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Metrics Transformations Preserving the Types of One-dimensional Minimal Fillings

S. Yu. Lipatov[a]

[a] *Lomonosov Moscow State University*

Abstract. Given a class F of metric spaces and a family of transformations T of a metric, one has to describe a family of transformations $T' \subset T$ that transfer F into itself and preserve some types of minimal fillings. The article considers four cases. First, when F is the class of all finite metric spaces, $T = \left\{(M, \rho) \to (M, f \circ \rho) \mid f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}\right\}$, and the elements of T' preserve all non-degenerate types of minimal fillings of four-point metric spaces and finite non-degenerate stars, and we prove that $T' = \left\{(M, \rho) \to (M, \lambda\rho + a) : a > \lambda a_\rho\right\}$. Second, when F is the class of all finite metric spaces, the class T consists of the maps $\rho \to N\rho$, where the matrix N is the sum of a positive diagonal matrix A and a matrix with the same rows of non-negative elements. The elements of T' preserve all minimal fillings of the type of non-degenerate stars. It has been proven that T' consists of maps $\rho \to N\rho$, where A is scalar. Third, when F is the class of all finite additive metric spaces, T is the class of all linear mappings given by matrices, and the elements of T' preserve all non-degenerate types of minimal fillings, and we proved that for metric spaces consisting of at least four points T' is the set of transformations given by scalar matrices. Fourth, when F is the class of all finite ultrametric spaces, T is the class of all linear mappings given by matrices, and we proved that for three-point spaces the matrices have the form $A = R(B + \lambda E)$, where B is a matrix of identical rows of positive elements, and R is a permutation of the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

1. Introduction

The concept of minimal filling first appeared in the papers of M. L. Gromov [3] in the following form. Let $\mathcal{M} = (M, \rho)$, where M is a closed Riemannian manifold with a distance function ρ at it, and $\mathcal{W} = (W, d)$, where a compact manifold W with the boundary M is such that d does not decrease the distances between points from M, then $\mathcal{W}$ is called a *filling of* $\mathcal{M}$. Gromov's problem consists in describing the greatest lower bound of the volumes of the fillings, as well as describing the spaces $\mathcal{W}$ at which the greatest lower bound is achieved, and which are referred to as *minimal fillings*.

This problem was preceded by another one. The Steiner problem is an optimal connection problem for a finite set of points in a metric space. We need to determine a Steiner minimal tree, i.e., a shortest network connecting a given finite set of points in the plane. A mapping $\Gamma : V \to X$ is called a *network* in a

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Email address: stepa.lipatov@yandex.ru (S. Yu. Lipatov)

pseudo-metric space $\mathcal{X} = (X, d)$ *parameterized* by a connected *graph* $G = (V, E)$, or a *network of type* G [1]. The *vertices* and *edges* of a network Γ are defined as restrictions of the mapping Γ onto the vertices and edges of the graph G. The *length of an edge* $\Gamma\colon vw \to X$ is the value $d\big(\Gamma(v), \Gamma(w)\big)$, and the *length* $d(\Gamma)$ *of the network* Γ is the sum of the lengths of all its edges. The *boundary* $\partial\Gamma$ *of the network* Γ is the restriction of the mapping Γ onto the boundary ∂G of the graph (an arbitrary subset of the vertex set). If $M \subset X$ is a finite subset, and $M \subset \Gamma(V)$, then we say that the *network* Γ *connects the set* M. The vertices of graphs and networks which are not boundary ones are referred to as *interior*. The value $\mathrm{smt}(M) = \inf\big\{d(\Gamma) : \Gamma \text{ is a network connecting } M\big\}$ is called the *length of a shortest network*. A network such that $d(\Gamma) = \mathrm{smt}(M)$ is called a *shortest network*, [2], or a Steiner minimal tree.

In the Steiner problem it is natural to consider finite metric spaces as the space X. In this case possible fillings are metric spaces having the structure of one-dimensional stratified manifolds (that can be considered as weighted graphs with non-negative weight functions). It was proved in [2] that the change of a metric ρ by the metric $\lambda\rho + a, \lambda > 0, a > \lambda a_\rho$, where a_ρ is a number dependent on the metric ρ, does not change the type of a minimal filling. We will prove the converse assertion and obtain the other results we described in the abstract.

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2. Preliminaries

Let M be an arbitrary finite set and $G = (V, E)$ a connected graph. We say that G *connects* M and M is *the boundary of the graph* G if $M \subset V$. The boundary of a graph G is denoted by ∂G. Now, let $\mathcal{M} = (M, \rho)$ be a finite pseudo-metric space, $G = (V, E)$ a connected graph connecting M, and $\omega : E \to \mathbb{R}_+$ a mapping into nonneggative real numbers usually called a *weight function* generating the *weighted graph* $\mathcal{G} = (G, \omega)$. The *weight* of a weighted graph $\mathcal{G}$ is the value $\omega(G)$ equal to the sum of weights of all edges of the graph. The function ω generates a pseudo-metric d_ω on V, namely: the distance between vertices of the graph $\mathcal{G}$ is the least weight of paths connecting those vertices. If for any points p and q from M the inequality $\rho(p, q) \le d_\omega(p, q)$ holds, then the weighted graph $\mathcal{G}$ is called a *filling* of the space $\mathcal{M}$ and the graph G is called the *type of this filling*. The number $\mathrm{mf}(\mathcal{M}) = \inf \omega(\mathcal{G})$, where the infimum is taken over all fillings $\mathcal{G}$ of the space $\mathcal{M}$, is called the *weight of minimal filling*, and a filling $\mathcal{G}$ such that $\omega(\mathcal{G}) = \mathrm{mf}(\mathcal{M})$ is called a *minimal filling*.

