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Anisotropic Image Evolution of Synge-Beil Type

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Abstract. The anisotropic Beltrami framework is introduced as an extended promising tool in image processing. In this framework, image surface evolution is governed by an anisotropic flow determined by an energy Lagrangian of Polyakov type. The Synge-Beil flow is derived, and applicative aspects illustrate the developed theoretical results.

1. Introduction

Image evolution is one of techniques in image processing which enhance an image throughout slow successive shifting its features. The evolution is commonly driven by a flow vector field, which impose a geometrical viewpoint on the subject. The adequate geometrical setting is the Beltrami framework, proposed in [24] as a valuable tool in computer vision and image processing.

An image is regarded as a two-dimensional surface embedded in appropriate (commonly Euclidean) space of sufficiently large dimension, whose tangent vectors are pointers to neighbor pixels. Grayscale image is modeled as a surface in 3-dimensional space, a color image is placed in a 5-dimensional one, while multichannel images need a space whose dimension is channel number plus two.

The paper recalls in short the classical Beltrami framework and presents its anisotropic extension, based on a directionally dependent metric structure of most general type, namely generalized Lagrangian metrics. We further derive the illustrative particular case of Synge-Beil image evolution. Applications of the anisotropic image evolution is discussed, with emphasis on the Synge-Beil evolution case.

2. The Beltrami framework

Theoretical foundations supporting the construction of Beltrami framework origin from differential geometry, but they are closely related to subsurfaces [1, 2], variational calculus and PDE on manifolds [3, 5], and harmonic maps [14, 23]. Comprehensive details can be found in [4, 16].

The Beltrami framework relies on two differentiable manifolds related by an embedding. It is uniquely determined by a triple $(X, (M, h), (\Sigma, g))$, where:

- the ambient space (M, h) - an m-dimensional differentiable manifold with a Riemannian metric structure, defined by the tensor field $h(x) = h_{ij}(x)dx^i \otimes dx^j$, $x \in M$;

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- the embedding $X : D \to M$ of a connected open domain $D \subset \mathbb{R}^n$, $(n < m)$, given by n smooth scalar functions

$$X : (x^1, \dots, x^n) \mapsto \left(X^1(x^1, \dots, x^n), \dots, X^m(x^1, \dots, x^n)\right); \tag{1}$$

- the image surface $\Sigma = X(D)$, a submanifold embedded in the ambient space M, and endowed with the Riemannian metric g, $g(x) = g_{\sigma\mu} dx^\sigma \otimes dx^\mu$,

and where the Greek and the Roman indices run, accordingly, within the ranges $\overline{1,n}$ and $\overline{1,m}$. The Beltrami framework uses the local intrinsic description of the manifolds M and Σ, and considers their metrics as dynamical variables. The image metric g is not necessarily induced from the ambient one. The parameters of the Beltrami framework are the embedding and the two metrics, and its unique global characteristic is the energy of the image surface, taking the form of a Polyakov action [20][1)],

$$S(\Sigma) := S(X, g, h) \quad = \quad \int_\Sigma f \langle gradX^i, gradX^j \rangle_g h_{ij} dV = \int_D L_\Sigma(x^\alpha)\, dx^1 \cdots dx^n, \tag{2}$$

$$\text{where} \quad L_\Sigma(x^\alpha) = f\sqrt{g}\, g^{\mu\nu} X^i_\mu X^j_\nu\, h_{ij}, \quad (x^\alpha) \in D. \tag{3}$$

where the scalar function f is the weight factor, which provides a directional impact on the desired image enhancement (see e.g. [3, 21, 27]), and the local energy of the image is expressed by the Lagrangian density[2)], $L \equiv L_\Sigma(x^\alpha)$. Within the Beltrami framework it is of the interest to minimize the Polyakov action, i.e., to solve the problem $argmin_\Sigma\{S(X)\}$[3)]. The standard methods of variational calculus confirm that the global and the local energies are simultaneously minimized. The equivalent form of the problem is the Euler-Lagrange PDE system stated in the ambient manifold[4)],

$$L_{,i} - L_{,\binom{i}{\alpha};\alpha} = 0. \tag{4}$$

The integral submanifold in M corresponding to the solution cannot be generally obtained by means of standard methods of variational calculus. In order to approach the solution submanifold, one may employ within the initial image submanifold Σ the descent flow technique [4],

$$\partial_t X = -S'(X(t)), \quad X(0) = X.$$

By taking t as an auxiliary variable, the equation is regarded as an ODE and the solution mapping $X = X(x, t)$ defines an one-parameter family of mappings $X_t := X_t + \partial_t X$ and the corresponding family of submanifolds Σ_t (called layers, in image processing). The validity of the technique is proved in [14], where the subject in focus is the mapping X and not its underlying submanifold.

