Filomat 33:5 (2019), 1403–1415
https://doi.org/10.2298/FIL1905403N

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized q-Laguerre Type Polynomials and q-Partial Differential Equations

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Abstract. In this paper we define the q-Laguerre type polynomials $U_n(x,y,z;q)$, which include q-Laguerre polynomials, generalized Stieltjes-Wigert polynomials, little q-Laguerre polynomials and q-Hermite polynomials as special cases. We also establish a generalized q-differential operator, with which we build the relations between analytic functions and $U_n(x,y,z;q)$ by using certain q-partial differential equations. Therefore, the corresponding conclusions about q-Laguerre polynomials, little q-Laguerre polynomials and q-Hermite polynomials are gained as corollaries. As applications, some generating functions and generalized Andrews-Askey integral formulas are given in the final section.

1. Introduction

The explicit form of q-Laguerre polynomials are

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^k\frac{q^{k^2+k\alpha}}{(q^{\alpha+1};q)_k}x^k,\alpha>-1. \tag{1}$$

q-Laguerre polynomials are a family of basic hypergeometric orthogonal polynomials in the basic Askey scheme [24, 33]. More detailed researches can be found in the papers [6, 14, 15, 17, 18, 22–25, 32, 33].

The little q-Laguerre (or Wall) polynomials are

$$p_n(x,a;q)={}_2\phi_1(q^{-n},0;aq;q,qx)=\frac{(-1)^nq^{-\binom{n}{2}}}{(aq;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}\frac{(aq;q)_n}{(aq;q)_{n-k}}x^{n-k},a\neq q^{-1},q^{-2},\cdots, \tag{2}$$

where ${}_r\phi_s$ are the basic hypergeometric series [19, Eq. (1.2.22)] defined by

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\b_1,b_2,\ldots,b_s\end{matrix};q,z\right)=\sum_{n=0}^{\infty}\frac{\left(a_1,a_2,\ldots,a_r;q\right)_n}{\left(q,b_1,b_2,\ldots,b_s;q\right)_n}\left[(-1)^nq^{\binom{n}{2}}\right]^{1+s-r}z^n, \tag{3}$$

2010 *Mathematics Subject Classification*. Primary 05A30; Secondary 11B65, 33D15, 33D45, 39A13
Keywords. q-series, q-derivative, q-partial differential equation, q-Laguerre polynomial, little q-Laguerre polynomial, q-Hahn polynomial, generating function
Received: 09 August 2018; Revised: 05 February 2019; Accepted: 09 February 2019
Communicated by Ljubiša D.R. Kočinac
This work is supported by the National Natural Science Foundation of China (grant number 11571114) and ECNU found for International Education (grant number 40600-20103-512200/002/001)
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which if $0 < |q| < 1$, converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s + 1$.

The q-Hahn (or Al-Salam-Carlitz [4]) polynomials [2, 12] are defined by

$$\phi_n^{(b)}(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (b;q)_k z^k \quad \text{and} \quad \psi_n^{(b)}(z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} (bq^{1-k};q)_k z^k. \tag{4}$$

In [9], Cao introduced a generalized version of (4):

$$\phi_n^{(a,b,c)}(x,y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a;q)_k (b;q)_k}{(c;q)_k} x^k y^{n-k} \tag{5}$$

and

$$\psi_n^{(a,b,c)}(x,y;q) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(a;q)_k (b;q)_k}{(c;q)_k} q^{-nk+\binom{k+1}{2}} x^k y^{n-k}. \tag{6}$$

For nonzero series c_k that are independent of n, we define a class of generalized q-Laguerre type polynomials

$$U_n(x,y,z;q) = \sum_{k=0}^{n} (-1)^k c_k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{rnk-r\binom{k+1}{2}} \frac{(aq;q)_n}{(aq;q)_{n-k}} x^{n-k} y^{n-k} z^k, \quad r \in \mathbb{R},\, a \neq q^{-1}, q^{-2}, \cdots, \tag{7}$$

particularly, we choose

$$c_k = \omega^k \lambda^{\binom{k}{2}} \frac{(\beta,d)_k (\eta,d)_k}{(\gamma;h)_k}, \quad \omega, \lambda, \beta, \eta, \gamma, d, h \in \mathbb{C},\, \gamma \neq 1, h^{-1}, h^{-2}, \cdots, \tag{8}$$

in the rest of the paper. Many konwn polynomials, such as the little q-Laguerre polynomials, q-Hahn polynomials, q-Laguerre polynomials and generalized Stieltjes-Wigert polynomials are special cases of (7).

