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Solvability of Optimization Problem for the Oscillation Processes with Optimal Vector Controls

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Abstract. The optimal control problem is investigated for oscillation processes, described by integro-differential equations with the Fredholm operator when functions of external and boundary sources nonlinearly depend on components of optimal vector controls. Optimality conditions having specific properties in the case of vector controls were found. A sufficient condition is established for unique solvability of the nonlinear optimization problem and its complete solution is constructed in the form of optimal control, an optimal process, and a minimum value of the functional.

1. Introduction

The fundamentals of optimal control theory for systems were laid in the works of A.G. Butkovskii [1], A.I. Egorov [2], T.K. Sirazetdtnov [10], and of other scientists in the 60s of the last century. This theory was widely developed and its methods began to penetrate into various fields of science and production, attracting new researchers. Development of optimal control theory is closely connected to the solution of applied problems described by functional equations. Control Problems for processes described by integro-differential equations with the Fredholm or Volterra integral operator are often used in practice [3, 9, 11, 12]. But methods for solving them are not sufficiently developed, particularly, when function of external or boundary source is nonlinear with respect to the control parameters. A number of studies were researched by authors in this scientific direction and interesting new results were obtained. For example, in [4], it was established that system of nonlinear integral equations, where solution is the optimal vector control, have the property of equal relations. It is a novelty in the optimal control theory for systems with distributed parameters.

In this paper, we investigate the solvability of the control problem for oscillation process described by integro-differential equation with the Fredholm operator when both function of the external source and function of the boundary source are vector controls of general form. In the process of investigation it is established that the system of nonlinear integral optimal control equations preserves the properties of equal relations for components of the external control vector as well as for components of the boundary control vector. Thus, it is established that the properties of equal relations in the case of vector control of general form is natural and it is noteworthy that the procedure for determining the components of optimal control is substantially simplified.

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2. Formulation of the Optimal Control Problem

Consider the optimal control problem for oscillation process described by following boundary value problem

$$V_{tt} - AV = \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau + g(t,x)f[t,u(t)], \quad x \in Q, \quad 0 < t \le T, \tag{1}$$

$$V(0,x) = \psi_1(x), \ V_t(0,x) = \psi_2(x), \quad x \in Q, \tag{2}$$

$$\Gamma V(t,x) \equiv \sum_{i,j}^{n} a_{ij}(x)V_{x_j}\cos(\nu, x_i) + a(x)V = b(t,x)p[t,\upsilon(t)], x \in \gamma, \ 0 < t < T. \tag{3}$$

Here A is the elliptic operator defined by the formula

$$AV(t,x) = \sum_{i,j=1}^{n} (a_{ij}(x)V_{x_j}(t,x))_{x_i} - c(x)V(t,x), \quad a_{i,j}(x) = a_{j,i}(x), \quad \sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \ge c_0 \times \sum_{i=1}^{n} \alpha_i^2, \quad c_0 > 0,$$

υ is a normal vector, outgoing from the point $x \in \gamma$; $K(t,\tau)$ is defined in domain $D = \{0 \le t \le T, 0 \le \tau \le T\}$ and satisfies the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty, \ K(t,\tau) \in H(D); \tag{4}$$

$g(t,x) \in H(Q_T)$, $Q_T = Q \times (0,T)$, $\psi_1(x) \in H_1(Q)$, $\psi_2(x) \in H(Q)$ are given functions; Q is the domain of space R^n bounded by piecewise smooth curve γ; $f[t,u(t)]$ is an external source function, non linearly depending on control function $u(t) = (u_1(t),...,u_m(t)) \in H^m(0,T)$, $u_i(t) \in H(0,T)$, $i = 1,...,m$; $p[t,\upsilon(t)]$ is a boundary source function depending on control function $\upsilon(t) = (\upsilon_1(t),...,\upsilon_r(t)) \in H^r(0,T)$, $\upsilon_j(t) \in H(0,T)$, $j = 1,...,r$; $H^k(0,T) = H(0,T) \times ... \times H(0,T)$ is a Cartesian product of k Hilbert spaces. Functions of external and boundary sources satisfy following monotonicity conditions:

$$f_{u_i}[t,u_i(t)] \ne 0, i = 1,...,m, \ p_{\upsilon_j}[t,\upsilon_j(t)] \ne 0, \ j = 1,...,r, \ \forall t \in [0,T]; \tag{5}$$

$H_1(Q)$ is a first order Sobolev space; λ is a parameter, T is a fixed moment of time.

The criterion of quality control is minimization of integral functional:

$$J[u(t),\upsilon(t)] = \int_0^T \int_Q \left\{[V(T,x) - \xi_1(x)]^2 + [V_t(T,x) - \xi_2(x)]^2\right\} dx + \beta \int_0^T \left(M^2[t,u(t)] + N^2[t,\upsilon(t)]\right) dt, \tag{6}$$

where $V(t,x)$ is a solution of boundary value problem (1)-(3); $\xi_1(x) \in H(Q)$, $\xi_2(x) \in H(Q)$ are known functions, given functions $M[t,u(t)]$ and $N[t,\upsilon(t)]$ are nonlinearly dependent on the functional variables and they are elements of the Hilbert space $H(0,T)$; β is a positive constant. The optimal control problem for the process $V(t,x)$ consists of transferring it from the initial state $(\psi_1(x), \psi_2(x))$ to the desired state $(\xi_1(x), \xi_2(x))$ in the given time T.

