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Lacunary Almost Convergence and Some New Sequence Spaces

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Abstract. In this paper we study some new sequence spaces of strongly lacunary almost summable sequences, which naturally come up for investigation and which will fill up a gap in the existing literature. Some topological results, a characterization of strongly lacunary almost regular matrices, uniqueness of generalized limits and inclusion relations of such sequences have been observed.

1. Introduction

Let w be the set of all real or complex sequences and let ℓ_∞ denote the Banach space of bounded sequences $x = \{x_k\}_{k=0}^{\infty}$ normed by $\|x\| = \sup_{k\geq 0}|x_k|$. Let D be the shift operator on w, that is, $Dx = \{x_k\}_{k=1}^{\infty}$, $D^2x = \{x_k\}_{k=2}^{\infty}$ and so on. It may be recalled that (see Banach [1]) Banach limit L is a nonnegative linear functional on ℓ_∞ such that L is invariant under the shift operator (that is, $L(Dx) = L(x) \forall x \in \ell_\infty$) and that $L(e) = 1$ where $e = \{1, 1, ...\}$. A sequence $x \in \ell_\infty$ is called almost convergent (see [12]), if all Banach limits of x coincide. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [12] proved that

$$\hat{c} = \left\{ x : \lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} x_{n+i} \text{ exists uniformly in } n \right\}.$$

Almost convergent sequences were studied by Lorentz [12], King [9], Duran [4], Nanda [15], Savas [17, 19, 20], and others.

It is natural to expect that the concept of lacunary almost convergence must give rise to three types of summability methods, lacunary almost, absolutely lacunary almost and strongly lacunary almost. The almost lacunary summable sequences have been discussed by Nuray [9] and some others. More recently, Savas and Karakaya [21] have considered absolute lacunary almost convergent and absolute lacunary almost summable sequences. Also, the strongly almost summable sequence was studied by Nanda [15]. The strongly summable sequences have been systematically investigated by Hamilton and Hill [7], Kuttner [10] and some others. The goal of this paper is to study the spaces of strongly lacunary almost summable sequences, which naturally come up for investigation and which will fill up a gap in the existing literature.

Let $\theta = (k_r)$ be the sequence of positive integers such that

i) $k_0 = 0$ and $0 < k_r < k_{r+1}$

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ii) $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$.

Then θ is called a lacunary sequence. The intervals determined by θ are denoted by $I = (k_r - k_{r-1}]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r (see, Freedman et al. [5]).

Recently, Das and Mishra [3] defined M_θ, the set of almost lacunary convergent sequences, as follows:

$$M_\theta = \left\{ x : \text{ there exists } l \text{ such that uniformly in } i \geq 0, \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} (x_{k+i} - l) = 0 \right\}.$$

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [6] as follows : A sequence (x_n) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$.

More results on this convergence can be seen from [8, 11, 18].

Throughout this paper let $B = (b_{nk})$ be an infinite matrix of nonnegative real numbers and $p = (p_k)$ be a sequence such that $p_k > 0$. We write $Bx = \{B_n(x)\}$ if $B_n(x) = \sum_k b_{nk} |x_k|^{p_k}$ converges for each n. We write.

$$t_{rn}(x) = \frac{1}{h_r} \sum_{i\in I_r} B_{n+i}(x) = \sum_k b(n,k,r) |x_k|^{p_k}$$

where

$$b(n,k,r) = \frac{1}{h_r} \sum_{i\in I_r} b_{n+i,k}.$$

We write

$$\begin{aligned} [B_\theta, p]_0 &= \{x : t_{rn}(x) \to 0 \text{ uniformly in } n\}; \\ [B_\theta, p] &= \{x : t_{rn}(x - l) \to 0 \text{ for some } l \text{ uniformly in } n\} \end{aligned}$$

and

$$[B_\theta, p]_\infty = \left\{ x : \sup_{rn} t_{rn}(x) < \infty \right\}.$$

The sets $[B_\theta, p]_0$, $[B_\theta, p]$ and $[B_\theta, p]_\infty$ will be respectively called the spaces of strongly lacunary almost summable to zero, strongly lacunary summable and strongly lacunary bounded sequences.

