonic transformation on (M, g).*

Killing vectors are a classical object of the Riemannian geometry. They are infinitesimal isometric by definition, i.e. the flow of such a vector field preserves a given metric. More precisely, a smooth vector field ξ on a Riemannian manifold (M, g) is said to be a Killing vector field if the Lie derivative of the metric tensor g with respect to ξ is zero, that is $L_\xi g = 0$. The following theorem on infinitesimal isometric transformations is a well known old result (see, for example, [5, p. 57] and [26, p. 44]). In addition, the following proposition is a corollary of our Theorem 2.1.

Corollary 3.2. *Let (M, g) be a Riemannian manifold and ξ be a vector field on (M, g). If ξ is an infinitesimal isometry of (M, g), it satisfies the following differential equations:*

$$(i) \quad \Delta_S \varphi := \Delta_H \varphi - 2\,Ric(\xi, \cdot) = 0, \qquad (ii) \quad \delta\varphi = 0$$

for the one-form φ corresponding to ξ under the duality defined by the metric g. Conversely, if (M, g) is compact and ξ satisfies (i) and (ii), then ξ is an infinitesimal isometry.

We proved in [42, 43] that the set of all infinitesimal harmonic transformations of a compact Riemannian manifold (M, g) is a finite-dimensional vector space over $\mathbb{R}$. From our Theorem 3.1 we conclude that the Lie algebra of infinitesimal isometric transformation $i(M)$ of (M, g) is a subspace of this vector space. It is well known that $\dim i(M) = 1/2\, n(n + 1)$ on an n-dimensional Riemannian manifold (M, g) of constant curvature (see, for example, [26, pp. 46-47]). Killing 1-forms are just the metric duals to Killing vector fields. Therefore, $\dim \mathbf{K}^1(M, \mathbb{R}) = 1/2\, n(n + 1)$ on an n-dimensional Riemannian manifold of constant curvature.

Remark. These statements concern the dimensions of the vector spaces of Killing one-forms and Killing vector fields in a neighborhood of an arbitrary point of the Riemannian manifold (see, for example, [26, pp. 55-59]).

If we define $\dim \operatorname{Ker} \Delta_S$ as the number of linearly independent (with constant real coefficients) one-forms which correspond to infinitesimal harmonic transformations of (M, g) under the duality defined by the metric g, then $\dim \operatorname{Ker} \Delta_S \geq 1/2\, n(n + 1)$ on an n-dimensional Riemannian manifold of constant curvature C. Therefore, we have the following local result.

Theorem 3.3. *The dimension of the kernel of the Sampson Laplacian* $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$ *on an n-dimensional Riemannian manifold of constant curvature is at least* $1/2\, n(n + 1)$.

Remark. We have proved that a holomorphic vector field on a nearly Kählerian manifold and the vector field that transforms a Riemannian metric into a Ricci soliton metric are examples of infinitesimal harmonic transformations (see [42, 43]). Therefore, all one-forms which correspond to these vector fields under the duality defined by the metric g belong to $\varphi \in \operatorname{Ker} \Delta_S$ for the Sampson Laplacian $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$.

3.2 Let φ be an arbitrary one-form such that $\varphi \in \operatorname{Ker} \Delta_S$. In accordance with the theory of *harmonic maps* we define the *energy density* of the flow on (M, g) generated by the vector field $\xi = \varphi^\sharp$ as the scalar function $e(\xi) = 1/2\, \|\xi\|^2$ where $\|\xi\|^2 = g(\xi, \xi)$. Then the *Beltrami Laplacian* $\Delta_B e(\xi) := -\bar\Delta\, e(\xi)$ for the energy density $e(\xi)$ of an infinitesimal harmonic transformation $\xi = \varphi^\sharp$ has the form (see [42])

$$\Delta_B\, e(\xi) = \|\nabla \varphi\|^2 - Ric(\xi, \xi). \tag{5}$$

We recall that the Ricci curvature of g is *quasi-negative* in a connected open domain if it is nonnegative everywhere in $U \subset M$ and it is strictly negative in all directions at some point of U. In this case, $e(\xi)$ is a *subharmonic function*. Then, using the *Hopf's maximum principle* (see [9]), we can formulate the following

Theorem 3.4. *Let (M, g) be a Riemannian manifold and $U \subset M$ be a connected open domain with the quasi-negative Ricci tensor Ric. If the energy density of the flow $e(\xi) = 1/2\, \|\xi\|^2$ generated by $\xi = \varphi^\sharp$ for an arbitrary one-form $\varphi \in \operatorname{Ker} \Delta_S$ has a local maximum in some point of U, then φ is identically zero everywhere in U.*

Remark. The last theorem is a direct generalization of the Theorem 4.3 presented in Kobayashi's monograph on transformation groups (see [26, p. 57]) and Wu's proposition on a Killing vector which length achieves a local maximum (see [46]).

We can formulate the following statement, which is a corollary of Theorem 3.4 (see also [42]).

Corollary 3.5. *The Sampson Laplacian* $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$ *has a trivial kernel on a compact Riemannian manifold (M, g) with quasi-negative Ricci curvature.*

3.3 A metric g is called *Einstein* if it satisfies the Einstein equation $Ric = n^{-1} s\, g$. A Riemannian manifold (M, g) is called *Einstein manifold* if the metric g is Einstein. In the case $n = \dim M \geq 3$, the scalar curvature s of (M, g) is constant (see [4, p. 44]).

Let (M, g) be an n-dimensional $(n \geq 3)$ Einstein manifold then from the formula $\Delta_S = \Delta_H - 2\, Ric$ we obtain $\Delta_H = 2n^{-1} s\varphi$ for an arbitrary one-form $\varphi \in \operatorname{Ker} \Delta_S$. Therefore, a nonzero one-form $\varphi \in \operatorname{Ker} \Delta_S$ is the eigenform of the Hodge-de Rham Laplacian with eigenvalue $\lambda = 2n^{-1} s$. The converse is also true. In particular, if (M, g) is compact (without boundary) then the scalar curvature s of (M, g) must be positive.