Definition 1. A finite pseudo-metric space $\mathcal{M} = (M, \rho)$ is called *additive* if M can be connected by a weighted tree $\mathcal{G} = (G, \omega)$ such that ρ coincides with the restriction of d_ω onto M (see [1]). The tree $\mathcal{G}$ in this case is said to be *generating*.

Statement 2.1 ([1]). *Let the minimal filling $\mathcal{G} = (G, \omega)$ of a space $\mathcal{M} = (M, \rho)$ be a star, where its interior vertex v is connected with all points $p_i \in M, 1 \le i \le n$, $n \ge 3$. In this case the space $\mathcal{M} = (M, \rho)$ is additive and its minimal fillings are its generating trees* (*i.e., the trees generating the distance function*).

Let $G = (V, E)$ be an arbitrary tree. Let $v \in V$ be its interior vertex of degree $(k + 1) \ge 3$ adjacent with k vertices $w_1, \ldots, w_k$ from ∂G. Then the vertex set $\{w_1, \ldots, w_k\}$ and also the edge set $\{vw_1, \ldots, vw_k\}$ are called *moustaches*. The number k is called the *degree* of the moustaches and the vertex v is called the *common vertex* of the moustaches.

A tree is called *binary* if the degree of any of its vertices is 1 or 3 and the set of boundary vertices consists of exactly the vertices of degree 1.

Statement 2.2 ([1]). *Let $M = \{p_1, p_2, p_3, p_4\}$ and ρ be an arbitrary pseudo-metric on M. Assume $\rho_{ij} = \rho(p_i, p_j)$. In this case the weight of a minimal filling $\mathcal{G} = (G, \omega)$ of the space $\mathcal{M} = (M, \rho)$ can be found by means of the formula* $\frac{1}{2}\Big(\min\{\rho_{12} + \rho_{34}, \rho_{13} + \rho_{24}, \rho_{14} + \rho_{23}\} + \max\{\rho_{12} + \rho_{34}, \rho_{13} + \rho_{24}, \rho_{14} + \rho_{23}\}\Big)$.

If the minimum in this formula is equal to $\rho_{ij} + \rho_{rs}$, then the type of the minimal filling is a binary tree with moustaches $\{p_i, p_j\}$ and $\{p_r, p_s\}$.

Statement 2.3. *For every additive space the only non-degenerate minimal filling is its non-degenerate generating tree.*

Statement 2.4 ([1]). *A criterion of additivity of f space is the 4 points rule: for any four points p_i, p_j, p_k, p_l the values $\rho(p_i, p_j) + \rho(p_k, p_l)$, $\rho(p_i, p_k) + \rho(p_j, p_l)$, $\rho(p_i, p_l) + \rho(p_j, p_k)$ are the lengths of sides of an isosceles triangle whose base does not exceed its other sides.*

3. Maps of the form $(M, \rho) \to (M, f \circ \rho)$

We consider the case when F is the class of all finite metric spaces, $T = \{(M, \rho) \to (M, f \circ \rho) : f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}\}$, where f is applied to the only positive part of metric $\rho(x, y)$, extended to $x = y$ by 0 and the elements of T' preserve all non-degenerate types of minimal fillings of four-point metric spaces and finite non-degenerate stars.

For any $a \in \mathbb{R}$ we denote $\mathbb{R}_{>a} = \{x \in \mathbb{R} : x > a\}$.

Theorem 1. *Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function such that for every metric space (M, ρ) the function $f \circ \rho$ is still a metric on M, and non-degenerate stars and types of minimal fillings of four-point spaces are preserved. Then there exists a real number C such that $f + C$ is linear on $\mathbb{R}_{>0}$.*

The proof of theorem 1 is based on two auxiliary results, lemmas 3.1 and 3.3.

Lemma 3.1. *Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be such a function that for every metric space (M, ρ), whose non-degenerate minimal filling type is a star, the function $f \circ \rho$ is still a metric on M and all non-degenerate types of all minimal fillings of the spaces (M, ρ) and $(M, f \circ \rho)$ are the same. Then there exists a real number C such that $f + C$ is additive on $\mathbb{R}_{>0}$.*

Proof. Show that there exists a number C such that for each $k \in \mathbb{N} \cup \{0\}$, $N = 2^k$, any $a, b \in \mathbb{R}_{>\frac{1}{N}}$, and $g = f + 2C$ the equality $g(a + b) = g(a) + g(b)$ holds. Let $\mathcal{M} = (M, \rho)$, $M = \{p_i\}_{i=0}^{7}$, be a metric space whose minimal filling $\mathcal{G} = (G, \omega)$ is a star with an interior vertex v, connected with all points p_i by the edges $e_i = vp_i$, and $\omega(e_0) = \omega(e_1) = \omega(e_2) = \frac{1}{N}$, $\omega(e_3) = a$, $\omega(e_4) = b$, $\omega(e_5) = a - \frac{1}{N}$, $\omega(e_6) = b - \frac{1}{N}$, $\omega(e_7) = \frac{2}{N}$.