3. Generalized Solution of Boundary Value Problem

During investigating optimal control problem it is purposeful to use the concept of a generalized solution of the boundary value problem.

Definition 3.1. A generalized solution of boundary value problem (1) - (3) is a function $V(t,x) \in H(Q_T)$, which satisfies the integral identity

$$\int_Q \Big[(V_t\Phi)\Big]_{t_1}^{t_2} dx \equiv \int_{t_1}^{t_2}\int_Q \Big[V_t(t,x)\Phi_t(t,x) - \sum_{i,j=1}^{n} a_{ij}(x)V_{x_j}(t,x)\Phi_{x_i}(t,x) - c(x)V(t,x)\Phi(t,x)\Big]dxdt +$$

$$+\int_{t_1}^{t_2}\int_\gamma [b(t,x)p(t,v(t)) - a(x)V(t,x)]\Phi(t,x)dxdt + \int_{t_1}^{t_2}\int_Q \Big[\lambda\int_0^T K(t,\tau)V(\tau,x)d\tau + g(t,x)f[t,u(t)]\Big]\Phi(t,x)dxdt,$$

for any $t_1,\ t_2\ (0 < t_1 \leq t \leq t_2 \leq T),\quad \Phi(t,x) \in H_1(Q_T)$ and it satisfies the initial conditions in the weak sense, i.e. for any functions $\phi_0(x) \in H(Q),\ \ \phi_1(x) \in H(Q)$ we have equalities

$$\lim_{t\to+0}\int_Q V(t,x)\phi_0(x)dx = \int_Q \psi_1(x)\phi_0(x)dx, \qquad \lim_{t\to+0}\int_Q V_t(t,x)\phi_1(x)dx = \int_Q \psi_2(x)\phi_1(x)dx. \tag{7}$$

The solution of problem (1) - (3) will be searched in the form:

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t)z_n(x), \tag{8}$$

where $V_n(t) = \langle V(t,x), z_n(x)\rangle = \int_Q V(t,x)z_n(x)dx$ are Fourier coefficients, $z_n(x)$ are generalized eigenfunctions of following boundary value problem [8] and they form a complete orthonormal system in the Hilbert space $H(Q)$,

$$D_n(\Phi, z_n) \equiv \int_Q\Big(\sum_{i,j=1}^{n} a_{i,j}(x)\Phi_{x_j}z_{nx_i} + c(x)z_n(x)\Phi(t,x)\Big)dx + \int_\gamma a(x)z_n(x)\Phi(t,x)dx = \lambda_n^2\int_Q z_n(x)\Phi(t,x)dx;$$

$$\Gamma z_n(x) = 0,\ \ x \in \gamma,\ \ 0 < t < T,\ \ n = 1,2,..., \tag{9}$$

corresponding eigenvalues λ_n satisfy the following conditions $\lambda_n \leq \lambda_{n+1}, \forall n = 1,2,3,..., lim_{n\to\infty}\lambda_n = \infty$.

Fourier coefficients $V_n(t)$ are defined as the solution of the linear Fredholm integral equation of the second kind:

$$V_n(t) = \lambda\int_0^T K_n(t,s)V_n(s)ds + a_n(t), n = 1,2,3..., \tag{10}$$

where

$$K_n(t,s) = \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-\tau)K(\tau,s)d\tau;\quad K_n(0,s) = 0,\ \ n = 1,2,3,...; \tag{11}$$

$$a_n(t) = \psi_{1n}\cos\lambda_n t + \frac{\psi_{2n}}{\lambda_n}\sin\lambda_n t + \frac{1}{\lambda_n}\int_0^t \sin\lambda_n(t-\tau)\Big[q_n(\tau)f[\tau,u(\tau)] + b_n(\tau)p[\tau,v(\tau)]\Big]d\tau. \tag{12}$$

The solution of integral equation (11) is found [6] by the formula

$$V_n(t) = \lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t); \tag{13}$$

where

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty}\lambda^{i-1}K_{n,i}(t,s),\ \ n = 1,2,3,..., \tag{14}$$

is a resolvent of kernel $K_n(t,s)$, and iterated kernels $K_{n,i}(t,s)$ are defined by the formulas:

$$K_{n,i+1}(t,s) = \int_0^T K_n(t,\eta)K_{n,i}(\eta,s)d\eta, \quad i = 1,2,3,..., \tag{15}$$

for each fixed $n = 1,2,3,...,$.