Theorem 3.6. *Let (M, g) be an n-dimensional $(n \geq 3)$ Einstein manifold with scalar curvature s. Then an arbitrary one-form $\varphi \in \operatorname{Ker} \Delta_S$ if and only is φ is an eigenform of the Hodge-de Rham Laplacian Δ_H with eigenvalue $\lambda = 2n^{-1} s$.*

In conclusion, we prove the following

Corollary 3.7. *For any an n-dimensional ($n \geq 3$) compact Einstein manifold (M, g) with positive scalar curvature one has the following $L^2(M, g)$-orthogonal decomposition*

$$\mathrm{Ker}\,\Delta_S = \mathrm{Ker}\,\Delta_S \cap (\mathrm{Im}\,d|_{C^\infty M}) \oplus \mathrm{Ker}\,\Delta_S \cap (\mathrm{Ker}\,\delta|_{C^\infty S^1 M})$$

where $\mathrm{Ker}\,\Delta_S \cap (\mathrm{Im}\,d|_{C^\infty M})$ *consists of one-forms which corresponding to gradient infinitesimal harmonic transformations under the duality defined by the metric g and* $\mathrm{Ker}\,\Delta_S \cap (\mathrm{Ker}\,\delta|_{C^\infty S^1 M})$ *consists of Killing one-forms.*

Proof. For the proof of this statement, we recall that the vector space $C^\infty \Lambda^q M$ has the following $L^2(M, g)$-orthogonal decomposition (2). In particular, for $q = 1$ we have $\varphi = \theta + df$ where $f \in C^\infty M$ and $\theta \in C^\infty \Lambda^1 M$ such that $\delta\theta = 0$. Then, according to our Theorem 3.3, the equation $\Delta_S \varphi = 0$ can be rewritten in the form $\Delta_H(\theta + df) = 2n^{-1}s(\theta + df)$. In turn, from this equation we obtain the formula $\bar{\Delta}(\bar{\Delta}f - 2n^{-1}sf) = 0$ that is based on the following equality $\Delta_H u = \bar{\Delta}u = \delta du$ for an arbitrary scalar function $u \in C^\infty M$. In this case, the scalar function $\bar{\Delta}f - 2n^{-1}sf$ is harmonic. Therefore, proceeding from the above formula and using the *Bochner maximum principle* (see [5, p. 30]), we conclude that $\bar{\Delta}f = 2n^{-1}sf + C$ for some constant C. In this case, we can rewrite the equation $\Delta_H(\theta + df) = 2n^{-1}s(\theta + df)$ in the following form $\Delta_H \theta - 2n^{-1}s\theta = -d(\bar{\Delta}f - 2n^{-1}s f) = 0$. By consequence we have the equations $\Delta_H \theta = 2n^{-1}s\,\theta$ and $\Delta_H df = 2n^{-1}s\,df$. These equations mean that the divergence-free one-form θ is a Killing one-form and the vector field $\xi = (df)^\sharp$ is a gradient infinitesimal harmonic transformation. The proof is complete.

3.4 Denote by the constant λ an arbitrary eigenvalue of the Sampson Laplacian acting on one-forms, i.e. $\Delta_S \varphi = \lambda\varphi$ for $\varphi \in C^\infty S^1 M$. Using the general theory of elliptic operators on a compact (without boundary) connected oriented Riemannian manifold (M, g), it can be proved that the Sampson Laplacian $\Delta_S : C^\infty S^1 M \to C^\infty S^1 M$ has a discrete spectrum, denoted by $\mathrm{Spec}\,\Delta_S$, consisting of real eigenvalues of finite multiplicity which accumulate only at infinity, i.e. in symbols, we have $\mathrm{Spec}\,\Delta_S = \{0 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots \to +\infty\}$ and $\dim V_{\lambda_a} < +\infty$ for an arbitrary eigenvalue λ_a. In addition, the following theorem about eigenvalues of Δ_S and their corresponding one-forms is valid. Then the following theorem is true (see [42])

Theorem 3.8. *Let (M, g) be an n-dimensional ($n \geq 2$) compact Riemannian manifold and $\Delta_S : C^\infty S^1 M \to C^\infty S^1 M$ be the Sampson Laplacian defined on one-forms.*

1. *Suppose the Ricci curvature is negative then an arbitrary eigenvalue λ_a of Δ_S is positive.*
2. *The eigenspaces of Δ_S are finite dimensional.*
3. *The eigentensors corresponding to distinct eigenvalues are orthogonal.*

Let φ be an eigenform of Δ_S with eigenvalue λ then from the Weitzenböck decomposition formula $\Delta_S \varphi = \bar{\Delta}\varphi - Ric(\xi, \cdot)$ for $\xi = \varphi^\sharp$ we obtain

$$\lambda\langle\varphi, \varphi\rangle = \langle\nabla\varphi, \nabla\varphi\rangle - \int_M Ric(\xi, \xi)\,dv_g \geq -\int_M Ric(\xi, \xi)\,dv_g. \tag{6}$$

Let $Ric \leq \mu \cdot g$ where μ is the upper bound of Ricci curvature of a compact (without boundary) Riemannian manifold (M, g). Then $\int_M Ric(\xi, \xi)\,dv_g \leq \mu\langle\varphi, \varphi\rangle$. In this case, we conclude from (6) that $\lambda \geq -\mu$. In particular, if (M, g) is an Einstein manifold with constant scalar curvature s then $\lambda \geq -n^{-1}s$. We proved the following