If x is strongly lacunary almost summable to l we write $x_k \to l[B_\theta, p]$. A pair (B, p) will be called strongly lacunary almost regular if

$$x_k \to l \Rightarrow x_k \to l[B_\theta, p].$$

Theorem 1.1. *Let $p \in \ell_\infty$. Then $[B_\theta, p]_0$ and $[B_\theta, p]_\infty$ $(\inf p_k > 0)$ are complete linear topological spaces paranormed by g. If*

$$\|B\| = \sup_r \sum_k b(n,k,r) < \infty. \tag{1}$$

and

$$\sum_k b(n,k,r) \to 0 \text{ uniformly in } n. \tag{2}$$

hold, then $[B_\theta, p]$ has the same property. If further $p_k = p \forall k$, they are Banach spaces for $1 \leqq p < \infty$ and $p-$normed spaces for $0 < p < 1$.

Proof. It is easy to prove, so we omit the details. □

2. Some Topological Results

We now study locally boundedness and $r-$convexity for the spaces of strongly almost summable sequences. We start with some definitions. For $0 < r \leqq 1$ a nonempty subset U of a linear space is said to be absolutely $r-$convex if $x, y \in U$ and $|\lambda|^r + \left|\mu\right|^r \leq 1$ together imply that $\lambda x + \mu y \in U$. It is clear that if U is absolutely $r-$convex, then it is absolutely $t-$ convex for $t < r$. A linear topological space X is said to be $r-$ convex if every neighbourhood of $0 \in X$ contains an absolutely $r-$convex neighbourhood of $0 \in X$. The $r-$convexity for $r > 1$ is of little interest, since X is $r-$convex for $r > 1$ if and only if X is the only neighbourhood of $0 \in X$ (ee Maddox and Roles [14]). A subset C of X is said to be bounded if for each neighbourhood U of $0 \in X$ there exists an integer $N > 1$ such that $C \subseteq NU$. X is called locally bounded if there is a bounded neighbourhood of zero.

We now prove

Theorem 2.1. *Let $0 < p_k \leqq 1$. Then $[B_\theta, p]_0$ and $[B_\theta, p]_\infty$ are locally bounded if $\inf p_k > 0$. If (1) holds, then $[B_\theta, p]$ has the same property.*

Proof. We shall only prove for $[B_\theta, p]_\infty$. Let $\inf p_k = \theta > 0$. If $x \in [B_\theta, p]_\infty$, then there exists a constant $K' > 0$ such that

$$\sum_k b(n,k,r) \left|x_k\right|^{p_k} \leqq K' \left(\forall r, n\right).$$

For this K' and given $\delta > 0$ choose an integer $N > 1$ such that

$$N^\theta \geqq \frac{K'}{\delta}$$

Since $\frac{1}{N} < 1$ and $p_k \geqq \theta$ we have

$$\frac{1}{N^{p_k}} \leqq \frac{1}{N^\theta} \ (\forall k)$$

Then for all r and n, we get

$$\begin{aligned}\sum_k b(n,k,r) \left|\frac{x_k}{N}\right|^{p_k} &\leqq \frac{1}{N^\theta} \sum_k b(n,k,r) \left|x_k\right|^{p_k} \\ &\leqq \frac{K'}{N_\theta} \leqq \delta.\end{aligned}$$

Therefore by taking supremum over n and r we get,

$$\{x : g(x) \leqq K'\} \subseteq N \{x : g(x) \leqq \delta\}.$$

For every $\delta > 0$ there exists $N > 1$ for which the above inclusion holds and so

$$\{x : g(x) \leqq K'\}$$

is bounded. This completes the proof. □

It is known that every locally bounded linear topological space is $s-$ convex for some s such that $0 < s \leqq 1$. But the following theorem gives exact conditions for $s-$convexity.