Note that Neumann series absolutely converges for values of the parameter λ satisfying

$$|\lambda| < \frac{\lambda_n}{T\sqrt{K_0}} \longrightarrow \infty, \tag{16}$$

for each $n = 1,2,3,....$ It is easy to see that the radius of convergence of Neumann series increases with increasing n and it converges for any $n = 1,2,3,...,$ only on the interval

$$|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}. \tag{17}$$

By direct calculations it can be established that there is following estimate

$$\int_0^T R_n^2(t,s,\lambda)ds = \frac{K_0 T}{\left(\lambda_n - |\lambda|\sqrt{K_0 T^2}\right)^2} \tag{18}$$

which will be repeatedly used in proving the convergence of series.

Thus, formal solution of the boundary value problem (1)-(3) has the form of

$$V(t,x) = \sum_{n=1}^{\infty} V_n(t)z_n(x) = \sum_{n=1}^{\infty}\Big(\lambda \int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t)\Big)z_n(x).$$

This solution will be rewritten in the form of

$$V(t,x) = \int_0^T \sum_{n=1}^{\infty}\left\{\psi_n(t,\lambda) + \frac{1}{\lambda_n}\int_0^T \varepsilon_n(t,\eta,\lambda)\left(q_n(\eta)f[\eta,u(\eta)] + b_n(\eta)p(\eta,v(\eta))\right)d\eta\right\}z_n(x), \tag{19}$$

where

$$\psi_n(t,\lambda) = \psi_{1n}\Big[\cos\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\cos\lambda_n s ds\Big] + \frac{\psi_{2n}}{\lambda_n}\Big[\sin\lambda_n t + \lambda\int_0^T R_n(t,s,\lambda)\sin\lambda_n s ds\Big],$$

$$\varepsilon_n(t,\eta,\lambda) = \begin{cases} \sin\lambda_n(t-\eta) + \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & 0 \le \eta \le t, \\ \lambda\int_\eta^T R_n(t,s,\lambda)\sin\lambda_n(s-\eta)ds, & t \le \eta \le T. \end{cases}$$

Lemma 3.2. *The solution of boundary value problem (1) - (3) defined by the formula (19) is an element of the Hilbert space $H(Q_T)$.*

Proof. The assertion of the lemma follows from the following inequality, established by direct calculations:

$$\int_0^T\int_Q V^2(t,x)dxdt = \int_0^T\int_Q\Big(\sum_{n=1}^{\infty}V_n(t)z_n(x)\Big)^2 dxdt = \int_0^T\sum_{n=1}^{\infty}V_n^2(t)dt \le \int_0^T\sum_{n=1}^{\infty}\Big(\lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds +$$

$$+a_n(t)\Big)^2 dt \le 2\int_0^T\sum_{n=1}^{\infty}\Big[\frac{\lambda^2 T K_0}{\left(\lambda_n - |\lambda|\sqrt{K_0T^2}\right)^2}\int_0^T a_n^2(s)ds + a_n^2(t)\Big]dt \le 6T\Big[\frac{\lambda^2 K_0 T}{\left(\lambda_n - |\lambda|\sqrt{K_0T^2}\right)^2}\Big]\Big\{\|\psi_1(x)\|^2_{H_1(Q)} +$$

$$+\frac{1}{\lambda_1^2}\|\psi_2(x)\|^2_{H(Q)} + \frac{2}{\lambda_1^2}\Big(\|g(t,x)\|^2_{H(Q_T)}\|f(t,u(t))\|^2_{H(0,T)} + \|b(t,x)\|^2_{H(\gamma_T)}\cdot\|p(t,v(\eta))\|^2_{H(0,T)}\Big)\Big\} \le \infty, \quad \gamma_T = \gamma\times(0,T).$$

□

By differentiating the formal solution (19) with respect to t, we obtain a generalized derivative of a function $V(t,x)$.

$$V_t(t,x) = \sum_{n=1}^{\infty}\Big(\lambda\int_0^T R'_{nt}(t,s,\lambda)a_n(s)ds + a'_n(t)\Big)z_n(x). \tag{20}$$

Lemma 3.3. *The function $V_t(t,x)$ defined by the formula (20) is an element of the Hilbert space $H(Q_T)$.*

Proof. Taking into account inequality

$$\int_0^T |R'_n(t,s,\lambda)|^2 ds \le \frac{\lambda_n^2 T K_0}{\Big(\lambda_n - |\lambda|\sqrt{K_0T^2}\Big)^2},$$

by direct calculation we can show that

$$\int_0^T\int_Q V_t^2(t,x)dxdt = \int_0^T\int_Q\Big(\sum_{n=1}^{\infty}\Big(\lambda\int_0^T R'_{nt}(t,s,\lambda)a_n(s)ds + a'_n(t)\Big)z_n(x)\Big)^2 dxdt \le 2\int_0^T\sum_{n=1}^{\infty}\Big[\Big(\lambda\int_0^T R'_{nt}(t,s,\lambda)\times$$
$$\times a_n(s)ds\Big)^2 + \Big(a'_n(t)\Big)^2\Big]dt \le 6T\Big[1 + \frac{\lambda^2K_0T^2}{\Big(\lambda_1 - |\lambda|\sqrt{K_0T^2}\Big)^2}\Big]\Big\{\|\psi_1(x)\|^2_{H_1(Q)} + \|\psi_2(x)\|^2_{H(Q)} + \|g(t,x)\|^2_{H(Q_T)}\|f(t,\overline{u}(t))\|^2_{H(0,T)} +$$
$$+\|b(t,x)\|^2_{H(\gamma_T)}\|p(t,v(\overline{u}(t))\|^2_{H(0,T)}\Big\} < \infty.$$