Theorem 3.9. *Let (M, g) be an n-dimensional ($n \geq 3$) compact Riemannian manifold and μ be the upper bound of its Ricci curvature. Then an arbitrary eigenvalue λ of the Sampson Laplacian acting on one-forms bounded from below by the number $-\mu$. In particular, if (M, g) is a compact Einstein manifold then $\lambda \geq -n^{-1}s$ for the constant scalar curvature s of (M, g).*

Let (M, g) be an n-dimensional ($n \geq 3$) Einstein manifold then $Ric = n^{-1}s\,g$ for the constant scalar curvature s of (M, g). Then from the formula $\Delta_S = \Delta_H - 2\,Ric$ we obtain $\Delta_S \varphi = -2n^{-1}s\,\varphi$ for an arbitrary one-form form $\varphi \in \mathbf{H}^1(M, \mathbb{R})$. This means that an arbitrary harmonic form is an eigenform of the Sampson Laplacian with eigenvalue $\lambda = -2n^{-1}s$. The converse is also true. We proved the following statement.

Theorem 3.10. *Let (M, g) be an n-dimensional ($n \geq 3$) Einstein manifold then an arbitrary one-form on (M, g) is harmonic if and only if it is an eigenform of the Sampson Laplacian with eigenvalue $\lambda = -2n^{-1}s$ for the scalar curvature s of (M, g).*

From this theorem we obtain the following

Corollary 3.11. *Let (M, g) be an n-dimensional ($n \geq 3$) compact and orientable Einstein manifold with negative constant scalar curvature s. If its first Betti number $b_1(M) \neq 0$, then the number of linearly independent (with constant real coefficients) eigenforms of the Sampson Laplacian $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$ with eigenvalue $-2n^{-1}s$ equals to $b_1(M)$.*

Proof. It is well known that in a compact (without boundary) and orientable Riemannian manifold (M, g), the number of linearly independent (with constant real coefficients) harmonic one-forms is equal to the first Betti number $b_1(M)$ of the manifold (M, g). Therefore, if $b_1(M) \neq 0$ for a compact (without boundary) Einstein manifold (M, g) then the Sampson Laplacian $\Delta_S \colon C^\infty S^1 M \to C^\infty S^1 M$ has $b_1(M)$ linearly independent eigenforms. In conclusion, we recall that if $Ric = n^{-1}s \, g > 0$, then $b_1(M) = 0$ (see [4, p. 57]). Therefore, the constant scalar curvature s of (M, g) must be negative.

In particular, if (M, g) is the hyperbolic space $(\mathbb{H}^n, g_0)$ with standard metric g_0 having constant sectional curvature which equals to -1, then we have $\Delta_S \varphi = \Delta_H \varphi - 2(n-1)\varphi$ for any one-form φ. In this case, the eigenvalue of Δ_S is $-2(n-1)$ for an arbitrary harmonic one-form φ.

Remark. More detailed information on the Sampson Laplacian can be found in our paper [42].

4. On the kernel and spectral properties of the Sampson Laplacian acting on symmetric two-tensors

4.1 In this section we consider the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields $S^2 M$ on M. In this case, we have the Weitzenböck decomposition formula

$$\Delta_S \varphi = \bar{\Delta} \varphi - \Gamma_2(\varphi), \tag{7}$$

where $\Gamma_2(\varphi) = (R_{ik}\varphi_j^k + R_{jk}\varphi_i^k) - 2 R_{ikjl}\varphi^{kl}$ for the local components φ_{ij} of $\varphi \in C^\infty S^2 M$ (see [47, p. 660]). Then by direct calculations from (7) we obtain the formula $\mathrm{trace}_g(\Delta_S \varphi) = \bar{\Delta} \, \mathrm{trace}_g \varphi$ for an arbitrary $\varphi \in C^\infty S^2 M$. Therefore the following lemma holds.

Lemma 4.1. *Let $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields $S^2 M$ over a Riemannian manifold (M, g), then $\mathrm{trace}_g(\Delta_S \varphi) = \bar{\Delta} \, \mathrm{trace}_g \varphi$.*

If φ is a harmonic symmetric 2-tensor field, then $\mathrm{trace}_g(\Delta_S \varphi) = \bar{\Delta} \, \mathrm{trace}_g \varphi = 0$. In this case, $\mathrm{trace}_g \varphi$ is a harmonic scalar function defined on (M, g). In addition, if (M, g) is a compact (without boundary) manifold, then by the *Bochner maximum principle* for harmonic functions we conclude that $\mathrm{trace}_g \varphi$ is a constant function (see [5, p. 30]). In particular, when we have $\Delta_S \varphi^\circ = 0$ for the traceless tensor $\varphi^\circ := \varphi - n^{-1}(\mathrm{trace}_g \varphi) \varphi$. We proved the following

Corollary 4.2. *Let $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields $S^2 M$ over a compact Riemannian manifold (M, g). Then the trace of an arbitrary $\varphi \in \mathrm{Ker}\, \Delta_S$ is a constant function and $\varphi^\circ := \varphi - n^{-1}(\mathrm{trace}_g \varphi) \varphi$ belongs to $\mathrm{Ker}\, \Delta_S$.*