Since $f \circ \rho$ is a metric on M and the star $\mathcal{G}_1 = (G, \omega_1)$ is a minimal filling of the space $\mathcal{M}_1 = (M, f \circ \rho)$ due to Statement 2.1, for $i \neq j$ the following equalities hold:

$$f\Big(\rho(p_i, p_j)\Big) = f\Big(\omega(e_i) + \omega(e_j)\Big) = \omega_1(e_i) + \omega_1(e_j).$$

Let us find all $\omega_1(e_i)$. We have

$$\omega_1(e_0) = \frac{\omega_1(e_0) + \omega_1(e_1) + \omega_1(e_0) + \omega_1(e_2) - \omega_1(e_1) - \omega_1(e_2)}{2} = \frac{f\Big(\rho(p_0, p_1)\Big) + f\Big(\rho(p_0, p_2)\Big) - f\Big(\rho(p_1, p_2)\Big)}{2} = \frac{f(\frac{2}{N})}{2},$$

and for $i \geq 1$ we obtain

$$\omega_1(e_i) = f\Big(\rho(p_0, p_i)\Big) - \omega_1(e_0) = f\Big(\omega(e_i) + \omega(e_0)\Big) - \omega_1(e_0) = f\Big(\omega(e_i) + \frac{1}{N}\Big) - \frac{f(\frac{2}{N})}{2}.$$

Further,

$$f\Big(a + \frac{1}{N}\Big) = f\Big(\rho(p_5, p_7)\Big) = \omega_1(e_5) + \omega_1(e_7) = f(a) + f\Big(\frac{3}{N}\Big) - f\Big(\frac{2}{N}\Big),$$
$$f\Big(b + \frac{1}{N}\Big) = f\Big(\rho(p_6, p_7)\Big) = \omega_1(e_6) + \omega_1(e_7) = f(b) + f\Big(\frac{3}{N}\Big) - f\Big(\frac{2}{N}\Big),$$

$$f(a + b) = f\Big(\rho(p_3, p_4)\Big) = \omega_1(e_3) + \omega_1(e_4) = f\Big(a + \frac{1}{N}\Big) + f\Big(b + \frac{1}{N}\Big) - f\Big(\frac{2}{N}\Big) = f(a) + f(b) + 2f\Big(\frac{3}{N}\Big) - 3f\Big(\frac{2}{N}\Big) = f(a) + f(b) + x,$$

where $x := 2f(\frac{3}{N}) - 3f(\frac{2}{N})$. We now show that for any $k \in \mathbb{N}$ the relation $2f(\frac{6}{N}) - 3f(\frac{4}{N}) = 2f(\frac{3}{N}) - 3f(\frac{2}{N})$ is valid. Let $a = b = \frac{3}{N}$ or $a = b = \frac{2}{N}$, then using the last formula we get

$$f\Big(\frac{6}{N}\Big) = 2f\Big(\frac{3}{N}\Big) + x, f\Big(\frac{4}{N}\Big) = 2f\Big(\frac{2}{N}\Big) + x,$$

$$2f\Big(\frac{6}{N}\Big) - 3f\Big(\frac{4}{N}\Big) = 4f\Big(\frac{3}{N}\Big) - 6f\Big(\frac{2}{N}\Big) - x = x = 2f\Big(\frac{3}{N}\Big) - 3f\Big(\frac{2}{N}\Big).$$

By induction on k we find that $x = 2f(3) - 3f(2)$. Thus, for any k the number C is equal to $\frac{2f(3)-3f(2)}{2}$ and hence it does not depend on a and b, which can be chosen arbitrary because for any $a, b \in \mathbb{R}_{>0}$ there exists a number $k \in \mathbb{N}, N = 2^k$ such that $a, b \in \mathbb{R}_{>\frac{1}{N}}$, and we can build an appropriate $\mathcal{M}$. The latter implies that the function $f + 2C$ is additive on any open ray $x > \frac{1}{N}$ and, consequently, it is such on the whole ray $x > 0$. □

Remark 1. If there are no additional restrictions on the additive function, then there are infinitely many nonlinear functions that satisfy the equation $f(a + b) = f(a) + f(b)$. This was proven in 1905 by Georg Hamel using a Hamel basis.