Theorem 2.2. *Let $0 < p_k \leqq 1$. Then $[B_\theta, p]_0$ and $[B_\theta, p]_\infty$ are $s-$convex for all s, where $0 < s < \liminf p_k$. Moreover, if $p_k = p \leqq 1$ for all k, then they are $p-$convex. $[B_\theta, p]$ has the same properties if (1) holds.*

Proof. We shall prove the theorem only for $[B_\theta, p]_\infty$. Let $[B_\theta, p]_\infty$ and $s \in (0, \liminf p_k)$. Then there exists k_0 such that $s \leqq p_k$ for all $k > k_0$. Now define

$$\hat{g}(x) = \sup_{r,n} \left[\sum_{k=1}^{k_0} b(n,k,r) |x_k|^s + \sum_{k=k_0+1}^{\infty} b(n,k,r) |x_k|^{p_k} \right].$$

Since $s \leqq p_k \leqq 1$ for all $k > k_0$, $\hat{g}$ is subadditive. Further, for $0 < |\lambda| \leqq 1$,

$$|\lambda|^{p_k} \leqq |\lambda|^s \text{ for all } k > k_0$$

Therefore for such λ we have

$$\hat{g}(\lambda x) \leqq |\lambda|^s \hat{g}(x).$$

Now for $0 < \delta < 1$,

$$U = \{x : \hat{g}(x) \leqq \delta\}$$

is an absolutely $s-$convex set, for $|\lambda|^s + |\mu|^s \leqq 1$ and $x, y \in U$ imply that

$$\begin{aligned} \hat{g}(\lambda x + \mu y) &\leqq \hat{g}(\lambda x) + \hat{g}(\mu y) \leqq |\lambda|^s \hat{g}(x) + |\mu|^s \hat{g}(y) \\ &\leqq \left(|\lambda|^s + |\mu|^s\right) \delta \leqq \delta. \end{aligned}$$

If $p_k = p$ for all k, then for $0 < \delta < 1$,

$$V = \{x : g(x) \leqq \delta\}$$

is an absolutely $p-$convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof. □

3. Some Further Results

Let X and Y be two nonempty subsets of the space w of sequences. If $x = (x_k) \in X$ implies that $\{\sum_k b_{nk} x_k\} \in Y$, we say that B defines a (matrix) transformation from X into Y, and we write $B : X \to Y$. $(X, Y.)$ denotes the class of matrices B such that $B : X \to Y$. Let c_0 and $(\hat{N}_\theta)_0$ respectively denote the linear spaces of null sequences and sequences lacunary almost convergent to zero.

We now characterize the class of strongly lacunary almost regular matrices.

To prove the next theorem we need the following lemma.

Lemma 3.1. ([13, p. 347]) *If $p_k, q_k > 0$, then $c_0(q) \subset c_0(p) \Leftrightarrow \liminf \frac{p_k}{q_k} > 0$.*

Theorem 3.2. *Let $0 < \eta \leqq p_k \leqq H < \infty$. Then (B, p) is strongly lacunary almost regular if and only if $B \in \left(c_0, (\hat{N}_\theta)_0\right)$, where*

$$(\hat{N}_\theta)_0 = \left\{ x : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} x_{n+i} = 0, \text{ uniformly in } n \right\}.$$

Remark 3.3. Necessary and sufficient conditions for $B \in \left(c_0, (\hat{N}_\theta)_0\right)$ are given by Nuray [16].

1. $\|B\| < \infty$;
2. $\lim_{n\to\infty} b(n,k,r) = 0$ uniformly in n, for all k.

Proof. Necessity. Suppose that (B, p) is strongly lacunary almost regular. Therefore

$$|x_k - l|^{1/p_k} \to 0 \Rightarrow \sum_k b(n,k,r)\,|x_k - l| \to 0$$

uniformly in n. $1/p_k \geqq \frac{1}{H} > 0$, by Lemma 3.1,

$$x_k \to l \Rightarrow |x_k - l|^{1/p_k} \to 0.$$

Thus

$$x_k \to l \Rightarrow \sum_k b(n,k,r)\,(x_k - l) \to 0$$

uniformly in n and therefore $B \in \left(c_0, (\hat{N}_\theta)_0\right)$.