□

4. Optimality Conditions for Vector Control and their Specifics

By monotonicity conditions (5), each set of controls $(u(t),v(t))$ uniquely determines the solution $V(t,x)$ of boundary value problem (1) - (3). When the set of controls receives an admissible increment $\big(\Delta u(t),\Delta v(t)\big)$, the solution $V(t,x)$ of boundary value problem (1) - (3) will receive the corresponding increment $\Delta V(t,x)$. Taking this circumstance into account, we calculate the increments of the functional

$$\Delta J(u(t),\vartheta(t)) = -\int_0^T \Delta\Pi(t,V(t,x),\omega(t,x),u(t),\vartheta(t))dt + \int_Q\Big[\Delta V^2(T,x) + \Delta V_t^2(T,x)\Big]dx,$$

where

$$\Delta\Pi[t,x,V(t,x),\omega(t,x),u(t),v(t)] = \Pi[t,x,V(t,x),\omega(t,x),u(t)+\Delta u(t),v(t)+\Delta v(t)] -$$
$$-\Pi[t,x,V(t,x),\omega(t,x),u(t),v(t)],$$
$$\Pi[t,x,V(t,x),\omega(t,x),u(t),v(t)] = \int_Q g(t,x)\omega(t,x)dx\cdot f(t,u(t)) + \int_\gamma b(t,x)\omega(t,x)dx\cdot p(t,v(t)), \tag{21}$$

here function $\omega(t,x)$ is a solution of the following adjoint boundary value problem

$$\omega_{tt} - A\omega = \lambda\int_0^T K(\tau,t)\omega(\tau,x)d\tau,\quad x\in Q,\quad 0\le t<T,$$
$$\omega(T,x) + 2[V_t(T,x) - \xi_2(x)] = 0,\quad \omega_t(T,x) - 2[V(T,x) - \xi_1(x)] = 0,\quad x\in Q,$$
$$\Gamma\omega(t,x) = 0,\quad x\in\gamma,\quad 0<t<T. \tag{22}$$

This problem is solved similarly to the basic boundary-value problem. Its solution has the form of

$$\omega(t,x) = -2\sum_{n=1}^{\infty}\Big\{-E_n^*(T-t)h_n + \int_0^T E_n^*(T-t)G_n(T-\tau)\{g_n(\tau)f[\tau,u(\tau)] + b_n(\tau)p[\tau,v(\tau)]\Big\}d\tau\Big\}z_n(x), \tag{23}$$

where the symbol (*) is a sign of transposition;

$$h_n = (h_{1n}, h_{2n}),\ \ h_{1n} = \xi_{2n} - \psi_{1n}\Big[-\lambda_n \sin\lambda_n T + \lambda\int_0^T R'_{nt}(T,s,\lambda)\cos\lambda_n s ds\Big] -$$
$$-\psi_{2n}\Big[\cos\lambda_n T + \frac{\lambda}{\lambda_n}\int_0^T R'_{nt}(T,s,\lambda)\sin\lambda_n s ds\Big];$$
$$h_{2n} = \xi_{1n} - \psi_{2n}\Big[\cos\lambda_n T + \lambda\int_0^T R_n(T,s,\lambda)\cos\lambda_n s ds\Big] -$$
$$-\frac{\psi_{2n}}{\lambda_n}\Big[\sin\lambda_n T + \lambda\int_0^T R_n(T,s,\lambda)\sin\lambda_n s ds\Big]; \tag{24}$$

$$G_n[T-t] = (G_{n1}[T-t], G_{n2}[T-t]),$$
$$G_{n1}[T-t] = \cos\lambda_n(T-t) + \frac{\lambda}{\lambda_n}\int_t^T R'_{nt}(T,s,\lambda)\sin\lambda_n(s-t)ds,$$
$$G_{n2}[T-t] = \frac{1}{\lambda_n}\Big(\sin\lambda_n(T-t) + \lambda\int_t^T R_n(T,s,\lambda)\sin\lambda_n(s-t)ds\Big); \tag{25}$$

$$E_n[T-t] = (E_{n1}[T-t], E_{n2}[T-t]),$$
$$E_{n1}[T-t] = \cos\lambda_n(T-t) + \lambda\int_0^T P_n(s,t,\lambda)\cos\lambda_n(T-s)ds,$$
$$E_{n2}[T-t] = \frac{1}{\lambda_n}\Big(\sin\lambda_n(T-t) + \lambda\int_0^T P_n(s,t,\lambda)\sin\lambda_n(T-s)ds\Big). \tag{26}$$

By direct calculation we prove that $\omega(t,x) \in H(Q_T)$.