Lemma 3.2. *The functions not changing the types of minimal fillings are monotone increasing.*

Proof. Show that if $0 < a < b$, then $f(a) < f(b)$. Take a set $X = \{p_i\}_{i=1}^4$ and a function $\rho : X \times X \to \mathbb{R}_{>0}$ such that $\rho(p_1, p_2) = \rho(p_2, p_1) = \rho(p_3, p_4) = \rho(p_4, p_3) = a$, for any $x \in X$ the relation $\rho(x, x) = 0$ holds, and for all remaining pairs of points the function ρ takes the value b. Evidently, $\big(X, \rho\big)$ is a metic space. In accordance with Statement 2.2, the filling of this space has the moustaches $\{p_1, p_2\}$ and $\{p_3, p_4\}$ because the minimum of the sum of the lengths is attained at the corresponding opposite edges (it is equal to $2a$). For a function f not changing the types of minimal fillings after changing ρ by $f \circ \rho$ the moustaches remain the same and hence $2f(a)$ is the minimum of the sum of the lengths of opposite edges, i.e. $f(a) < f(b)$. □

Lemma 3.3. *If $g\colon \mathbb{R}_{>0} \to \mathbb{R}$ is a monotone increasing additive function, then g is linear on $\mathbb{R}_{>0}$.*

Remark 2. Here and below, not the functions of the form $f(x) = kx + b$ but the restrictions of linear mappings of $\mathbb{R}$ into itself are called linear functions on $\mathbb{R}_{>0}$.

Proof. Let g be nonlinear, then there exist $x_1, x_2 > 0$ such that $\frac{g(x_1)}{x_1} = \alpha > \beta = \frac{g(x_2)}{x_2}$. Take $\epsilon > 0$ so that $\alpha x_1 > \beta(x_1 + \epsilon)$. In this case there exists $m \in \mathbb{N}$ such that $\frac{x_2}{m} < \epsilon$ and hence $\frac{x_2}{m} < x_1 + \epsilon$, therefore there exists a number $k \in \mathbb{N}$ such that $x_1 < \frac{kx_2}{m} < x_1 + \epsilon$.

But then the additivity of g implies that $g(\frac{kx_2}{m}) = \beta\frac{kx_2}{m} < \beta(x_1 + \epsilon) < \alpha x_1 = g(x_1)$, which contradicts the monotonicity. □

Proof. [Proof of theorem 1] By Lemma 3.1, there exists a number C such that the function $f + 2C$ is additive. By Lemma 3.2, the function f is monotone increasing and so $f + 2C$ is also monotone increasing, therefore, in accordance with Lemma 3.3, $f + 2C$ is linear. □

4. Linear maps

We put $\rho_{ij} = \rho(p_i, p_j)$ and $\rho = (\rho_{12}, \rho_{13}, \dots, \rho_{n-1,n})$ for any metric space (M, ρ), where $M = \{p_1, \dots, p_n\}$.

Notation 4.1. Denote by N the sum of a positive diagonal matrix $A = \mathrm{diag}(\lambda_{12}, \lambda_{13}, \dots, \lambda_{n-1,n})$ and a matrix B made of identical rows of nonnegative elements, and by $C(\rho)$ the scalar product of a row of the matrix B with the vector ρ.

Remark 3. Matrix N of the form $A + B$ in notation 4.1 generates a map $\rho \mapsto \rho'$ such that $\rho'_{ij} = \lambda_{ij}\rho_{ij} + C(\rho)$.

We consider the case when F is the class of all finite metric spaces, the class T consists of the maps $\rho \to N\rho$, where the matrix N is a matrix of the form 4.1, and the elements of T' preserve all minimal fillings of the type of non-degenerate stars.

Lemma 4.1. *Matrix N of the form A+B in Notation 4.1 where $A = \lambda E$, generates a metrics transformation preserving metrics and the types of non-degenerate minimal fillings.*

Proof. By Remark 3, the transformation has the form $\rho \mapsto \lambda\rho + C(\rho)$; therefore, it preserves metrics and non-degenerate types of minimal fillings. □

Theorem 2. *A matrix N in notation 4.1 preserves metrics and minimal fillings whose types are non-degenerate stars if and only if A is a scalar matrix.*

Proof. If A is scalar, then Lemma 4.1 implies that the matrix N preserves metrics and the types of non-degenerate minimal fillings.

We now prove the converse assertion. Let A be not scalar. Show that the matrix N does not always preserve the metrics and types of minimal fillings that are non-degenerate stars.

Suppose that the matrix N preserves all the metrics. Since A is not scalar, there exist a, b and c such that $\lambda_{ab} \neq \lambda_{ac}$. Using Statement 2.3, we construct a metric space $M = \{1, \ldots, n\}$ with metric ρ so that its type of minimal filling is a non-degenerate star, where M is the set of all vertices of degree 1, and the type of minimal filling of its image is not a star. Denote by (G, ω) the minimal filling, and let o be the interior vertex of G. We set $\lambda_i = \omega(oi)$, where i is a boundary vertex. Since G is a star, we have $\rho_{ij} = \lambda_i + \lambda_j$, where $\rho_{ij} = \rho(i, j)$. It follows that $\rho_{ab} + \rho_{cd} = \rho_{ac} + \rho_{bd}$, hence $\rho_{ac} = \rho_{ab} + \rho_{cd} - \rho_{bd}$.