In fact, taking $r = 0$ and $c_k = q^{\binom{k}{2}}$ in (7) yields generalized little q-Laguerre polynomials

$$\mathcal{P}_n(x,y,z;q) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq;q)_n}{(aq;q)_{n-k}} x^{n-k} y^{n-k} z^k. \tag{9}$$

It is clear that

$$p_n(x,a;q) = \frac{(-1)^n q^{-\binom{n}{2}}}{(aq;q)_n} \mathcal{P}_n(x,1,1;q).$$

Choosing $r = 0$ and $c_k = (-1)^k (b;q)_k$ in (7), we get generalized q-Hahn polynomials

$$\Phi_n^{(a,b)}(x,y,z;q) = \sum_{k=0}^{n} (b;q)_k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq;q)_n}{(aq;q)_{n-k}} x^{n-k} y^{n-k} z^k, \tag{10}$$

which become $\phi_n^{(b)}(z;q)$ in (4) by letting $a = 0$ and $x = y = 1$ in (10).

Set $r = -1$ and $c_k = (-1)^k q^{\binom{k}{2}} (aq^{1-k};q)_k$ in (7) to get

$$\Psi_n^{(a,b)}(x,y,z;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} (bq^{1-k};q)_k \frac{(aq;q)_n}{(aq;q)_{n-k}} x^{n-k} y^{n-k} z^k. \tag{11}$$

Obviously, polynomials (11) reduce to $\psi_n^{(b)}(z;q)$ in (4) by letting $a = 0$ and $x = y = 1$ in (11).

Taking $r = -2$ and $c_k = (bq)^{-k}$ in (7), we get generalized q-Laguerre polynomials

$$\mathcal{L}_n^{(a,b)}(x,y,z;q) = \sum_{k=0}^{n} (-1)^k q^{k^2-2nk} b^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq;q)_n}{(aq;q)_{n-k}} x^{n-k} y^{n-k} z^k. \tag{12}$$

Set $a = b = q^\alpha$ and $y = z = 1$ in (12) to get

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \mathcal{L}_n^{(q^\alpha,q^\alpha)}(x,1,1;q).$$

Set $b = \sqrt{q}$ and $y = z = 1$ in (12) to get the generalized Stieltjes-Wigert polynomials ([19], p. 214)

$$S_n(x;aq;q) = \sum_{k=0}^{n} (-1)^k q^{k^2-2nk-\frac{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq;q)_n}{(aq;q)_{n-k}} x^{n-k}.$$

Letting $r = a = 0$, $c_k = (-1)^k (\alpha;q)_k (\eta;q)_k / (\gamma;q)_k$ and $y = 1$ in (7) gives (5). Choosing $r = -1$, $a = 0$, $c_k = (\alpha;q)_k (\eta;q)_k / (\gamma;q)_k$ and $y = 1$ in (7) yields (6).

In recent years, by using the theory of analytic functions of several complex variables, Liu published a series of papers to prove that if an analytic function in several variables satisfies a system of q-partial (or partial) differential equations, then it can be expanded in terms of certain important polynomials. Many orthogonal polynomials are studied and their applications are obtained, please see [5, pp. 445-461], [26–30] for details. In [7–11], Cao applied Liu's methods of q-partial difference equations to various q-orthogonal polynomials and proved many q-identities and q-integrals.