Maximum principle. In order to the vector-valued functions $\{u_1(t),...,u_m(t)\}$ and $\{v_1(t),...,v_r(t)\}$ were optimal, it is necessary and sufficient that following relation is satisfied almost everywhere on $[0,T]$

$$\Pi(\cdot, u_1^0(t),...,u_m^0(t), v_1^0(t),...,v_r^0(t)) \Longleftrightarrow$$
$$\Longleftrightarrow Sup_{(u_i\in U_i, i=1,..,m, v_j\in S_j, j=1,...,r)}\Pi(\cdot, u_1,...,u_m, v_1,...,v_r),$$

where U_i, S_j are sets of admissible values to controls u_i and v_j respectively.

As a consequence of maximum principle, we obtain following system of equalities

$$\int_Q g(t,x)\omega(t,x)dx = \frac{2\beta M[t,u]M_{u_1}(t,u)}{f_{u_1}(t,u)} = ... = \frac{2\beta M[t,u]M_{u_m}(t,u)}{f_{u_m}(t,u)},$$
$$\int_\gamma b(t,x)\omega(t,x)dx = \frac{2\beta N[t,v]N_{v_1}(t,v)}{p_{v_1}(t,v)} = ... = \frac{2\beta N[t,v]N_{v_r}(t,v)}{p_{v_r}(t,v)} \tag{27}$$

and according to the Sylvester criterion, we obtain the system of determinant inequalities

$$\Big|f_{u_i}\Big(\frac{MM_{u_i}}{f_{u_i}}\Big)_{u_k}\Big| > 0,\ \ i,k = 1,2,..,m,\ \ \Big|f_{v_j}\Big(\frac{MM_{v_j}}{f_{v_j}}\Big)_{v_k}\Big| > 0,\ \ j,k = 1,2,..,r, \tag{28}$$

which should hold simultaneously. The set of relations (27) and (28) are called the *optimality conditions.*

The obtained optimality conditions in form of equalities have the property of equal relations. It is a specific property that hold only in the case of vector control of general form.

We introduce the following notation

$$\frac{\beta M[t,u]M_{u_i}(t,u)}{f_{u_i}(t,u)}=q(t),\ \ i=1,...,m, \tag{29}$$

$$\frac{\beta N[t,v]N_{v_j}(t,v)}{p_{v_j}(t,v)}=s(t),\ \ j=1,...,r. \tag{30}$$

Hence, according to the theorem about implicit function [7], taking into account the optimality condition in form of inequalities, we have the following equalities

$$u_i(t)=\varphi_i(t,q(t),\beta),\ \ i=1,...,m, \tag{31}$$

$$v_j(t)=\delta_j(t,s(t),\beta),\ \ j=1,...,r, \tag{32}$$

where functions $\varphi_i(t,q(t))$ and $\delta_j(t,s(t))$ are uniquely determined. By force of these relations, in order to determine the optimal vector controls, it is necessary to construct a solution of the following system of two equations

$$2q(t)=\int_Q g(t,x)\omega(t,x)dx,\ \ x\in Q, \tag{33}$$

$$2s(t)=\int_\gamma b(t,x)\omega(t,x)dx,\ \ x\in \gamma. \tag{34}$$

This circumstance essentially simplifies the procedure for constructing optimal vector controls,when functions $q(t)$ and $s(t)$ are known, their components can be found by formulas (31) and (32).

Note that second optimality condition in the form of inequalities essentially restricts the class of external $\{f(t,u(t))\}$ and boundary $\{p(t,v(t))\}$ sources functions. As shown by other studies [5], these properties of the optimal vector control problem are natural and only optimal vector control problems of general form have such properties.

5. Construction of Optimal Vector Controls

Taking into account equality (31), (32) and equality (23), we rewrite system (33), (34) in the form of

$$\begin{cases} q(t)=-\sum_{n=1}^{\infty}g_n(t)\Big\{-E_n^*(T-t)h_n+\int_0^T E_n^*(T-t)G_n(T-\tau)(g_n(\tau)f[\tau,\varphi_1(\tau,q(\tau),\beta),...,\varphi_m(\tau,q(\tau),\beta)] \\ +b_n(\tau)p[\tau,\delta_1(\tau,s(\tau),\beta),...,\delta_r(\tau,s(\tau),\beta)])d\tau\Big\}, \\ s(t)=-\sum_{n=1}^{\infty}b_n(t)\Big\{-E_n^*(T-t)h_n+\int_0^T E_n^*(T-t)G_n(T-\tau)(g_n(\tau)f[\tau,\varphi_1(\tau,q(\tau),\beta),...,\varphi_m(\tau,q(\tau),\beta)] \\ +b_n(\tau)p[\tau,\delta_1(\tau,s(\tau),\beta),...,\delta_r(\tau,s(\tau),\beta)])d\tau\Big\}. \end{cases} \tag{35}$$

We introduce the notations

$$\begin{aligned} W(t)&=(q(t),s(t)),Y_n(t)=(g_n(t),b_n(t)),\\ F[t,W(\tau),\beta]&=(f[t,\overline{\varphi}(t,q(t),\beta)],p[t,\overline{\delta}(t,s(t),\beta)]),\\ \overline{\varphi}(t,q(t),\beta)&=\Big\{\varphi_1(t,q(t),\beta),...,\varphi_m(t,q(t),\beta)\Big\};\\ \overline{\delta}(t,s(t),\beta)&=\Big\{\delta_1(t,s(t),\beta),...,\delta_r(t,s(t),\beta)\Big\}. \end{aligned} \tag{36}$$