To ensure that the type of minimal filling of the image of M is not a star, it is sufficient that $\rho'_{ab} + \rho'_{cd} \neq \rho'_{ac} + \rho'_{bd}$. Using Remark 3, we obtain $\lambda_{ab}\rho_{ab} + C(\rho) + \lambda_{cd}\rho_{cd} + C(\rho) \neq \lambda_{ac}\rho_{ac} + C(\rho) + \lambda_{bd}\rho_{bd} + C(\rho)$, hence $\lambda_{ab}\rho_{ab} + \lambda_{cd}\rho_{cd} \neq \lambda_{ac}\rho_{ac} + \lambda_{bd}\rho_{bd}$, i.e., $\lambda_{ab}\rho_{ab} + \lambda_{cd}\rho_{cd} \neq \lambda_{ac}(\rho_{ab} + \rho_{cd} - \rho_{bd}) + \lambda_{bd}\rho_{bd}$. Combining similar terms, we obtain

$$\begin{aligned}(\lambda_{ab} - \lambda_{ac})\rho_{ab} \neq (\lambda_{bd} - \lambda_{ac})\rho_{bd} - (\lambda_{cd} - \lambda_{ac})\rho_{cd} &= (\lambda_{bd} - \lambda_{ac})(\lambda_b + \lambda_d) - (\lambda_{cd} - \lambda_{ac})(\lambda_c + \lambda_d) \\ &= (\lambda_{bd} - \lambda_{ac})\lambda_b + (\lambda_{ac} - \lambda_{cd})\lambda_c + (\lambda_{bd} - \lambda_{cd})\lambda_d.\end{aligned}$$

We take $\lambda_a = \lambda_b = 1$ and choose λ_d and λ_c. If $\lambda_{bd} - \lambda_{cd} = \lambda_{ac} - \lambda_{cd} = 0$, then in the resulting inequality we get 0 on the right and non-zero on the left for any λ_d and λ_c. Now, let $\lambda_{ac} - \lambda_{cd} \neq 0$ or $\lambda_{bd} - \lambda_{cd} \neq 0$. In the first case, the inequality can be solved with respect to λ_c for $\lambda_d = 1$, $\lambda_c \neq \frac{(\lambda_{ab}-\lambda_{ac})\rho_{ab}-(\lambda_{bd}-\lambda_{ac})\lambda_b-(\lambda_{bd}-\lambda_{cd})\lambda_d}{\lambda_{ac}-\lambda_{cd}}$, and we can take a positive solution, for example:

$$\lambda_c = \left|\frac{(\lambda_{ab} - \lambda_{ac})\rho_{ab} - (\lambda_{bd} - \lambda_{ac})\lambda_b - (\lambda_{bd} - \lambda_{cd})\lambda_d}{\lambda_{ac} - \lambda_{cd}}\right| + 1.$$

In the second case, the inequality can be solved with respect to λ_d for $\lambda_c = 1$, and we can take a positive solution. Thus, for N with non-scalar A it is always possible to construct a metric space such that one of its minimal fillings has the type of a non-degenerate star, and for an N-image of this space, its minimal filling is not a star. □

5. Additive spaces

We consider the case when F is the class of all finite additive metric spaces, T is the class of all linear maps given by matrices, and the elements of T' preserve all non-degenerate types of minimal fillings.

Let us consider the set of pseudometrics $K(n)$ in $\mathbb{R}^{n(n-1)/2}$. Each non-negativity condition and the triangle inequality gives a (closed) half-space bounded by a hyperplane passing through the origin of coordinates O, therefore $K(n)$ is a convex closed cone with vertex at O. Note that the metrics correspond exactly to all points of the cone $K(n)$ that do not lie on the coordinate hyperplanes. In particular, all the interior points of this cone correspond to metrics.

Definition 2. By a *union of k-dimensional faces* of the set $X \subset \mathbb{R}^N$ we mean a subset $E_k(X)$ of X such that for any $x \in E_k(X)$ there exists a ball of dimension k in X with the centre at x, but there does not exist a ball of dimension $k+1$ lying in X with the centre at x. We put $E_1(X) = E(X)$ and call it *the union of edges*. It is easy to see that $E\big(K(n)\big)$ is the union of rays starting at the origin of coordinates. Each of them will be called *an edge*.

Lemma 5.1. *Let $x \in K(n)$. Then $x \in E\Big(K(n)\Big)$ if and only if any non-zero vector $a \in \mathbb{R}^{n(n-1)/2}$ for which there exists $\delta > 0$ such that $x + a\epsilon \in K(n)$ holds for any $\epsilon, -\delta < \epsilon < \delta$, is proportional (collinear) to x, that is, there exists $\lambda \in \mathbb{R}$ such that $\lambda a = x$.*

Proof. Let $x \in E\Big(K(n)\Big)$, then if there existed a noncollinear non-zero vector a, then, together with the ray $\{\lambda x : \lambda \in \mathbb{R}_+\}$ and the segment $\{x + a\epsilon : -\delta < \epsilon < \delta\}$, the cone $K(n)$ would contain their convex hull due to the convexity of $K(n)$. This hull contains a two-dimensional ball (circle) with centre at x, which contradicts $x \in E\Big(K(n)\Big)$.