Liu's method shows its universality when applied to many q-orthogonal polynomials or classical orthogonal polynomials. However we find it hardly be used directly to q-Laguerre and more complicated polynomials. In [34], we introduced a modified q-differential operator and obtained relations between a special form of q-Laguerre polynomials and q-differential equations. In this paper, we define q-Laguerre type polynomials $U_n(x,y,z;q)$ and then the q-Laguerre, little q-Laguerre, q-Hahn (or Al-Salam-Carlitz) polynomials become special cases of $U_n(x,y,z;q)$. By introducing a generalized q-differential operator, using Liu's method, we find that when a analytic function satisfies certain q-partial differential equation with generalized q-differential operator, then it can be expressed in terms of q-Laguerre type polynomials $U_n(x,y,z;q)$. Finally, we obtain generating functions for $U_n(x,y,z;q)$ and generalized Andrews-Askey integral formulas as applications.

The q-differential operators D_x and θ_x ([9]) are defined by

$$D_x\{f(x)\} = \frac{f(x) - f(qx)}{x} \quad \text{and} \quad \theta_x\{f(x)\} = \frac{f(xq^{-1}) - f(x)}{xq^{-1}}. \tag{13}$$

When $f(x)$ is differentiable at x, we have

$$\lim_{q\to 1} \frac{D_x\{f(x)\}}{1-q} = f'(x).$$

We give a more general q-differential operator including both D_x and θ_x as special cases as follows.

Definition 1.1. Let $a > 0$ and r be real number, for any function $f(x)$ of one variable, the Generalized q-derivative of $f(x)$ with respect to x is defined as

$$_{(r,a)}\mathcal{D}_x\{f(x)\} = \begin{cases} \frac{f(xq^r) - af(xq^{r+1})}{xq^r}, & f(x) \text{ is not a constant function,} \\ 0, & f(x) \text{ is a constant function.} \end{cases} \tag{14}$$

We define $_{(r,a)}\mathcal{D}_x^0\{f(x)\} = f(x)$ and $_{(r,a)}\mathcal{D}_x^n\{f\} = {}_{(r,a)}\mathcal{D}_x\{_{(r,a)}\mathcal{D}_x^{n-1}\{f\}\}$.

Remark 1.2. It's obvious that ${}_{(0,1)}\mathcal{D}_x\{f(x)\} = D_x\{f(x)\}$ and ${}_{(-1,1)}\mathcal{D}_x\{f(x)\} = \theta_x\{f(x)\}$.

For the sake of simplicity, we use $\delta_x\{f(x)\} \triangleq {}_{(0,1)}\mathcal{D}_x\{f(x)\}$, $\partial_x\{f(x)\} \triangleq {}_{(r,1)}\mathcal{D}_x\{f(x)\}$, $\tau_{r,x}\{f(x)\} \triangleq {}_{(r,a)}\mathcal{D}_x\{f(x)\}$ for $a > 0$ in the following of this paper.

The q-shift operator $\eta^r_{x_i}$ for a function $f(x_1, x_2, \cdots, x_k)$ is defined by

$$\eta^r_{x_i}\{f(x_1, x_2, \cdots, x_k)\} = f(x_1, x_2, \cdots, x_{i-1}, q^r x_i, x_{i+1}, \cdots, x_k) \quad i = 1, 2, \cdots, k, r \in \mathbb{R}.$$

We now give the q-Leibniz formula for ${}_{(r,a)}\mathcal{D}_x$.

Theorem 1.3. *For positive integer n and $g(x)$ not a constant, we have*

$${}_{(r,a)}\mathcal{D}^n_x\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k q^{(1+r)k(k-n)} \partial^k_x\{f(xq^{rn-rk})\} \cdot {}_{(r,a)}\mathcal{D}^{n-k}_x\{g(xq^{rk+k})\} \tag{15}$$

Remark 1.4. Taking $r = 0$ in (15) yields the q-Leibniz formula obtained in [34]. Setting $a = 1$, $r = 0$ in (15), we get the ordinary q-Leibniz formula ([19], p. 27). Choosing $a = 1, r = -1$ in (15) leads to the q-Leibniz formula for θ_x in [13].

The following lemma 1.5 is useful in the proof of Theorem 1.3.