We rewrite system (35) in vector form

$$W(t)=\sum_{n=1}^{\infty}Y_n(t)E_n^*(T-t)h_n-\sum_{n=1}^{\infty}Y_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)Y_n^*(\tau)F[\tau,W(\tau),\beta]d\tau. \tag{37}$$

Next, we introduce vector function

$$\sigma(t)=\sum_{n=1}^{\infty}Y_n(t)E_n^*(T-t)h_n \Rightarrow \begin{pmatrix}\sigma_1(t)\\ \sigma_2(t)\end{pmatrix}=\sum_{n=1}^{\infty}\begin{pmatrix}g_n(t)\\ b_n(t)\end{pmatrix}\langle E_n(T-t),h_n\rangle_{R_2}, \tag{38}$$

where symbol $\langle R_2\rangle$ is the scalar product in R_2, and the operator

$$L[W(t)]=-\sum_{n=1}^{\infty}Y_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)Y_n^*(\tau)F[\tau,W(\tau),\beta]d\tau. \tag{39}$$

Equation (37) can be rewritten in the operator form of

$$W(t)=L[W(t)]+\sigma(t). \tag{40}$$

Lemma 5.1. *Function $\sigma(t)$ determined by (38) is an element of space $H^2(0,T)=H(0,T)\times H(0,T)$.*

Proof. By force of inequalities

$$\int_0^T\sigma_1^2(t)dt=\int_0^T\Big(\sum_{n=1}^{\infty}g_n(t)\langle E_n(T-t),h_n\rangle_{R_2}\Big)^2dt\le\int_0^T\Big(\sum_{n=1}^{\infty}g_n(t)\|E_n(T-t)\|_{R_2}\|h_n\|_{R_2}\Big)^2dt\le$$
$$\le\int_0^T\Big(\sum_{n=1}^{\infty}g_n^2(t)\|E_n(T-t)\|_{R_2}^2\sum_{n=1}^{\infty}\|h_n\|_{R_2}^2dt\le E_0\|g(t,x)\|_{H(Q_T)}^2\sum_{n=1}^{\infty}\|h_n\|_{R_2}^2<\infty,$$
$$\int_0^T\sigma_2^2(t)dt=\int_0^T\Big(\sum_{n=1}^{\infty}b_n(t)\langle E_n(T-t),h_n\rangle_{R_2}\Big)^2dt\le\int_0^T\Big(\sum_{n=1}^{\infty}b_n(t)\|E_n(T-t)\|_{R_2}\|h_n\|_{R_2}\Big)^2dt\le$$
$$\le\int_0^T\Big(\sum_{n=1}^{\infty}b_n^2(t)\|E_n(T-t)\|_{R_2}^2\sum_{n=1}^{\infty}\|h_n\|_{R_2}^2dt\le E_0\|b(t,x)\|_{H(\gamma_T)}^2\sum_{n=1}^{\infty}\|h_n\|_{R_2}^2<\infty,$$

where E_0 is a smallest of the numbers satisfying the estimate

$$\|E_n(T-t)\|_{R_2}^2<E_0,\ E_0>0.$$

We obtain inequality

$$\|\sigma(t)\|_{H^2(0,T)}^2=\int_0^T\|\sigma(t)\|_{R_2}^2\,dt=\int_0^T\Big(\sigma_1^2(t)+\sigma_2^2(t)\Big)dt\le\infty,$$

from which it follows that

$$\sigma(t)\in H^2(0,T)=H(0,T)\times H(0,T).$$

□

Lemma 5.2. *Operator L maps the space $H^2(0,T)$ into itself, i.e. $L[W(t)]$ is an element of space $H^2(0,T)$ for each $W(t)\in H^2(0,T)$.*

Proof. Suppose that $W(t)\in H^2(0,T)$, i.e. $\{q(t)\in H(0,T),s(t)\in H(0,T)\}$. Then following equality is hold

$$\|L[W(t)]\|_{H^2(0,T)}^2=\int_0^T\Big\{\Big(\sum_{n=1}^{\infty}g_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)\Big(g_n(\tau)f[\tau,\bar\varphi(\tau,q(\tau),\beta)]+b_n(\tau)\times$$
$$\times p[\tau,\bar\delta(\tau,s(\tau),\beta)]\Big)d\tau\Big)^2+\sum_{n=1}^{\infty}b_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)\Big(g_n(\tau)f[\tau,\bar\varphi(\tau,q(\tau),\beta)]+$$
$$+b_n(\tau)p[\tau,\bar\delta(\tau,s(\tau),\beta)]d\tau\Big)^2\Big\}dt. \tag{41}$$

From this equality, by force of following inequalities the assertion of Lemma 5.2 is proved.