The Beltrami flow shifts the image surface to the state of minimal Polyakov action:

$$\partial_t X^r = -\frac{1}{2}\frac{1}{\sqrt{g}} h^{ir}\left(L_{,i} - L_{,\binom{i}{\alpha};\alpha}\right), \quad r = 1, \dots, m. \tag{5}$$

The multiplier ensures the invariance with respect to reparametrization, hence the flow defines a global vector field over the image surface. The geometrical properties of the Beltrami flow are discussed in [23]. The Beltrami flow supports interaction between all components of the embedding, which is extremely important in image processing, where vector-valued features are often considered.

If the metrics h and g are fixed Rimannian ones, and the mapping X is an immersion, then the Beltrami minimization is equivalent to minimization in harmonic maps theory. Namely, the Polyakov action, for the

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1)For brevity, the same notation will be used for the metric tensor and its determinant: $\det(g_{\mu\nu}) \rightsquigarrow g$, and $\det(\gamma_{\mu\nu}) \rightsquigarrow \gamma$.

2)The standard notation for partial differentials will be further used in shortened form like, e.g., $X^i_\mu =: \frac{\partial X^i}{\partial x^\mu}$.

3)Here, the minimization with respect to the embedding is considered; for alternative approaches, see Sochen et al. [23, 24].

4)We use the brief notation: $\Phi_{;\alpha} := \dfrac{\partial \Phi}{\partial x^\alpha}$ $\quad \Phi_{,i} := \dfrac{\partial \Phi}{\partial X^i}$ $\quad \Phi_{,\binom{i}{\alpha}} := \dfrac{\partial \Phi}{\partial X^i_\alpha}$ $\quad \Phi_{,\binom{i}{\alpha};\alpha} = \dfrac{\partial}{\partial x^\alpha}\left(\dfrac{\partial \Phi}{\partial X^i_\alpha}\right)$

case of trivial weight $f \equiv 1$, is the energy of the mapping that is to be minimized. The extremal mappings are characterized by the vanishing of the tension field $\tau(X) = (\tau^1(X), \dots, \tau^m(X))$,

$$\tau^r(X) = g^{\sigma\mu} X^r_{\sigma\mu} - g^{\sigma\mu}\Gamma^\nu_{\sigma\mu} X^r_\nu + g^{\sigma\mu}\Gamma^r_{kl} X^k_\sigma X^l_\mu,$$

where $\Gamma^r_{kl}, \Gamma^\sigma_{\rho\theta}$ are the Christoffel symbols of h and g. However, the mapping can be adjusted by the tension flow which is equal (up to a factor) with the Beltrami flow. For the Euclidean ambient metric, the tension coincides with the Laplace-Beltrami operator of the image metric.

Particular choices of parameters in the Beltrami framework produce different Beltrami flows, and consequent various enhancements of the digital image, like: smoothing, denoising, segmentation, registration, etc. [12, 16, 17, 22, 24, 25].

Classical Beltrami flow may also depend on direction through the weight function, while the anisotropic Beltrami flow avoids the weight and involves directional dependence of the metric structure on the image surface.

3. The Synge-Beil metric structure

The Synge-Beil type metric is an anisotropic structure and provides the base differentiable manifold with a generalized Lagrangian space structure. It lives on the tangent space of the base manifold as distinguished tensor field [6, 7].

Let (Σ, g) be a Riemannian manifold[5)]. A vector field $B \in \chi(T\Sigma)$ compatible with reparametrizations on the base manifold Σ (a d-tensor field) has its dual d-object a corresponding flat vector field having the components $B_\sigma = g_{\sigma\mu} B^\mu$. The deformation of the Riemannian metric by the d-vector field B and by the scalar function $c : T\Sigma \to \mathbb{R}$

$$\gamma(x, y) = \gamma_{\sigma\mu}(x, y) dx^\sigma \otimes dx^\mu, \quad \gamma_{\sigma\mu}(x, y) = g_{\sigma\mu}(x) + c(x, y) B_\sigma(x, y) B_\mu(x, y), \tag{6}$$

is a d-tensor field called *the Synge-Beil metric structure* [8], widely used in theory of relativistic optics and unified field theory [7, 11]. This is a special case of Beil metric that additively deforms a Finsler metric [6]. According to the algebra of tensors, particular properties of the Synge-Beil metric can be emphasized.