Lemma 1.5. ([16, 21]) *Let A and B be two linear operators such that $BA = qAB$, then we have*

$$(A + B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^k B^{n-k}.$$

Proof. [Proof of Theorem 1.3] For convenience, the q-shift operator η_x acting on function $f(x)$ is denoted by η_f. The operator ${}_{(r,a)}\mathcal{D}_x$ acting on $f(x)$ is denoted by ${}_{(r,a)}\mathcal{D}_f$, the operator ∂_x acting on $f(x)$ is denoted by ∂_f.

Let $A = {}_{(r,a)}\mathcal{D}_g \eta^r_f$ and $B = a\eta^{r+1}_g \partial_f$, it is easy to verify $q^r \eta^r_f \partial_f = \partial_f \eta^r_f$ and $q^{r+1}\eta^{r+1}_g \cdot {}_{(r,a)}\mathcal{D}_g = {}_{(r,a)}\mathcal{D}_g \eta^{r+1}_g$. Then we have

$$\begin{aligned} BA\{f(x)g(x)\} =& a\eta^{r+1}_g \partial_f \eta^r_f\{f(x)\} \cdot {}_{(r,a)}\mathcal{D}_g\{g(x)\} = a\eta^{r+1}_g q^r \eta^r_f \partial_f\{f(x)\} \cdot {}_{(r,a)}\mathcal{D}_g\{g(x)\} \\ =& aq^r \eta^r_f \partial_f\{f(x)\} q^{-r-1} \cdot {}_{(r,a)}\mathcal{D}_g \eta^{r+1}_g\{g(x)\} = q^{-1} AB\{f(x)g(x)\}. \end{aligned}$$

If $f(x)g(x)$ is not a constant, by Definition 14, we have

$$\begin{aligned} {}_{(r,a)}\mathcal{D}_x\{f(x)g(x)\} =& \frac{f(xq^r)g(xq^r) - af(xq^{r+1})g(xq^{r+1})}{xq^r} \\ =& f(xq^r)\frac{g(xq^r) - ag(q^{1+r}x)}{x} + ag(q^{r+1}x)\frac{f(xq^r) - f(q^{r+1}x)}{xq^r} \\ =& ({}_{(r,a)}\mathcal{D}_g \eta^r_f + a\eta^{r+1}_g \partial_f)\{f(x)g(x)\} \\ =& (A + B)\{f(x)g(x)\}. \end{aligned} \tag{16}$$

If $f(x)g(x)$ is a constant, equation ${}_{(r,a)}\mathcal{D}_x\{f(x)g(x)\} = 0 = (A + B)\{f(x)g(x)\}$ is valid too.

Using Lemma 1.5 and the fact of $\eta_f^{r(n-k)}\partial_f^k = q^{kr(n-k)}\partial_f^k\eta_f^{r(n-k)}$, we have

$$\begin{aligned}
{}_{(r,a)}\mathcal{D}_x^n\{f(x)g(x)\} &=(A+B)^n\{f(x)g(x)\} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} A^{n-k}B^k\{f(x)g(x)\} \\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \eta_f^{r(n-k)} \cdot {}_{(r,a)}\mathcal{D}_g^{n-k} a^k \eta_g^{k(r+1)}\partial_f^k\{f(x)g(x)\} \\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} {}_{(r,a)}\mathcal{D}_g^{n-k}\{g(xq^{k(r+1)})\}a^k q^{kr(k-n)}\partial_f^k\{f(xq^{r(n-k)})\} \\
&= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{q} a^k q^{(1+r)k(k-n)}\partial_x^k\{f(xq^{rn-rk})\} \cdot {}_{(r,a)}\mathcal{D}_x^{n-k}\{g(xq^{rk+k})\}.
\end{aligned}$$

The proof of Theorem 1.3 is completed. □

The next equality (17) ([1]) and two Propositions 1.6 and 1.7 will be used in this paper:

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q (1-q^k) = \begin{bmatrix} \alpha \\ k-1 \end{bmatrix}_q (1-q^{\alpha-k+1}), \quad \alpha \in \mathbb{R}, \tag{17}$$

Proposition 1.6. [Hartogs' theorem [20, p. 15]] *If a complex-valued function is holomorphic (analytic) in each variable separately in an open domain $D \subseteq \mathbb{C}^n$, then it is holomorphic (analytic) in D.*