1)

$$\int_0^T \Big(\sum_{n=1}^{\infty} g_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)(g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]+b_n(\tau)p[\tau,\overline{\delta}(\tau,s(\tau),\beta)])d\tau\Big)^2 dt \leq$$
$$\leq \int_0^T \Big(\sum_{n=1}^{\infty} g_n(t)\Big\langle E_n^*(T-t), \int_0^T G_n(T-\tau)\Big(g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]+b_n(\tau)p[\tau,\overline{\delta}(\tau,s(\tau),\beta)]\Big)d\tau\Big)\Big\rangle_{R_2}\Big)^2 dt \leq$$
$$\leq \int_0^T \Big(\sum_{n=1}^{\infty} g_n^2(t)\|E_n(T-t)\|_{R_2}^2 \sum_{n=1}^{\infty}\Big\|\int_0^T G_n(T-\tau)\Big(g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]+b_n(\tau)p[\tau,\overline{\delta}(\tau,s(\tau),\beta)]\Big)d\tau\Big\|_{R_2}^2 dt \leq$$
$$\leq E_0\|g(t,x)\|_{H(Q_T)}^2 \sum_{n=1}^{\infty}\Big(\int_0^T G_n(T-\tau)g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]d\tau + \int_0^T G_n(T-\tau)b_n(\tau)p[\tau,\overline{\delta}(\tau,s(\tau),\beta)])d\tau\Big)^2 dt \leq$$
$$\leq 2E_0\|g(t,x)\|_{H(Q_T)}^2\Big(E_0\|g(t,x)\|_{H(Q_T)}^2\|f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]\|_{H(0,T)}^2 + E_0\|b(t,x)\|_{H(\gamma_T)}^2\|p[\tau,\overline{\delta}(\tau,s(\tau),\beta)])\|_{H(0,T)}^2\Big) \leq$$
$$\leq 2E_0^2\|g(t,x)\|_{H(Q_T)}^2\Big(\|g(t,x)\|_{H(Q_T)}^2\|f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]\|_{H(0,T)}^2 + \|b(t,x)\|_{H(\gamma_T)}^2\|p[\tau,\overline{\varphi}(\tau,s(\tau),\beta)]\|_{H(0,T)}^2\Big) < \infty;$$

2)

$$\int_0^T \Big(\sum_{n=1}^{\infty} b_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)(g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]+b_n(\tau)p[\tau,\overline{\delta}(\tau),s(\tau)\beta)])d\tau\Big)^2 dt \leq$$
$$\leq 2E_0^2\|b(t,x)\|_{H(\gamma_T)}^2(\|g(t,x)\|_{H(Q_T)}^2\|f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]\|_{H(Q_T)}^2 + \|(b(t,x)\|_{H(\gamma_T)}^2\|p[\tau,\overline{\delta}(\tau,s(\tau),\beta)]\|_{H(Q_T)}^2\Big) < \infty.$$

□

Lemma 5.3. *Suppose that following conditions are satisfied*

$$\|f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)] - f[\tau,\overline{\varphi}(\tau,\tilde{q}(\tau),\beta)]\|_{H(0,T)}^2 \leq f_0^2\|\overline{\varphi}\ \ [t,q(t),\beta] - \overline{\varphi}\ \ [t,\tilde{q}(t),\beta]\|_{H(0,T)}^2 \leq$$
$$\leq f_0^2 m\varphi_0^2\ \ (\beta)\|q(t)-\tilde{q}(t)\|_{H(0,T)}^2, f_0>0,$$
$$\|p[t,\overline{\delta}(t,s(t),\beta)] - p[t,\overline{\delta}(t,\tilde{s}(t),\beta)]\|_{H(0,T)}^2 \leq p_0^2\|\overline{\delta}(t,s(t),\beta)-\delta(t,\tilde{s}(t),\beta)\|_{H(0,T)} \leq$$
$$\leq p_0^2 r\delta_0^2(\beta)\|s(t)-\tilde{s}(t)\|_{H(0,T)},\ \ p_0>0. \tag{42}$$

Then if the conditions

$$\gamma = E_0\alpha_1\Big(2\Big[\|g(t,x)\|_{H(Q_T)}^2 + \|b(t,x)\|_{H(\gamma_T)}^2\Big]\Big)^{1/2} < 1,\ \ \alpha_1 = max\Big(\|g(t,x)\|_{H(Q_T)}^2 f_0^2\varphi_0^2(\beta)m;\ \|b(t,x)\|_{H(\gamma_T)}^2 p_0^2 q_0^2(\beta)r\Big) \tag{43}$$