We obtain the formula by direct calculations from the Weitzenböck decomposition formula (7):

$$1/2 \, \Delta_B \|\varphi\|^2 = -g(\Delta_S \varphi, \varphi) + \|\nabla \varphi\|^2 - g(\Gamma_2(\varphi), \varphi) \tag{8}$$

where $\Delta_B = \mathrm{div} \circ \mathrm{grad}$ is the *Beltrami Laplacian* on functions. In addition, for any point $x \in M$ there exists an orthonormal eigenframe $e_1, \ldots, e_n$ of $T_x M$ such that $\varphi_x(e_i, e_j) = \mu_i \, \delta_{ij}$ for the Kronecker delta δ_{ij}. Then we have (see [4, p. 436], [3, p. 388])

$$g(\Gamma_2(\varphi_x), \varphi_x) = 2 \sum_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2. \tag{9}$$

where $\sec(e_i \wedge e_j) = R(e_i, e_j, e_i, e_j)$ is the *sectional curvature* $\sec \sigma_x$ of (M, g) in the direction of the two-plane $\sigma_x = \text{span}\{e_i, e_j\}$ at $x \in M$. In this case, the formula (8) can be rewritten in the form

$$1/2 \, \Delta_B \|\varphi\|^2 = -g(\Delta_S \varphi, \varphi) + \|\nabla \varphi\|^2 - 2 \sum_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2). \tag{10}$$

In particular, if φ is a covariant harmonic symmetric 2-tensor, then from the formula (10) we obtain

$$1/2 \, \Delta_B \|\varphi\|^2 = \|\nabla \varphi\|^2 - 2 \sum_{i<j} \sec(e_i \wedge e_j)(\mu_i - \mu_j)^2. \tag{11}$$

Therefore, proceeding from the above formula and using the Hopf maximum principle (see [9]), we can conclude that if the section curvature of (M, g) is nonpositive at any point of a connected open domain $U \subset M$ and is in addition negative (in all directions σ_x) at a point x of U, then $\|\varphi\|^2$ is constant and $\nabla \varphi = 0$ in U. If $C > 0$, then φ is nowhere zero. Now, at a point x where the section curvature $\sec(e_i \wedge e_j)$ is negative, the left side of (11) is zero while the right side is nonnegative. This contradiction shows $\mu_1 = \cdots = \mu_n = \mu$ and hence $\varphi = \mu \cdot g$ for some constant μ everywhere in U.

On the other hand, the fact that $\nabla \varphi = 0$ means that φ is invariant under parallel translation. In this case, if the holonomy of (M, g) is irreducible, then the tensor φ has a one eigenvalue, i.e. $\varphi = \mu \cdot g$ for some constant μ at each point of U. As a result, we have the following

Theorem 4.3. *Let U be a connected open domain of a Riemannian manifold (M, g) and φ be a harmonic symmetric 2-tensor field defined on U. If the section curvature of (M, g) is negative semi-define at any point of U and the scalar function $\|\varphi\|^2$ has a local maximum at some point of U, then $\|\varphi\|^2$ is a constant function and φ is invariant under parallel translation in U. Moreover, if $\sec < 0$ at some point of U or if the holonomy of (M, g) is irreducible, then φ is constant multiple of g at all points of U.*

Let us consider a *Hadamar manifold* which is a complete simply connected nonpositively curved manifold (M, g) by definition (see [29, p. 381]). For this case we can prove the following

Corollary 4.4. *Let (M, g) be a Hadamar manifold or, in particular, a Riemannian symmetric manifold (M, g) of the non-compact type. If φ is a harmonic symmetric 2-tensor on (M, g) such that $\int_M \|\varphi\|^q \, d\,Vol_g < +\infty$ at least for one $q \geq 1$, then it is invariant under parallel translation. In addition, if the volume of (M, g) is infinite, then $\varphi \equiv 0$.*

Proof. Let φ is a non-zero harmonic symmetric 2-tensor on a Riemannian manifold with nonpositive sectional curvature, then from the formula (11) we obtain

$$1/2 \, \Delta_B \|\varphi\|^2 \geq \|\nabla \varphi\|^2 \geq 0. \tag{12}$$

This means that $\|\varphi\|^2$ is a *subharmonic function*. On the other hand, the following theorem was proved in [49, p. 663]: *Let u be a nonnegative subharmonic function on a complete manifold (M, g), then $\int_M u^q \, dv_g = \infty$ for $q > 1$, unless u is a constant function C.* In our case, this means that $C \cdot \int_M d\,Vol_g < +\infty$ for $\|\varphi\|^2 = C$. From this we conclude that if the volume of (M, g) is infinite, then the harmonic symmetric 2-tensor φ is identically zero on (M, g).

Let (M, g) be a compact manifold with nonpositive sectional curvature. Then based on (11) and the Bochner maximum principle, we can conclude that the kernel of the Sampson Laplacian $\Delta_S: C^\infty S^2 M \to C^\infty S^2 M$ consists of parallel symmetric 2-tensor tensor fields on (M, g), i.e. from the condition $\varphi \in \text{Ker} \Delta_S$ we obtain $\nabla \varphi = 0$. From this implies topological restrictions namely if a Riemannian manifold (M, g) admits a parallel symmetric 2-tensor field then (M, g) is locally the direct product of a number of Riemannian manifolds (see [18]). Another situation where the parallelism of φ is involved appears in the theory of affine mappings, namely, as is point out in [34], $\nabla \varphi = 0$ is equivalent with the fact that the identity map $Id_M: (M, g) \to (M, \varphi)$ is an affine map. Therefore, we can formulate the following statement.