Let $x \notin E\Big(K(n)\Big)$, then there is a ball with center at x and dimension $k > 1$ lying in $K(n)$. As a, we can take any noncollinear x vector in this ball. □

Lemma 5.2. *The set $K(n)$ is the convex hull of the set $E\Big(K(n)\Big)$.*

Proof. Let x_i be coordinates in the space $\mathbb{R}^{n(n-1)/2}$. The set $W = K(n) \cap \{\sum x_i = 1\}$ is a convex polytope, and $K(n)$ is a cone over W with vertex O, since $K(n)$ lies in the positive orthant, and each ray in this orthant intersects $\{\sum x_i = 1\}$. The rays from $E(K(n))$ are rays starting at O and passing through the vertices of W. Since a convex polyhedron is a convex combination of its set of vertices, $K(n)$ is a convex combination of the set $E(K(n))$. □

Lemma 5.3. *In order for a linear map A to transform a pseudometric into a pseudometric, it is necessary and sufficient that*

$$A\Big(E\Big(K(n)\Big)\Big) \subset K(n).$$

Proof. Necessity. If there exists $x \in E\Big(K(n)\Big)$ such that $A(x) \notin K(n)$, then this is the pseudometric that transformed not into a pseudometric.

Sufficiency. By Lemma 5.2, any vector x of the cone $K(n)$ can be represented by a convex combination of vertices of the polytope multiplied by a nonnegative number, that is, by linear combination of edge vectors with nonnegative coefficients. Any linear mapping transforms this combination into a combination of edge images with the same coefficients, so the vector v remains in the cone $K(n)$.

□

Lemma 5.4. *Pseudometrics in which the set of n points is divided into 2 nonempty subsets U and V so that the distances between points of the same subset vanish, belong to $E\Big(K(n)\Big)$.*

Proof. Consider such an $x \in E\Big(K(n)\Big)$ and a non-zero vector $a \in \mathbb{R}^{n(n-1)/2}$ for which there exists $\delta > 0$ such that $x + a\epsilon \in K(n)$ holds for any $\epsilon, -\delta < \epsilon < \delta$. Note that if $x_{ij} = 0$, then $a_{ij} = 0$, since otherwise for any $\delta > 0$ for $\epsilon = -\frac{\delta}{2}\frac{a_{ij}}{|a_{ij}|}$ the inequalities $-\delta < \epsilon < \delta$ are valid, but $x_{ij} + \epsilon a_{ij} < 0$, so $x + a\epsilon \notin K(n)$. Since the vector a is non-zero, there exist $i, j \in \{1 \dots n\}$ such that $a_{ij} = d \neq 0$. It follows from the triangle inequalities that $x_{u_1v_1} = x_{u_2v_2}$ holds for any $u_1, u_2 \in U, v_1, v_2 \in V$.

In the transition from x to $x + a\epsilon$, the zero distances transform into zero ones, that is, the partition into 2 subsets is preserved, and all nonzero distances remain equal. Since $a_{ij} = 0$ follows from $x_{ij} = 0$, then $x_{ij} \neq 0$ also follows from $a_{ij} \neq 0$, and $x_{kl} + a_{kl}\epsilon = x_{ij} + a_{ij}\epsilon$ for any $x_{kl} \neq 0$ because in the transition from x to $x + a\epsilon$ the non-zero distances remain equal to each other. Since $x_{ij} \neq 0$ and $x_{kl} \neq 0$, then $x_{ij} = x_{kl}$ and $a_{ij} = a_{kl} = d$. That is, for any $i, j \in \{1 \dots n\}$, if $x_{ij} = 0$ then $a_{ij} = 0$, and if $x_{ij} \neq 0$ then $a_{ij} = d$, therefore x is collinear to a. □

Notation 5.1. We denote by $\begin{smallmatrix} i_{1,1} \\ i_{1,2} \\ \dots \\ i_{1,n_1} \end{smallmatrix} - \begin{smallmatrix} i_{2,1} \\ i_{2,2} \\ \dots \\ i_{2,n_2} \end{smallmatrix} - \dots - \begin{smallmatrix} i_{k,1} \\ i_{k,2} \\ \dots \\ i_{k,n_k} \end{smallmatrix}$ (or $A_1 - A_2 - \dots - A_k$, $A_j = \{i_{j,1}, i_{j,2}, \dots, i_{j,n_j}\}$) the set of n-points pseudometric spaces, $n = n_1 + \dots + n_k$, for which there are k consecutive points on the real line such that the map π taking $i_{j,1}, i_{j,2}, \dots, i_{j,n_j}$ into jth point of the line is isometric.

Remark 4.

1. Some of the corresponding points of the line may coincide. In particular, the set $A_1 - A_2 - \ldots - A_k$ includes $A_1 - A_2 - \ldots - (A_j \cup A_{j+1}) - \ldots - A_k$ as such a subset where $\pi(A_j) = \pi(A_{j+1})$.
2. The fact that the points are successive means that they form a non-strictly increasing or non-strictly decreasing sequence.
3. In the metric space $M = \{p_1, \ldots, p_n\} = \bigsqcup_{i=1}^{k} A_i$ all its elements are considered as distinct. Further, we will identify them with their numbers, that is, for any $i \in \{1 \ldots n\}$ we put $p_i = i$. When the points of a metric space are rearranged, the sequence $A_1 - A_2 - \ldots - A_k$ changes if and only if there is a number $i \in \{1 \ldots k\}$ such that the set A_i changes.