Proposition 3.1. *The relation (6) defines a symmetric 2-covariant d-tensor field over the slit tangent bundle $\widetilde{T\Sigma}$. If $c(x, y) \geq 0$, then the bilinear form $\gamma_{(x,y)}(u, v)$, $u, v \in T_x\Sigma$ is regular and positive definite, for all $y \in T_x\Sigma \setminus \{0\}$.*

Related geometrical objects of the Synge-Beil metric are also characterized, as follows:

Proposition 3.2. *The inverse metric of (6) has the following components*

$$\gamma^{\sigma\mu} = g^{\sigma\mu} - \frac{c}{1 + c||B||^2_g} B^\sigma B^\mu,$$

where $||B(x, y)||^2_g = g_{\sigma\mu}(x) B^\sigma(x, y) B^\mu(x, y)$, and the corresponding determinant value is:

$$\gamma = \left(1 + c||B||^2_g\right) \cdot g.$$

3.1. *Synge-Beil structures in the Beltrami framework*

Let $(X, (M, h)$, and $(\Sigma, g))$ be a Beltrami framework with the Riemannian ambient metric h and the induced image metric

$$g_{\sigma\mu} = h_{ij} X^i_\sigma X^j_\mu. \tag{7}$$

5) Though we assume the section to be an embedded image surface, the whole construction is valid in the general case.

Within the Beltrami framework, the simplest Synge-Beil structure on the image surface is the canonically deformed induced metric,

$$\gamma_{\sigma\mu}(x,y) = g_{\sigma\mu}(x) + c(x,y) y_\sigma y_\mu, \quad c(x,y) \geq 0,\ y \in T_x\Sigma,\ x \in \Sigma. \tag{8}$$

The squared induced length of a vector $y \in T_x\Sigma$ is briefly denoted by $V(x,y) = ||y||_g^2 = g_{\sigma\mu}(x) y^\sigma y^\mu$.

Corollary 3.3. *The canonically deformed induced metric (8) determines a positive definite generalized Lagrangian structure, having the following inverse metric components and the determinant value*

$$\gamma^{\sigma\mu} = g^{\sigma\mu} + S \cdot y^\sigma y^\mu, \qquad \gamma = K \cdot g,$$

where the scalar fields over the tangent bundle $T\Sigma$ are

$$S(x,y) = -\frac{c}{1+cV}, \tag{9}$$

$$K(x,y) = 1 + cV. \tag{10}$$

Similar to the isotropic case (3), the Synge-Beil metric defines an anisotropic local energy over the image surface. The Lagrangian density is expressed by $L(x,y) = f\,\sqrt{\gamma}\,\gamma^{\mu\nu} X^i_\mu X^j_\nu\, h_{ij}$, and direct computation based on the previous Corollary produces the following result:

Proposition 3.4. *The Lagrangian density in the Beltrami framework (X, M, Σ) with canonically deformed induced metric is the scalar field over the tangent bundle $T\Sigma$ given by the expression*

$$L(x,y) = f\,\sqrt{1 + c||y||_g^2}\left(n - \frac{c||y||_g^2}{1 + c||y||_g^2}\right)\sqrt{g}.$$

The Polyakov action is the function $S : \chi(\Sigma) \to \mathbb{R}$,

$$S(Y) = \int_D L(x, Y(x))dx^1 \dots dx^n.$$

4. The anisotropic image evolution

The directional dependence of the evolution of curves is developed in [13, 15, 19] for object extractions and segmentation in image processing, while the analogous 2-dimensional problem is considered in [25–27] for the minimization of the deformation field in image registration. Initial theoretical aspects of the subject can be found in [9, 10, 18, 21].

The anisotropic evolution is accomplished within the anisotropic Beltrami framework $(X, (M, h), (\Sigma, \gamma))$ containing the embedded image surface regarded as generalized Lagrangian space. The embedding X and the ambient metric of Riemannian type are assumed to be fixed, and only the image metric will be allowed to vary. The anisotropic nature of the image metric γ causes a directional dependence of the Lagrangian density (3), and further of the Polyakov action (2), and the corresponding Beltrami flow (5) further depend on tangent vectors as well. Therefore, the weight factor (which enables a directional impact) is not needed and hence, is omitted.