are met, the operator $L[W(t)]$ *is contractive.*

Proof. By direct calculation we have the inequality

$$\|L[W(t)]-L[\tilde{W}(t)\|^2_{H^2(0,T)}=\int_0^T\Big\{\sum_{n=1}^{\infty}g_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)(g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]+b_n(\tau)\times$$
$$\times p[\tau,\overline{\delta}(\tau,s(\tau),\beta)])d\tau+\sum_{n=1}^{\infty}b_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)(g_n(\tau)f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]+b_n(\tau)p[\tau,\overline{\delta}(\tau,s(\tau),\beta)]d\tau)-$$
$$-\sum_{n=1}^{\infty}g_n(t)E_n^*(T-t)\int_0^T G_n(T-\tau)(g_n(\tau)f[\tau,\overline{\varphi}(\tau,\tilde{q}(\tau),\beta)]+b_n(\tau)p[\tau,\delta(\tau,\tilde{s}(\tau),\beta])d\tau+\sum_{n=1}^{\infty}b_n(t)E_n^*(T-t)\times$$
$$\times\int_0^T G_n(T-\tau)(g_n(\tau)f[\tau,\overline{\varphi}(\tau,\tilde{q}(\tau),\beta)]+b_n(\tau)p[\tau,\overline{\delta}(\tau,\tilde{s}(\tau),\beta)]d\tau)\Big\}^2dt\le 2E_0^2\Big(\|g(t,x)\|^2_{H(Q_T)}+$$
$$+\|b(t,x)\|^2_{H(\gamma_T)}\Big)\Big\{\|g(t,x)\|^2_{H(Q_T)}\Big[\|f[\tau,\overline{\varphi}(\tau,q(\tau),\beta)]-f[\tau,\overline{\varphi}(\tau,\tilde{q}(\tau),\beta)]\|^2_{H(Q_T)}+\|b(t,x)\|^2_{H(\gamma_T)}\times$$
$$\times\|p[\tau,\overline{\delta}(t,s(t),\beta)]-p[t,\overline{\delta}(t,\tilde{s}(t),\beta)]\|^2_{H(0,T)}\Big\}\le 2E_0^2\ \Big(\|g(t,x)\ \|^2_{H(Q_T)}\ +\|b(t,x)\ \|^2_{H(\gamma_T)}\Big)\ \alpha_1^2\ \ \|W(t)-\tilde{W}(\tau)\|^2_{H(0,T)}\ ,$$

from which the lemma follows. □

Theorem 5.4. *Suppose that conditions (5), (23), (42), (43) are satisfied, then operator equation (40) has a unique solution* $W(t)=(q(t),s(t))\in H^2(0,T)$.

Proof. According to Lemmas 5.1 and 5.2, operator equation (40) can be considered in the space $H^2(0,T)$. According to Lemma 5.3 operator L is contractive. Since the Hilbert space $H^2(0,T)$ is a complete metric space, by the theorem on contraction mappings [7] the operator L has a unique fixed point. □

The solution of operator equation (40) can be found by the method of successive approximations by the formulas

$$W_n(t)=L[W_{n-1}(t)]+\sigma(t), n=1,2,3,....$$

Zero approximation $W_0(t)$ can be any vector- function, in particular $W_0(t)=\sigma(t)$. Exact solution $\overline{W}(t)$ is defined as the limit of approximate solutions, i.e.

$$\overline{W}^0(t)=lim_{n\to\infty}W_n(t), \tag{44}$$

and we have the estimate

$$\|\overline{W}^0(t)-W_n(t)\|^2_{H^2(0,T)}\le\frac{\alpha^n}{1-\alpha}\|L[W_0(t)]+\sigma(t)-W_0(t)\|^2_{H^2(0,T)}.$$

Next, substituting found vector-function $\overline{W}^0(t)=(\overline{q}^0(t),\overline{s}^0(t))$ into (31) and (32) we obtain the components $u_i^0(t),v_i^0(t)$ of the optimal vector controls $u^0(t),v^0(t)$

$$u_i^0(t)=\varphi_i[t,\overline{q}^0(t),\beta],\ \ i=1,...,m,\ v_i^0(t)=q_i[t,\overline{s}^0(t),\beta],\ \ j=1,...,r. \tag{45}$$

Next we will find an optimal process by the formula

$$V^0(t,x)=\sum_{n=1}^{\infty}V_n^0(t)z_n(x);\quad V_n^0(t)=\lambda\int_0^T R_n(t,s,\lambda)a_n^0(s)ds+a_n^0(t),$$

$$a_n^0(t)=\psi_{1n}cos\lambda_nt+\frac{\psi_{2n}}{\lambda_n}sin\lambda_{nt}+\frac{1}{\lambda_n}\int_0^t sin\lambda_n(t-\tau)\Big\{g_n(\tau)f[\tau,u_1^0(\tau),...,u_m^0(\tau)]+b_n(\tau)p[\tau,v_1^0(\tau),...,\vartheta_r^0(\tau)]\Big\}d\tau;$$

And minimum value of the functional $J[u^0(t), v^0(t)]$ defined by the formula (6)

$$J[u^0(t), v^0(t)] = \int_Q \left\{ \left[V^0(T,x) - \xi_1(x) \right]^2 + \left[V_t^0(T,x) - \xi_2(x) \right]^2 \right\} dx +$$

$$+\beta \int_T^0 \left\{ M^2 \left[t, u_1^0(t), ..., u_m^0(t) \right] + M^2 \left[t, v_1^0(t), ..., v_r^0(t) \right] \right\} dt \tag{46}$$

The complete solution of the nonlinear optimization problem is found in the form of a triple

$$\left(\left(u^0(t), v^0(t) \right), V^0(t,x), J\left[u^0(t), v^0(t) \right] \right)$$

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