Definition 3. The direction vector $U - V$ in the notation of 5.1 is a pseudometric space from $U - V$ in which all nonzero distances are 1.

Statement 5.1. *The matrix A preserving $\frac{n(n-1)}{2}$ linearly independent edges of a cone in $\frac{n(n-1)}{2}$-dimensional space is diagonal in coordinates codirected with the direction vectors of these edges.*

Theorem 3. *For $n \geq 4$, the linear map A transforms additive metric spaces into additive metric spaces with the same non-degenerate type of minimal filling if and only if A has the form $\rho \to \alpha\rho$.*

Proof. Note that the set of all additive spaces in whose non-degenerate types of minimal fillings the vertices $i_1, i_2, \ldots, i_n$ are connected sequentially, coincides with $i_1 - i_2 - \ldots - i_n$, therefore all sets of the form $A_1 - A_2 - \ldots - A_n$, where $\#A_i = 1$ holds for any $i \in \{1 \ldots n\}$, are preserved. Also, for each $i \in \{1 \ldots n-1\}$, we have $A_1 - \ldots - (A_i \cup A_{i+1}) - \ldots - A_n = (A_1 - \ldots - A_i - A_{i+1} - \ldots - A_n) \cap (A_1 - \ldots - A_{i+1} - A_i - \ldots - A_n)$. That is, the preservation of all sets of the form $A_1 - A_2 - \ldots - A_n$ implies the preservation of all sets of the form $A_1 - A_2 - \ldots - A_{n-1}$, where $A_1, A_2, \ldots, A_{n-1}$ is a partition of M into $n-1$ non-empty subsets.

Similarly, the conservation of all sets of the form $A_1 - A_2 - \ldots - A_k$ implies the preservation of all sets of the form $A_1 - A_2 - \ldots - A_{k-1}$, where $A_1, A_2, \ldots, A_{k-1}$ is a partition of M into $k-1$ non-empty subsets. The result is the preservation of all sets of the form $U - V$, where U, V is a partition of M into 2 non-empty subsets.

Let $n \geq 5$. Consider the matrix X, in each ijth column of which the coordinates of the direction vector $\{i, j\} - (M \setminus \{i, j\})$ are written. By Definition 3, it holds $X_{ijkl} = 1$ if $\#(\{i, j\} \cap \{k, l\}) = 1$, otherwise $X_{ijkl} = 0$. It is not difficult to see that $X^T = X$. To show that these vectors are linearly independent, we find the inverse matrix Y, $XY = YX = E$, in the form

$$Y_{ijkl} = \begin{cases} x, & \#\big(\{i, j\} \cap \{k, l\}\big) = 1, \\ y, & \#\big(\{i, j\} \cap \{k, l\}\big) = 2, \\ z, & \#\big(\{i, j\} \cap \{k, l\}\big) = 0. \end{cases}$$

We have

$$1 = (XY)_{ijij} = \sum_{(a,b), \#(\{i,j\} \cap \{a,b\}) = 1} Y_{abij} = 2(n-2)x,$$

hence, $x = \frac{1}{2(n-2)}$;

$$0 = (XY)_{ijkl} = X_{ijik} Y_{ikkl} + X_{ijil} Y_{ilkl} + X_{ijjk} Y_{jkkl} + \\ + X_{ijjl} Y_{jlkl} + \sum_{(a,b), \#(\{i,j\} \cap \{a,b\}) = 1, \{k,l\} \cap \{a,b\} = \varnothing} Y_{abkl} = 4x + 2z(n-4),$$

thus, $\frac{1}{n-2} = -z(n-4)$, whence $z = -\frac{1}{(n-2)(n-4)}$;

$$0 = (XY)_{ijil} = x\#\Big(\big\{\{a, b\} : i \in \{a, b\}\big\} \setminus \{i, j\} \setminus \{i, l\} \cup \{j, l\}\Big) + \\ + X_{ijil} Y_{ilil} + z\#\Big(\big\{\{a, b\} : j \in \{a, b\}\big\} \setminus \{i, j\} \setminus \{j, l\}\Big) = x(n-2) + y + z(n-3),$$

therefore, $\frac{1}{2} + y - \frac{n-3}{(n-2)(n-4)} = 0$, so $y = \frac{n-3}{(n-2)(n-4)} - \frac{1}{2}$.

Thus, there exists an inverse matrix $X^{-1} = Y$, i.e., X is nondegenerate and, thus, its columns are linearly independent.

Consider the $1 - \begin{smallmatrix}2\\3\\\dots\\n\end{smallmatrix}$ direction vector, its first $n-1$ coordinates $(12, \dots, 1n)$ are 1, the remaining are 0. We find its coordinates in the basis of the direction rays vectors of the form $\{i, j\} - (M \setminus \{i, j\})$, multiplying by the matrix Y. The first $n-1$ coordinates are $(n-2)x + y = \frac{1}{2} + y = \frac{n-3}{(n-2)(n-4)}$, the remaining $(n-3)z + 2x = \frac{1}{n-2} - \frac{n-3}{(n-2)(n-4)} = -\frac{1}{(n-2)(n-4)}$. In this new basis of $\frac{n(n-1)}{2}$ linearly independent vectors, the preservation of rays of the form $\{i, j\} - (M \setminus \{i, j\})$ and Statement 5.1 imply that the matrix A is diagonal. All the coordinates of the direction vector $1 - \begin{smallmatrix}2\\3\\\dots\\n\end{smallmatrix}$ are non-zero, so the scalarity of the matrix A follows from the preservation of this edge.