The form of the Lagrangian density L imposes the simultaneous consideration of two metrics on the image surface: the anisotropic one, $\gamma = \gamma_{\sigma\mu}(x,y)\, dx^\sigma \otimes dx^\mu$, and the induced one $g = g_{\sigma\mu}(x)\, dx^\sigma \otimes dx^\mu$ given by (7). Without loss of generality, we may assume that the anisotropic metric can be regarded as an additive extension of the induced one,

$$\gamma_{\sigma\mu}(x,y) = g_{\sigma\mu}(x) + \varphi_{\sigma\mu}(x,y),$$

where $\varphi_{\sigma\mu}$ are the components of an arbitrarily chosen d-tensor field on $T\Sigma$. The induced metric is related to the Beltrami framework, and its derivatives show its dependency on the embedding.

Proposition 4.1. *Let X be an embedding (1) into the Riemannian space (M, h_{ij}), which produces the submanifold Σ, and let $g_{\sigma\mu}$ be the induced metric. Then the derivatives of the metric components involved in the evolution process are*

$$\begin{aligned} g_{\sigma\mu;\alpha} &= h_{kl,j}X^j_\alpha X^k_\sigma X^l_\mu + h_{kl}X^k_{\sigma\alpha}X^l_\mu + h_{kl}X^k_\sigma X^l_{\mu\alpha}, \\ g_{\sigma\mu,i} &= h_{kl,i}X^k_\sigma X^l_\mu, \\ g_{\sigma\mu,\binom{i}{\alpha}} &= h_{il}\delta^\alpha_\sigma X^l_\mu + h_{ik}X^k_\sigma \delta^\alpha_\mu, \\ g_{\sigma\mu,\binom{i}{\alpha};\alpha} &= h_{il,j}X^j_\sigma X^l_\mu + h_{ki,j}X^j_\mu X^k_\sigma + 2h_{ij}X^j_{\sigma\mu}. \end{aligned}$$

The anisotropic Lagrangian density may be written in the following simple form $L(x, y) = f\sqrt{\gamma}\gamma^{\sigma\mu}g_{\sigma\mu}$, which is more appropriate for further considerations.

The most general case of anisotropic Beltrami flow, also called generalized Lagrangian flow, is considered in [10]. This represents a proper valid generalization of the classic Beltrami flow.

Theorem 4.2. *The explicit form of the anisotropic Beltrami flow which minimizes the non-weighted Polyakov action with generalized Lagrange image metric γ, is*

$$\begin{aligned} \partial_t X^r = \ & \tau^r(X) + \frac{1}{2}h^{ir}\Big\{g_{\sigma\mu;\alpha}\Big[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha}} + \gamma^{\sigma\mu}(\ln\sqrt{\gamma})_{,\binom{i}{\alpha}}\Big] + \\ & g_{\sigma\mu}\Big[(\gamma^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} + (\gamma^{\sigma\mu})_{,\binom{i}{\alpha}}(\ln\sqrt{\gamma})_{;\alpha} + (\gamma^{\sigma\mu})_{;\alpha}(\ln\sqrt{\gamma})_{,\binom{i}{\alpha}} + \gamma^{\sigma\mu}\frac{1}{\sqrt{\gamma}}(\sqrt{\gamma})_{,\binom{i}{\alpha};\alpha} - (\gamma^{\sigma\mu})_{,i} - \gamma^{\sigma\mu}(\ln\sqrt{\gamma})_{,i}\Big]\Big\}, \end{aligned}$$

where the tension field components $\tau^r(X)$ are also anisotropic, i.e., are defined by the metrics $\gamma(x, y)$ and h, and by their Christoffel symbols $\Gamma^\rho_{\sigma\mu}(x, y)$ and Γ^r_{kl}.

The features of the chosen image metric in a particular anisotropic Beltrami framework determine the implementation of the generalized Lagrangian flow.

To achieve image evolution of the Synge-Beil type, it is necessary to reconsider the previous Theorem by reconsidering the form of the metric structure (8). Namely, the single terms from the flow equation determined in Theorem 4.2 had to be considered with the induced metric and with the additional tensor given in (8). Various types of derivatives of the inverse and of the determinant term corresponding to the Synge-Beil image metric need to be specified by means of the following auxiliary results.

Lemma 4.3. *Let γ be a metric tensor of the Synge-Beil type, with its components given by (8) on the image surface Σ. Then, the derivatives of the corresponding inverse tensor are given by*

$$(\gamma^{\sigma\mu})_\star = (g^{\sigma\mu})_\star + S_\star y^\sigma y^\mu, \qquad (\gamma^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} = (g^{\sigma\mu})_{,\binom{i}{\alpha};\alpha} + S_{,\binom{i}{\alpha};\alpha}y^\sigma y^\mu,$$

where $\star$ is an unified notation for the three types of the first order partial derivatives, $\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial X^i}, \frac{\partial}{\partial X^i_\alpha}$.