Theorem 4.5. *Let (M, g) be a compact and connected Riemannian manifold with nonpositive sectional curvature and $\Delta_S: C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting C^∞-sections of the bundle of covariant symmetric two-tensor fields $S^2 M$ on (M, g). If $\varphi \in Ker\Delta_S$ then $Id_M: (M, g) \to (M, \varphi)$ is an affine map.*

4.2 Let us consider here the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ on a compact manifold (M, g). Denote by the constant λ an arbitrary eigenvalue of the Sampson Laplacian acting on symmetric 2-tensor fields, i.e. $\Delta_S \varphi = \lambda \varphi$ for some $\varphi \in C^\infty S^2 M$. We recall that all nonzero eigentensor $\varphi \in C^\infty S^2 M$ corresponding to a fixed eigenvalue λ form a vector subspace of $C^\infty S^2 M$ denoted by $V_\lambda(M)$ and called the eigenspace corresponding to the eigenvalue λ. Using the general theory of elliptic operators on a compact (without boundary) connected oriented Riemannian manifold (M, g), it can be proved that the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ has a discrete spectrum, denoted by $\operatorname{Spec} \Delta_S$, consisting of real eigenvalues of finite multiplicity which accumulate only at infinity, i.e. in symbols, we have $\operatorname{Spec} \Delta_S = \{ 0 \le |\lambda_1| \le |\lambda_2| \le \cdots \to +\infty \}$ and $\dim V_{\lambda_a} < +\infty$ for an arbitrary eigenvalue λ_a, $a = 1, 2, \ldots$. Now we prove the following theorem.

Theorem 4.6. *Let (M, g) be an n-dimensional ($n \ge 2$) compact and connected Riemannian manifold and $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields $S^2 M$.*

1. *Suppose the section curvature is negative then an arbitrary eigenvalue λ_a of Δ_S is positive.*
2. *The eigenspaces of Δ_S are finite dimensional.*
3. *The eigentensors corresponding to distinct eigenvalues are orthogonal.*

Proof. **1.** Let $\varphi \in V_{\lambda_a}(M)$ be a non-zero eigentensor corresponding to the eigenvalue λ_a, that is $\Delta_S \varphi = \lambda_a \varphi$, then we can rewrite the formula (7) in the form $\lambda_a \varphi = \bar{\Delta} \varphi - \Gamma_2(\varphi)$. In addition, for any point $x \in M$ there exists an orthonormal basis $e_1, \ldots, e_n$ of $T_x M$ such that $\varphi_x(e_i, e_j) = \mu_i \delta_{ij}$ for the Kronecker delta δ_{ij}. Then, using (6), we obtain the integral formula

$$\lambda_a \langle \varphi, \varphi \rangle = -2 \int_M \sum_{i<j} \sec(e_i \wedge e_j) \, (\mu_i - \mu_j)^2 \, dv_g + \langle \nabla \varphi, \nabla \varphi \rangle. \tag{13}$$

Now we suppose that the section curvature of (M, g) is negative, then from (13) we obtain $\lambda_a > 0$. Therefore, if the section curvature of (M, g) is negative then the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ has the spectrum $\operatorname{Spec} \Delta_S = \{ 0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty \}$.

2. The eigenspaces of Δ_S are finite dimensional because it is an elliptic operator.

3. Let $\lambda_a \ne \lambda_b$ and φ_a, φ_b be the corresponding eigenforms. Then $\langle \Delta_S \varphi_a, \varphi_b \rangle = \lambda_a \langle \varphi_a, \varphi_b \rangle$ and $\langle \Delta_S \varphi_b, \varphi_a \rangle = \lambda_b \langle \varphi_b, \varphi_a \rangle$. Therefore $0 = (\lambda_a - \lambda_b) \langle \varphi_a, \varphi_b \rangle$ and since $\lambda_a \ne \lambda_b$ it follows that $\langle \varphi_a, \varphi_b \rangle = 0$, that φ_a and φ_b are orthogonal. This completes the proof of our Theorem 4.5.

If we assume that $\operatorname{trace}_g \varphi = C$ for some constant $C \ne 0$ then from the equation $\Delta_S \varphi = \lambda \varphi$ we obtain $\lambda = 0$, since the identity $\operatorname{trace}_g(\Delta_S \varphi) = \bar{\Delta} \operatorname{trace}_g \varphi$ holds for an arbitrary $\varphi \in C^\infty S^2 M$ (see our Lemma 4.1). Therefore, if $\varphi \in C^\infty S^2 M$ is an eigentensor corresponding to the eigenvalue $\lambda \ne 0$ of the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ on a compact manifold (M, g) then $\operatorname{trace}_g \varphi$ is not a nonzero constant function.

Proceeding from the above, we will distinguish two cases. Firstly, we will consider the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ acting on covariant symmetric two-tensor fields with nonzero traces. Secondary, we will consider the Sampson Laplacian $\Delta_S \colon C^\infty S_0^2 M \to C^\infty S_0^2 M$ acting on C^∞-sections of the bundle of covariant symmetric traceless two-tensor fields $S_0^2 M$ on M.

Let $\varphi \in C^\infty S^2 M$ be an eigentensor of Δ_S corresponding to the eigenvalue λ and $\operatorname{trace}_g \varphi$ be a non-constant function. Then by direct calculations we obtain the formula $\bar{\Delta} \operatorname{trace}_g \varphi = \lambda \operatorname{trace}_g \varphi$. The investigation of the Laplace equation $\bar{\Delta} f = \lambda f$ for the non-constant scalar function $f = \operatorname{trace}_g \varphi$ is a priori a problem of analysis. In particular, we recall the following well-known classical results. If (M, g) is a compact and connected Riemannian manifold, then there exists a sequence $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_a \le \ldots$ of non-negative negative real numbers with finite multiplicities, and $L^2(M, g)$-orthonormal basis $\{ f_1, f_2, \ldots, f_a, \ldots \}$ of real scalar C^∞-functions such that $\bar{\Delta} f_a = \lambda_a f_a$ (see [28, p. 77]). This assertion agrees with our Theorem 4.6. Moreover, from above, we can conclude that if an eigenvalue λ of the Sampson Laplacian $\Delta_S \colon C^\infty S^2 M \to C^\infty S^2 M$ corresponds to some covariant symmetric two-tensor field φ with nonzero trace, then it is non-negative.