Let $n = 4$. Then A preserves $\begin{smallmatrix}1\\2\end{smallmatrix} - \begin{smallmatrix}3\\4\end{smallmatrix}$, $\begin{smallmatrix}1\\3\end{smallmatrix} - \begin{smallmatrix}2\\4\end{smallmatrix}$, $\begin{smallmatrix}1\\4\end{smallmatrix} - \begin{smallmatrix}2\\3\end{smallmatrix}$, $1 - \begin{smallmatrix}2\\3\\4\end{smallmatrix}$, $2 - \begin{smallmatrix}1\\3\\4\end{smallmatrix}$, $3 - \begin{smallmatrix}1\\2\\4\end{smallmatrix}$, and $4 - \begin{smallmatrix}2\\3\\1\end{smallmatrix}$. Thus, A preserves 7 edges of $K(4)$, therefore, by Statement 5.1, the matrix A is diagonal in coordinates directed along any 6 linearly independent edges of the cone, for example, all, except $(1, 1, 1, 0, 0, 0)$ (linear independence is established by testing the matrix of these six vectors). Since all the coordinates of this vector in the basis of the remaining six ones are nonzero, the matrix A is scalar in these coordinates and, therefore, has the form $\rho \to \alpha\rho$. □

Remark 5. For $n = 3$, Theorem 3 is not true. In this case all spaces are additive. For $n = 3$, the set of metrics is a 3-faceted cone in 3-dimensional space; it is sufficient for the matrix to be only diagonal in the coordinates directed along all its edges. For $n = 2$ there is only one distance, and the matrix degenerates into a scalar.

Consider the case when F is the class of all finite ultrametric spaces, T is the class of all linear maps given by matrices.

Statement 5.2. *Ultrametric spaces are additive.*

Proof. Consider an arbitrary subset of 4 points. The condition of ultrametry implies that in any triangle the two largest distances are equal to each other, so we can choose the notation so that $\rho_{12} = \rho_{13} = a$ and $\rho_{23} = b$, $a \geq b$. Let us check the 4 points condition (2.4) for this subset.

1) Let $\rho_{14} = c < a$, then $\rho_{24} = \rho_{34} = a$ and $\rho_{12} + \rho_{34} = \rho_{13} + \rho_{24} = 2a > b + c = \rho_{23} + \rho_{14}$.

2) Let $\rho_{14} = a$, then in an isosceles 234 all sides are not greater than a. The 3 cases are possible:

- $\rho_{34} = \rho_{24} = c$, $a \geq c \geq b$, so $\rho_{12} + \rho_{34} = \rho_{13} + \rho_{24} = a + c \geq a + b = \rho_{23} + \rho_{14}$,
- $\rho_{34} = c$, $\rho_{24} = b$, $a \geq b \geq c$, so $\rho_{13} + \rho_{24} = \rho_{23} + \rho_{14} = a + b \geq a + c = \rho_{12} + \rho_{34}$,
- $\rho_{34} = b$, $\rho_{24} = c$, $a \geq b \geq c$, this case is symmetric to the previous one, so it is treated exactly in the same way.

3) Let $\rho_{14} = c > a$, then $\rho_{24} = \rho_{34} = c$ and $\rho_{12} + \rho_{34} = \rho_{13} + \rho_{24} = a + c \geq c + b = \rho_{23} + \rho_{14}$.

The 4 points condition is satisfied for any subset, and, according to Statement 2.4, the space is additive. □

Theorem 4. *The matrix of a one-to-one linear map that maps any ultrametric space of* 3 *points to an ultrametric point has the form $A = R(B + \lambda E)$, where B is a matrix of identical rows of positive elements, $\lambda \in \mathbb{R}$, and R is the permutation of the points* $(1, 0, 0)$, $(0, 1, 0)$ *and* $(0, 0, 1)$.

Proof. The set of all three-point ultrametric spaces is the union of the parts of the planes consisting of points of the form (a, a, b), (a, b, a) and (b, a, a), so we can choose R so that $S = R^{-1}A$ translates these planes into themselves.

Since $S = (s_{ij})$ transforms the first plane into itself, for any vector (a, b), $a \geq b$, we have $(a, b)(s_{11} + s_{12}, s_{13}) = (a, b)(s_{21} + s_{22}, s_{23})$, hence $s_{13} = s_{23} = z$, $s_{11} + s_{12} = s_{21} + s_{22} = c$. Considering the remaining two planes, we obtain

$$S = \begin{pmatrix} c-y & y & z \\ x & c-x & z \\ x & y & c-y-x+z \end{pmatrix} = \begin{pmatrix} x & y & z \\ x & y & z \\ x & y & z \end{pmatrix} + (c-x-y)E = B + \lambda E.$$

□

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