Lemma 4.4. *Let γ be a metric tensor of the Synge-Beil type with its components given by (8) on the image surface Σ. Then, the derivatives of the term $\ln\sqrt{\gamma}$ are*

$$\begin{aligned} \left(\ln\sqrt{\gamma}\right)_\star &= \frac{1}{2\gamma}\left(K_\star g + K g_\star\right), \\ \frac{1}{\sqrt{\gamma}}\left(\sqrt{\gamma}\right)_{,\binom{i}{\alpha};\alpha} &= \frac{1}{4K^2}\left(2KK_{,\binom{i}{\alpha};\alpha} - K_{,\binom{i}{\alpha}}K_{;\alpha}\right) \\ &\quad + \frac{1}{4g^2}\left(2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\right) + \frac{1}{4\gamma}\left(K_{,\binom{i}{\alpha}}g_{;\alpha} + K_{;\alpha}g_{,\binom{i}{\alpha}}\right), \end{aligned}$$

where g is the determinant of the induced metric.

Lemma 4.5. *For the scalar functions K and S defined on the image surface $T\Sigma$ by (10) and (9), the $\star$-derivatives are*

$$K_\star = c_\star V + cV_\star, \qquad S_\star = \frac{1}{K^2}\left(c^2 V_\star - c_\star\right)$$

and the mixed derivatives are

$$\begin{aligned}
K_{,\binom{i}{\alpha};\alpha} &= c_{,\binom{i}{\alpha};\alpha}V + c_{,\binom{i}{\alpha}}V_{;\alpha} + c_{;\alpha}V_{,\binom{i}{\alpha}} + cV_{,\binom{i}{\alpha};\alpha}\\
S_{,\binom{i}{\alpha};\alpha} &= \frac{1}{K^3}\Big[c^3\left(V_{,\binom{i}{\alpha};\alpha}V - 2V_{,\binom{i}{\alpha}}V_{;\alpha}\right) - c_{,\binom{i}{\alpha};\alpha}\\
&\quad + 2c_{,\binom{i}{\alpha}}cV_{;\alpha} + 2c_{;\alpha}cV_{,\binom{i}{\alpha}} + c^2V_{,\binom{i}{\alpha};\alpha} - V(c_{,\binom{i}{\alpha};\alpha}c - 2c_{,\binom{i}{\alpha}}c_{;\alpha})\Big].
\end{aligned}$$

The merging of the previous Theorem and Lemmas yields the main result of this article:

Theorem 4.6. *The PDEs of the Synge-Beil evolution flow, which provide the minimality of the Polyakov energy on the image surface Σ embedded into the Riemannian manifold (M, h) by the mapping (1), are given by*