Moreover, if λ is an eigenvalue of $\bar{\Delta}$ and $f \in C^\infty M$, $f \ne 0$, is an associated eigenfunction, i.e. $\bar{\Delta} f = \lambda f$, then $\lambda = R(f)$ for the *Rayleigh quotient* $R(f)$ on a scalar function $f \ne 0$ which is defined by the equality

$R(f) = \langle df, df \rangle \cdot \langle f, f \rangle^{-1}$ (see [28, p. 91]). Therefore, if $\varphi \in C^\infty S^2 M$ is an eigentensor of Δ_S such that $\text{trace}_g \varphi$ is a non-constant scalar function and λ is an associated eigenvalue, then $\lambda = R(\text{trace}_g \varphi)$. As a result we have the following

Theorem 4.7. *Let (M, g) be a compact and connected Riemannian manifold and $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting C^∞-sections of the bundle of covariant symmetric two-tensor fields $S^2 M$ on (M, g). If $\varphi \in C^\infty S^2 M$ is an eigenfunction of Δ_S such that $\text{trace}_g \varphi$ is a non-constant scalar function and λ is an associated eigenvalue, then $\lambda = R(\text{trace}_g \varphi) \geq 0$.*

Further, we will estimate the first eigenvalue λ_1 of the Sampson Laplacian $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$ in the case when its eigentensor φ has $\text{trace}_g \varphi \neq 0$. In this case, the following theorem is true.

Theorem 4.8. *Let (M, g) be a compact and connected Riemannian manifold with the diameter $d(M)$ and the Ricci curvature $\text{Ric} \geq (n - 1) K$. If the eigentensor $\varphi \in C^\infty S^2 M$ which corresponds to the first nonzero eigenvalue λ_1 of the Sampson Laplacian $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$ has the non-constant $\text{trace}_g \varphi$, then $\lambda_1 \geq nK$ for the case $K > 0$,*
$$\frac{\pi^2}{4d(M)^2} \leq \lambda_1 \leq \frac{n\pi^2}{d(M)^2} \quad \text{for the case } K = 0 \text{ and } \lambda_1 \geq \frac{exp - (1 + (1 - 4(n-1)^2 \, d(M)^2 \, K)^{1/2})}{2(n-1)^2 \, d(M)^2} \quad \text{for the case } K < 0.$$
In particular, if (M, g) is a compact hyperbolic manifold $(\mathbb{H}^2, g_0)$ with $n \geq 3$ and standard metric g_0 having constant sectional curvature which equals to -1, then there exists a constant $C = C(n) > 0$ such that $\lambda_1 \geq C \cdot \text{Vol}(M)^{-2}$.

Proof. For the proof our theorem it is sufficient to recall the following well-known facts. Suppose that $d(M)$ denotes the diameter of a compact and connected Riemannian manifold (M, g) with the Ricci curvature $\text{Ric} \geq (n - 1) K$ for some constant K. Then we have the following three cases (see [28, pp. 114, 116]). The first, if $K < 0$ then $\lambda_1 \geq \dfrac{exp - (1 + (1 - 4(n-1)^2 \, d(M)^2 \, K)^{1/2})}{2(n-1)^2 \, d(M)^2}$. The second, if $K > 0$, then $\lambda_1 \geq n K$. The third, if $K = 0$, then $\lambda_1 \geq \dfrac{\pi^2}{4d(M)^2}$ for the first eigenvalue λ_1 of $\bar{\Delta}$.

We recall here that the third result belongs to P. Li and S.-T. Yau (see [34]) and must be compared with a result of S. T. Chen: $\lambda_1 \leq \dfrac{n\pi^2}{d(M)^2}$ (see [11]). In particular, if (M, g) is a compact n-dimensional $(n \geq 3)$ hyperbolic manifold $(\mathbb{H}^n, g_0)$ with standard metric g_0, then there exists a constant $C = C(n) > 0$ such that $\lambda_1 \geq C \cdot \text{Vol}(M)^{-2}$ (see [12]).

Next, we will consider the Sampson Laplacian acting on the smooth sections of the vector bundle of trace-free symmetric 2-tensor fields $S_0^2 M$ on a compact Riemannian manifold (M, g). The following obvious statement is true.

Theorem 4.9. *The Sampson Laplacian Δ_S maps $S_0^2 M$ to itself.*

Let (M, g) be an n-dimensional compact and oriented Riemannian manifold with strictly negative sectional curvature and $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting on trace-free symmetric 2-tensor fields. If we denote by $K_{\max}$ the maximum of the negative defined sectional curvature of (M, g), i.e. $\sec(\sigma_x) \leq K_{\max}$ in all directions σ_x at each point $x \in M$, then from (13) we obtain the integral inequality

$$\lambda_a \langle \varphi, \varphi \rangle \geq -2n K_{\max} \int_M \sum_{i<j} (\mu_i - \mu_j)^2 \, dv_g + \langle \nabla \varphi, \nabla \varphi \rangle \geq 0 \tag{14}$$

for an arbitrary eigenvalue λ_g corresponding to a non-zero eigentensor $\varphi \in C^\infty S_0^2 M$ of Δ_S. If the condition $\text{trace}_g \varphi = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2 = 0$ holds then it is not difficult to prove the following equality

$$\|\varphi\|^2 = \mu_1^2 + \mu_2^2 + \cdots + \mu_n^2 = \frac{1}{n} \sum_{i<j} (\mu_i - \mu_j)^2.$$

In this case, from (14) one can obtain the integral inequality

$$(\lambda_a + 2n \, K_{\max}) \int_M \|\varphi\|^2 \, dv_g \geq \|\nabla \varphi\|^2 \geq 0 \tag{15}$$

Then from (15) we conclude that $\lambda_a \geq -2n\,K_{\max}$ for an arbitrary eigenvalue λ_a. In turn, if the first eigenvalue $\lambda_1 = -2n\,K_{\max}$, then its corresponding 2-tensor field φ is invariant under parallel translation. In this case, if the holonomy of (M, g) is irreducible, then at each point of (M, g) the tensor φ must have the form $\varphi = \mu \cdot g$ for some constant μ. But in our case, $\mathrm{trace}_g \varphi = 0$ and, consequently, we have $\mu = 0$. The following statement is true.