Sufficiency. Since $p_k \geqq \eta > 0$, by Lemma 3.1,

$$x_k \to l \Rightarrow |x_k - l|^{p_k} \to 0.$$

Again we have $B \in \left(c_0, (\hat{N}_\theta)_0\right)$. Therefore, $x_k \to l\,[B_\theta, p]$ and this concludes the proof. Note that $p_k \leqq H$ superfluous in the sufficiency and $\eta \leqq p_k$ is superfluous in the necessity. □

We next consider the uniqueness of generalized limits.

Theorem 3.4. *Suppose that $B \in \left(c_0, (\hat{N}_\theta)_0\right)$ and $p = (p_k)$ converges to a positive limit. Then $x = \{x_k\} \to l \Rightarrow x_k \to l\,[B_\theta, p]$ uniquely if and only*

$$\sum_k b(n,k,r) \nrightarrow 0 \text{ uniformly in } n. \tag{3}$$

Proof. Necessity. Suppose that $B \in \left(c_0, (\hat{N}_\theta)_0\right)$ and (p_k) are bounded. Let $x_k \to l$ imply that $x_k \to l\,[B_\theta, p]$ uniquely. We have $e \to 1\,[B_\theta, p]$. Therefore the condition (3) must hold. For, otherwise $e \to 0\,[B_\theta, p]$ which contradicts the uniqueness of l.

Note that the restriction on $\{p_k\}$ (except boundedness) is superfluous for the necessity.

Sufficiency. Suppose that the condition (3) holds and $B \in \left(c_0, (\hat{N}_\theta)_0\right)$ and that $p_k \to r > 0$. Further assume that $x_k \to l$ imply that $x_k \to l\,[B_\theta, p]$ and $x_k \to \acute{l}\,[B_\theta, p]$ where $\left|l - \acute{l}\right| = a > 0$.Then we get

$$\lim_{n\to\infty} \sum_k b(n,k,r) u_k = 0 \text{ (uniformly in } n), \tag{4}$$

where

$$u_k = |x_k - l|^{p_k} + \left|x_k - \acute{l}\right|^{p_k}.$$

By the assumption we have $u_k \to a^t$. Since $A \in \left(c_0, (\hat{N}_\theta)_0\right)$, $u_k \to a^t$ implies that

$$\sum_k b(n,k,r)\left|u_k - a^t\right| \to 0 \text{ (uniformly in } n) \tag{5}$$

But we have

$$a^t \sum_k b(n,k,r) \quad \leqq \quad \sum_k b(n,k,r) u_k + \sum_k b(n,k,r)\left|u_k - a^t\right|. \tag{6}$$

Now by (4), (5) and (6) it follows that

$$\lim_{n\to\infty} \sum_k b(n,k,r) = 0 \text{ (uniformly in } n)$$

Since this contradicts (3), we must have $l = \acute{l}$. This completes the proof. □

Suppose that $0 < p_k \leqq q_k$. We conclude this note by showing that $[B_\theta, q] \subset [B_\theta, p]$ is not true in general. However the inclusion holds for a special class. We prove

Theorem 3.5. *Suppose that $\|B\| < \infty$ and $\frac{q_k}{p_k}$ is bounded, then $[B_\theta, q] \subset [B_\theta, p]$.*

Proof. Write $w_k = |x_k - l|^{q_k}$ and $p_k/q_k = \lambda_k$. So that $0 < \lambda \leqq \lambda_k \leqq 1$ (λ constant). Let $x \in [B_\theta, q]$. Then

$$\sum_k b(n,k,r)w_k \to 0 \text{ (uniformly in } n)$$

Define $u_k = w_k$ $(w_k \geqq 1)$, $= 0$ $(w_k < 1)$ and $v_k = 0$ $(w_k \geqq 1)$, $= w_k$ $(w_k < 1)$. So that $w_k = u_k + v_k$, $w_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that $u_k^{\lambda_k} \leqq u_k \leqq w_k$, $v_k^{\lambda_k} < v_k^{\lambda}$. We have the inequality [see Maddox [13], p. 351].

$$\sum_k b(n,k,r)w_k^{\lambda_k} \leqq \sum_k b(n,k,r)w_k + \left(\sum_k b(n,k,r)v_k\right)^{\lambda} \|B\|^{1-\lambda} .$$

Therefore $x \in [B_\theta, p]$ and this completes the proof. □

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