$$\begin{aligned}
\partial_t X^r &= \tau^r(X) + \frac{1}{2}h^{ir}\left[(g^{\sigma\mu})_{,\binom{i}{\alpha};\alpha}g_{\sigma\mu} + (g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{\sigma\mu;\alpha} + g^{\sigma\mu}g_{\sigma\mu,i}\right]\\
&+\frac{1}{2}h^{ir}\Big[c\frac{1}{2\gamma}\left(g^{\sigma\mu}g_{\sigma\mu;\alpha}(V_{,\binom{i}{\alpha}}g + Vg_{,\binom{i}{\alpha}}) - n(V_{,i}g + Vg_{,i})\right) + c_{,\binom{i}{\alpha}}\frac{1}{2\gamma}Vg^{\sigma\mu}g_{\sigma\mu;\alpha}g\\
&\quad - c_{,i}\frac{n}{2\gamma}Vg + \frac{1}{2\gamma}\left(g^{\sigma\mu}g_{\sigma\mu;\alpha}g_{,\binom{i}{\alpha}} - ng_{,i}\right)\Big]\\
&+\frac{1}{2}h^{ir}\frac{1}{2K^2g}\Big[c^2\left((V_{,\binom{i}{\alpha}}g - Vg_{,\binom{i}{\alpha}})g_{\sigma\mu;\alpha} + (Vg_{,i} - V_{,i}g)g_{\sigma\mu}\right) - cc_{,\binom{i}{\alpha}}Vg_{\sigma\mu;\alpha}g\\
&\quad + cc_{,i}Vg_{\sigma\mu}g - 2c_{,\binom{i}{\alpha}}g_{\sigma\mu;\alpha}g + 2c_{,i}g_{\sigma\mu}g - cg_{\sigma\mu;\alpha}g_{,\binom{i}{\alpha}} + cg_{\sigma\mu}g_{,i}\Big]v^\sigma v^\mu\\
&+\frac{1}{2}h^{ir}\frac{1}{4\gamma^2}\Big[c^2\Big(2V\left\{V_{;\alpha}(g^{\sigma\mu})_{,\binom{i}{\alpha}} + V_{,\binom{i}{\alpha}}(g^{\sigma\mu})_{;\alpha}\right\}g_{\sigma\mu}g^2\\
&\quad +2V^2\left\{(g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{;\alpha} + (g^{\sigma\mu})_{;\alpha}g_{,\binom{i}{\alpha}}\right\}g_{\sigma\mu}g + n\left\{2VV_{,\binom{i}{\alpha};\alpha} - V_{,\binom{i}{\alpha}}V_{;\alpha}\right\}g^2\\
&\quad +nV^2\left\{2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\right\} + nV\left\{V_{,\binom{i}{\alpha}}g_{;\alpha} + V_{;\alpha}g_{,\binom{i}{\alpha}}\right\}g\Big)\\
&\quad + n\left(2cc_{,\binom{i}{\alpha};\alpha} - c_{,\binom{i}{\alpha}}c_{;\alpha}\right)V^2g^2 + cc_{,\binom{i}{\alpha}}V\Big(2V(g^{\sigma\mu})_{;\alpha}g_{\sigma\mu}g + nV_{;\alpha}g + nVg_{;\alpha}\Big)g\\
&\quad +cc_{;\alpha}V\left(2V(g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{\sigma\mu}g + nV_{,\binom{i}{\alpha}}g + nVg_{,\binom{i}{\alpha}}\right)g + 2nc_{,\binom{i}{\alpha};\alpha}Vg^2\\
&\quad +c_{,\binom{i}{\alpha}}\Big(2V(g^{\sigma\mu})_{;\alpha}g_{\sigma\mu}g + 2nV_{;\alpha}g + nVg_{;\alpha}\Big)g + c_{;\alpha}\left(2V(g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{\sigma\mu}g + 2nV_{,\binom{i}{\alpha}}g + nVg_{,\binom{i}{\alpha}}\right)g\\
&\quad +c\left(2\Big\{V_{;\alpha}g + 2Vg_{;\alpha}\Big\}(g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{\sigma\mu}g + 2\Big\{V_{,\binom{i}{\alpha}}g + 2Vg_{,\binom{i}{\alpha}}\Big\}(g^{\sigma\mu})_{;\alpha}g_{\sigma\mu}g\right.\\
&\quad \left. +2nV_{,\binom{i}{\alpha};\alpha}g^2 + 2nV\Big\{2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\Big\} + n\Big\{V_{,\binom{i}{\alpha}}g_{;\alpha} + V_{;\alpha}g_{,\binom{i}{\alpha}}\Big\}g\right)\\
&\quad +2\left((g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{;\alpha} + (g^{\sigma\mu})_{;\alpha}g_{,\binom{i}{\alpha}}\right)g_{\sigma\mu}g + n\Big(2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\Big)\Big]\\
&+\frac{1}{2}h^{ir}\frac{V}{K^3g}\Big[c^3\Big(\left\{2VV_{,\binom{i}{\alpha};\alpha} - V_{,\binom{i}{\alpha}}V_{;\alpha}\right\}g^2 - V^2\left\{2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\right\} - 2V_{,\binom{i}{\alpha}}V_{;\alpha}g^2\\
&\quad + V\left\{V_{,\binom{i}{\alpha}}g_{;\alpha} + V_{;\alpha}g_{,\binom{i}{\alpha}}\right\}g\Big) + c^2c_{,\binom{i}{\alpha}}V\left(V_{;\alpha}g - Vg_{;\alpha}\right)g + c^2c_{;\alpha}V\left(V_{,\binom{i}{\alpha}}g - Vg_{,\binom{i}{\alpha}}\right)g\\
&\quad -c\left(2cc_{,\binom{i}{\alpha};\alpha} - c_{,\binom{i}{\alpha}}c_{;\alpha}\right)V^2g^2 + 2\left(2c_{,\binom{i}{\alpha}}c_{;\alpha} - 3cc_{,\binom{i}{\alpha};\alpha}\right)Vg^2\\
&\quad +c^2\left(V_{,\binom{i}{\alpha};\alpha}g^2 + \left\{V_{,\binom{i}{\alpha}}g_{;\alpha} + V_{;\alpha}g_{,\binom{i}{\alpha}}\right\}g - 2V\left\{2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\right\}\right)\\
&\quad +cc_{,\binom{i}{\alpha}}\left(4V_{;\alpha}g - 3Vg_{;\alpha}\right)g + cc_{;\alpha}\left(4V_{,\binom{i}{\alpha}}g - 3Vg_{,\binom{i}{\alpha}}\right)g - c\left(2gg_{,\binom{i}{\alpha};\alpha} - g_{,\binom{i}{\alpha}}g_{;\alpha}\right)\\
&\quad -4c_{,\binom{i}{\alpha};\alpha}g^2 - 2c_{,\binom{i}{\alpha}}g_{;\alpha}g - 2c_{;\alpha}g_{,\binom{i}{\alpha}}g\Big],
\end{aligned} \tag{11}$$