Theorem 4.10. *Let (M, g) be an n-dimensional ($n \geq 2$) compact Riemannian manifold and $\Delta_S\colon C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian acting on trace-free symmetric 2-tensor fields. Then the first eigenvalue of Δ_S satisfies the inequality $\lambda_1 \geq -2n\,K_{\max}$ for the maximum $K_{\max}$ of the strictly negative sectional curvature of (M, g). If $\lambda_1 = -2n\,K_{\max}$, then the trace-free symmetric 2-tensor field φ corresponding to λ_1 is invariant under parallel translation. In particular, if the holonomy of (M, g) is irreducible, then this relation means that $\varphi \equiv 0$.*

Remark. Suppose now that $(\mathcal{H}^n, g_0)$ is a compact n-dimensional ($n \geq 3$) hyperbolic manifold with standard metric g_0 having constant sectional curvature which equals to -1. Then the first eigenvalue λ_1 of the Sampson Laplacian $\Delta_S\colon C^\infty S_0^2 M \to C^\infty S_0^2 M$ satisfies the inequalities $\lambda_1 \geq 2n$.

5. Spectrum of the Sampson Laplacian acting on TT-tensors

We recall here the definition of the well known *Lichnerowicz Laplacian* Δ_L acting on $C^\infty(\otimes^p T^*M)$. It is an elliptic linear differential operator of second order $\Delta_L\colon C^\infty(\otimes^p T^*M) \to C^\infty(\otimes^p T^*M)$ for $p \geq 0$ which is determined by $\Delta_L T = \bar{\Delta} T + \Gamma_p(T)$ for an arbitrary $T \in C^\infty(\otimes^p T^*M)$ (see [4, p. 54]). It is self-adjoint with respect to the $L^2(M, g)$-product, and coincides on $S^0 M = C^\infty M$ with ordinary Laplacian on C^∞-functions.

Let us consider here the Lichnerowicz Laplacian $\Delta_L\colon C^\infty S^2 M \to C^\infty S^2 M$ acting on symmetric 2-tensor fields that has the form $\Delta_L \varphi = \bar{\Delta} \varphi + \Gamma_2(\varphi)$ for an arbitrary $\varphi \in C^\infty S^2 M$ (see, for example, [47]). Then we have the following equation

$$\Delta_S \varphi = \Delta_L \varphi - 2\Gamma_2(\varphi). \tag{16}$$

Let (M, g) be a manifold of the constant sectional curvature C, then $R_{ikjl} = C\,(g_{ij}g_{kl} - g_{il}g_{jk})$ and $R_{ik} = (n-1)\,C\,g_{ik}$. In this case, the equation (16) can be rewritten in the form

$$\Delta_S \varphi = \Delta_L \varphi - 4n\,C\,\varphi. \tag{17}$$

for any $\varphi \in C^\infty S^2 M$.

We recall here that a TT-tensor (*Transverse Traceless tensor*) is by definitions a symmetric divergence free and traceless covariant 2-tensor (see, for instance, [22]). Such tensors are of fundamental importance in *stability analysis* in General Relativity (see, for instance, [19, 20, 37]) and in Riemannian geometry (see [4]). In turn, Boucetta has proved in [6] that the eigenvalues of the Lichnerowicz Laplacian acting on TT-tensors which is defined on a Euclidian unit sphere S^n are given by $\mu_a = a(n + a - 1) + 2(n-1)$ for $a \geq 2$. Then using (17) we conclude that the following theorem is true.

Theorem 5.1. *The eigenvalues of the Sampson Laplacian Δ_S defined on the Euclidian unit sphere S^n and acting on TT-tensors are given by $\lambda_a = a(n + a - 1) - 2(n + 1)$ for $a \geq 2$.*

References

[1] A. Barnes, B. Edgar, R. Rani, Killing tensors from conformal Killing vectors, Gravitation and cosmology. Proceedings of the Spanish Relativity Meting, Edicions Universitat Barcelona, 2003, 248–252.
[2] P.H. Bérard, From vanishing theorems to estimating theorems: the Bochner technique revisited, Bull. of AMS 19:2 (1988) 371–406.
[3] M. Berger, D. Ebine, Some decomposition of the space of symmetric tensors of a Riemannian manifold, J. Diff. Geom. 3 (1969) 379–392.
[4] A.L. Besse, Einstein Manifolds, Springer-Verlag, Berlin and Heidelberg, 1987.
[5] S. Bochner, K. Yano, Curvature and Betti Numbers, Princeton University Press, Princeton, 1953.
[6] M. Boucetta, Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on S^n, Osaka J. Math. 46 (2009) 235–254.
[7] M. Boucetta, Spectre des Laplaciens de Lichnerowicz sur les sphères et les projectifs réels, Publ. Mat. 43 (1999) 451–483.