where $\tau(X)$ is the tension of the embedding X, g is the induced metric tensor field, and $c = c(X)$ is the scalar field which defines the Synge-Beil metric.

Corollary 4.7. *Within the anisotropic Beltrami framework (X, M, Σ) endowed with Synge-Beil image metric with components $\gamma_{\sigma\mu}(x,y) = g_{\sigma\mu}(x) + c\cdot y_\sigma y_\mu$ where $c = const$, the image surface evolves toward the minimal Polyakov action by means of the flow:*

$$\begin{aligned}\partial_t X^r =& \tau^r(X) + \frac{1}{2}h^{ir}\cdot\Big\{\frac{c}{2K}\Big[\frac{c}{K}V_{,\binom{i}{\alpha}}g_{\sigma\mu;\alpha} - \frac{1}{g}g_{\sigma\mu;\alpha}g_{,\binom{i}{\alpha}}\Big]y^\sigma y^\mu + (g^{\sigma\mu})_{,\binom{i}{\alpha};\alpha}g_{\sigma\mu} + g_{\sigma\mu;\alpha}(g^{\sigma\mu})_{,\binom{i}{\alpha}} - g_{\sigma\mu}(g^{\sigma\mu})_{,i}\\ &+\frac{1}{2g}(n-\frac{cV}{K})\Big(g_{,\binom{i}{\alpha};\alpha} - g_{,i}\Big) - \frac{1}{4g^2}(n-\frac{cV}{K})g_{,\binom{i}{\alpha}}g_{;\alpha} + \frac{1}{2g}g_{\sigma\mu}(g^{\sigma\mu})_{,\binom{i}{\alpha}}g_{;\alpha} + \frac{c}{4Kg}(n+\frac{cV}{K})\Big(V_{,\binom{i}{\alpha}}g_{;\alpha} + V_{;\alpha}g_{,\binom{i}{\alpha}}\Big)\\ &+\frac{c}{2K}g_{\sigma\mu}(g^{\sigma\mu})_{,\binom{i}{\alpha}}V_{;\alpha} - \frac{c}{2K}(n-\frac{cV}{K})V_{,i} + \frac{c}{2K}(n+\frac{cV}{K})V_{,\binom{i}{\alpha};\alpha} - \frac{c^2}{4K^2}(n+\frac{3cV}{K})V_{,\binom{i}{\alpha}}V_{;\alpha}\Big\}.\end{aligned}$$

5. Anisotropic Beltrami framework in image processing

Within the Beltrami framework, a digital image is commonly modeled as 2-dimensional surface determined by an embedding $X : (x^1, x^2) \to (x^1, x^2, I(x^1, x^2))$. The mapping I usually stands for an image feature, and can be vector valued.

In the case of monochrome digital images, this function is a scalar one, and the surface is of Monge type. The surface evolves within a Beltrami framework, while the flow has only one nonzero component, $\partial_t I := \partial_t X^3$. Within a 3-dimensional Euclidean ambient space having the metric $(h_i j) = Diag(1, 1, \beta^2)$, the canonically deformed metric induced on the embedded surface has the following components

$$(\gamma_{\sigma\mu}(x,y)) = \begin{pmatrix} 1+\beta^2 I^2_{x^1} & \beta^2 I_{x^1}I_{x^2} \\ \beta^2 I_{x^1}I_{x^2} & 1+\beta^2 I^2_{x^2}\end{pmatrix} + c\cdot\begin{pmatrix}(y_1)^2 & y_1y_2\\ y_1y_2 & (y_2)^2\end{pmatrix}. \tag{12}$$

The evolution of the image surface relies on Theorem 4.6 and on Corollary 4.7, and the following derivatives are essential in the implementation process:

Corollary 5.1. *Within the Beltrami framework, the induced metric tensor on the Monge surface has the following nontrivial derivatives*

$$\begin{aligned} g_{\sigma\mu;\alpha} &= \beta^2 I_{\sigma\alpha}I_\mu + \beta^2 I_\sigma I_{\mu\alpha}\\ g_{\sigma\mu,\binom{3}{\alpha}} &= \beta^2\delta^\alpha_\sigma I_\mu + \beta^2 I_\sigma\delta^\alpha_\mu\\ g_{\sigma\mu,\binom{3}{\alpha};\alpha} &= 2\beta^2 I_{\sigma\mu}.\end{aligned}$$

Corollary 5.2. *The induced g-quadratic form is*

$$V = (y^1)^2 + (y^2)^2 + \beta^2\left(I^2_{x^1}(y^1)^2 + 2I_{x^1}I_{x^2}y^1y^2 + I^2_{x^2}(y^2)^2\right).$$

and the corresponding derivatives are

$$\begin{aligned} V_{,i} &= 0,\\ V_{;\alpha} &= 2\beta^2\left(I_{x^1}I_{x^1x^\alpha}(y^1)^2 + (I_{x^1x^\alpha}I_{x^2} + I_{x^1}I_{x^2x^\alpha})y^1y^2 + I_{x^2}I_{x^2x^\alpha}(y^2)^2\right),\\ V_{,\binom{3}{1}} &= 2\beta^2\left(I_{x^1}(y^1)^2 + I_{x^2}y^1y^2\right),\\ V_{,\binom{3}{2}} &= 2\beta^2\left(I_{x^1}y^1y^2 + I_{x^2}(y^2)^2\right),\\ V_{,\binom{3}{\alpha};\alpha} &= 2\beta^2\left(I_{x^1x^1}(y^1)^2 + 2I_{x^1x^2}y^1y^2 + I_{x^2x^2}(y^2)^2\right).\end{aligned}$$

The implementation of the Beltrami framework in image processing is enabled through the discretization of the image surface and of the Beltrami flow. The discretization is induced by the discretization of the domain, hence a monochrome digital image is presented as an image matrix $\Sigma = (I(i,j))$, where i and j are nonnegative integers. The elements $I(i,j)$ are in correspondence with the locations of the pixels of the image $(i,j) = (x^1, x^2) =: x$, and their values $I(i,j) \in \{0, 1, \ldots, 255\}$ represent the level of their grey color intensity. The matrix dimensions are determined by the image resolution. The tangent vectors in the model point to the neighboring pixels, and the partial derivatives of the feature I are determined by

$$\begin{aligned}
&I_{x^1}(i,j) = I(i+1,j) - I(i,j), \qquad I_{x^2}(i,j) = I(i,j+1) - I(i,j),\\
&I_{x^1x^1}(i,j) = I(i+2,j) + I(i,j) - 2I(i+1,j), \quad I_{x^2x^2}(i,j) = I(i,j+2) + I(i,j) - 2I(i,j+1).\\
&I_{x^1x^2}(i,j) = I(i+1,j+1) + I(i,j) - I(i+1,j) - I(i,j+1).
\end{aligned}$$

The gradient vector is discretized by the shift tangent vector computed as max-abs of the shifts towards the pixels of the eight nearest neighbors of the current pixel. The image surface $I(x^1, x^2)$ evolves as a geometric active surface by the successive shifting of the image feature

$$I(i,j) \rightsquigarrow I(i,j) + \triangle I(i,j),$$

where $\triangle I(i,j)$ discretizes the Beltrami flow $\partial_t I$. For details on the discretization within the frame of the level set formulation we mainly refer to [4].

The Beltrami-induced evolution of the image $\Sigma = (I(i,j))$ is achieved by successive shifting, where each iteration implies the following steps:

1. raster-sequentially passing through all pixels (i,j);
2. determining the shift tangent vector $y(i,j)$;
3. applying the flow expression to obtain the feature variation $\triangle I(i,j)$;
4. altering the feature value to $I \rightsquigarrow I + \triangle I$.

The main difference in the implementation of both classic and anisotropic Beltrami evolution deals with the steps 2 and 3. Namely, in the classic case, a shift tangent vector is fixed, commonly the gradient one, and substituted into the flow expression to obtain the shift $\triangle I$. The anisotropic evolution enables to interchange the order of steps. The flow expression produces eight shifting values at each pixel, which allow extra shifting alternatives. The feature value at the pixel may be further modified by a particular one, afterwards by additional task criterions. This processing is more sensitive than the classic one, but also it is substantially slower, and requires higher computing performance.

An appropriate embedding, ambient space, and image metric, are chosen according to the purpose of the processing. A comprehensive overview of Beltrami frameworks can be found in [3, 4].

The further works on the subject will extend the study of the Beltrami flow applications for specific direction-dependent anisotropic metric structures.

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