[8] M. R. Bridson, A. Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, Berlin, Heidelberg (1999).
[9] E. Calabi, An extension of E. Hop's maximum principle with an application to Riemannian geometry, Duke Math. J. 25 (1957) 45–56.
[10] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press. INC, Orlando, 1984.
[11] S.Y. Chen, Eigenvalue comparison theorems and its geometric applications, Math. Z. 143:3 (1975) 289–297.
[12] B. Colbois, A. El Sou, Extremal eigenvalues of the Laplacian in a conformal class of metrics: The "Conformal Spectrum", Ann. Glob. Anal. Geom. 24 (2003) 337–349.
[13] C.D. Collinson, L. Howarth, Generalized Killing tensors, General Relativity and Gravitation 32:9 (2000) 1767–1776.
[14] M. Craioveanu, M Puta., T.M. Rassias, Old and new Aspects in Spectral Geometry, Kluwer Academic Publishers, London, 2001.
[15] N.S. Dairbekov, V.A. Sharafutdinov, On conformal Killing symmetric tensor fields on Riemannian manifold, Siberian Adv. Math. 21:1 (2011) 1–41.
[16] P Dolan., A. Kladouchou, C. Card, On the significance of Killing tensors, General Relativity and Gravitation 21:4 (1989) 427–437.
[17] J. Eells, J.H. Sampson, Harmonic mappings of Riemannian manifolds, American J. Math. 86:1 (1964) 109–160.
[18] L.P. Eisenhart, Riemannian Geometry, Princeton University, Princeton, 1949.
[19] R. Garattini, Self sustained tranversable wormholes?, Class. Quant. Grav. 22:6 (2005) 2673–2682.
[20] G.W. Gibbons, M.J. Perry, Quantizing gravitational instantons, Nuclear Phys. B 146 (1978) 90–108.
[21] G.W. Gibbons, S.A. Hartnoll, A gravitational instability in higher dimensions, Phys. Rev. D 66 (2002) 064024, 43 pp.
[22] R. Gicquaud, Q.A. Ngŏ, A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor, Class. Quant. Grav. 31:19 (2014) 195014, 20 pp.
[23] K. Heil, A. Moroianu, U. Semmelmann, Killing and conformal Killing tensors, J. Geom. Phys. 106 (2016) 383–400.
[24] A. Kimaczyńska, Some differential operators in the symmetric bundle, J. Appl. Math. and Comp. Mech. 16:3 (2017) 27–36.
[25] A. Kimaczyńska, A. Pierzchalski, Elliptic operators in the bundle of symmetric tensors, Banach Center Publ. 113 (2017) 193–218.
[26] S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag, Berlin and Heidelberg, 1995.
[27] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol. II, Int. Publishers, New York-London-Sydney, 1969.
[28] O. Lablée, Spectral theory in Riemannian geometry, European Mathematical Society (EMS), Zürich, 2015.
[29] P. Li, Geometric Analysis, Cambridge University Press, Cambridge, 2012.
[30] P. Li, S.-T. Yau, Estimate of eigenvalue of a compact Riemannian manifold, Proc. Symposia in Pure Mathematics 30 (1980) 205–239.
[31] J. Mikeš, E.S. Stepanova, A five-dimensional Riemannian manifold with an irreducible SO(3)-structure as a model of abstract statistical manifold, Ann. Glob. Anal. Geom. 45:2 (2014) 111–128.
[32] J. Mikeš, S.E. Stepanov, I.I. Tsyganok, From infinitesimal harmonic transformations to Ricci solitons, Math. Bohem. 138:1 (2013) 25–36.
[33] J. Mikeš, S.E. Stepanov, I.I. Tsyganok, On a Laplacian which acts on symmetric tensors, Preprint: arXiv: 1406.2829 [math.DG], 1 (2014) 14 pp.
[34] J. Mikeš, A. Vanžurová, I. Hinterleitner, Geodesic mappings and some generalizations, Palacky University Press, Olomouc, 2009.
[35] R. Penrose, M. Walker, On quadratic first integrals of geodesic equations for type (22) space-times, Comm. Math. Phys. 18:4 (1970) 265–274.
[36] P. Petersen, Riemannian Geometry, Springer Science, New York (2006).
[37] K. Pilch, A.N. Schellekens, Formulas for the eigenvalues of the Laplacian on tensor harmonics on symmetric coset spaces, J. Math. Phys. 25 (1984) 3455–3459.
[38] J.H. Sampson, On a theorem of Chern, Transactions of the American Mathematical Society 177 (1973) 141–153.
[39] S.E. Stepanov, Fields of symmetric tensors on a compact Riemannian manifold, Math. Notes 52:4 (1992) 1048–1050.
[40] S.E. Stepanov, Affine differential geometry of Killing tensors, Russian Math. (Iz. VUZ) 48:11 (2005) 74–78.
[41] S.E. Stepanov, Vanishing theorems in affine, Riemannian and Lorentz geometries, J. Math. Sci. (New York) 141: 1 (2007) 929–964.
[42] S.E. Stepanov, J. Mikeš, The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom. 48 (2015) 37–46.
[43] S.E. Stepanov, I.G Shandra., Geometry of infinitesimal harmonic transformations, Ann. Global Anal. Geom. 24:3 (2003) 291–299.
[44] S.E. Stepanov, I.I. Tsyganok, I.A. Aleksandrova, A remark on the Laplacian operator which acts on symmetric tensors, Preprint: arXiv: 1411.1928 [math.DG], 4 (2014) 8 pp.
[45] S.E. Stepanov, I.I. Tsyganok, M.B. Khripunova, The Killing tensors on an n-dimensional manifold with SL(n,R)-structure, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 55:1 (2016) 121–131.
[46] H. Wu, A remark on the Bochner technique in differential geometry, Proc. Amer. Math. Soc. 78:3 (1980) 403–408.
[47] K. Tandai, T. Sumitomo, Killing tensor fields of degree 2 and spectrum of SO(n + 1)/(SO(n − 1) × SO(2)), Osaka J. Math. 17 (1980) 649–675.
[48] K. Yano, Integral Formulas in Riemannian Geometry, Marcel Dekker, New York, 1970.
[49] S.-T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J. 25:7 (1976) 659–670.