Proposition 1.7. ([31, p. 5]) *If function $f(x_1, x_2, \cdots, x_k)$ is analytic at origin $(0, 0, \cdots, 0) \in \mathbb{C}^k$, then f can be expanded in an absolutely convergent power series*

$$f(x_1, x_2, \cdots, x_k) = \sum_{n_1,n_2,\cdots,n_k=0}^{\infty} \lambda_{n_1,n_2,\cdots,n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

We have the main theorem based on Proposition 1.7:

Theorem 1.8. *Let $f(x, z)$ be a 2-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then function $f(xy, z)$ can be expanded in terms of $U_n(x, y, z; q)$ with c_k defined by (8) if and only if $f(xy, z)$ satisfies the q-partial differential equation*

$$\delta_z\{f(xy,z) - \gamma h^{-1} f(xy, zh)\} = -\omega\delta_x\tau_{r,y}\{f(xy, \lambda z) - (\beta+\eta) f(xy, \lambda dz) + \beta\eta f(xy, \lambda d^2 z)\}, \quad h \neq 0, \tag{18}$$

or

$$\delta_z\{f(xy,z)\} = -\omega\delta_x\tau_{r,y}\{f(xy, \lambda z) - (\beta+\eta) f(xy, \lambda dz) + \beta\eta f(xy, \lambda d^2 z)\}, \quad h = 0. \tag{19}$$

By taking $r = 0$, $c_k = q^{\binom{k}{2}}$ and $c_k = (-1)^k (b; q)_k$ in Theorem 1.8 respectively, we obtain the next two corollaries.

Corollary 1.9. *Let $f(x, z)$ be a 2-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then $f(xy, z)$ can be expended in terms of $\mathcal{P}_n(x, y, z; q)$ if and only if $f(xy, z)$ satisfies the q-partial differential equation*

$$\delta_z f(xy, z) = -\delta_x\tau_{0,y} f(xy, qz). \tag{20}$$

Corollary 1.10. *Let $f(x, z)$ be a 2-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then $f(xy, z)$ can be expended in terms of $\Phi_n^{(a,b)}(x, y, z; q)$ if and only if $f(xy, z)$ satisfies the q-partial differential equation*

$$\delta_z f(xy, z) = \delta_x\tau_{0,y}\{f(xy, z) - b f(xy, qz)\}. \tag{21}$$

Setting $r = -1$ and $c_k = (-1)^k q^{\binom{k}{2}} (aq^{1-k}; q)_k = a^k (a^{-1}; q)_k$ in (18) implies

Corollary 1.11. *Let $f(x, z)$ be a 2-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then $f(xy, z)$ can be expended in terms of $\Psi_n^{(a,b)}(x, y, z; q)$ if and only if $f(xy, z)$ satisfies the q-partial differential equation*

$$\delta_z f(xy, z) = \delta_x \tau_{-1,y} \{f(xy, qz) - a f(xy, z)\}. \tag{22}$$

Similarly, taking $r = -2$ and $c_k = (bq)^{-k}$ in (18), we have

Corollary 1.12. *Let $f(x, z)$ be a 2-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then $f(xy, z)$ can be expended in terms of $\mathcal{L}_n^{(a,b)}(x, y, z; q)$ if and only if $f(xy, z)$ satisfies the q-partial differential equation*

$$bq\delta_z f(xy, z) = -\delta_x \tau_{-2,y} \{f(xy, z)\}. \tag{23}$$

Taking $a = r = 0$, $y = 1$, $c_k = (-1)^k (\beta; q)_k (\eta; q)_k / (\gamma; q)_k$ and $a = 0$, $r = -1$, $y = 1$, $c_k = (\beta; q)_k (\eta; q)_k / (\gamma; q)_k$ in Theorem 1.8, respectively, we have

Corollary 1.13. *Let $f(x, z)$ be a 2-variable analytic function at $(0, 0) \in \mathbb{C}^2$, then $f(xy, z)$ can be expended in terms of $\phi_n^{(\beta,\eta,\gamma)}(z, xy; q)$ and $\psi_n^{(\beta,\eta,\gamma)}(z, xy; q)$ $(q \neq 0)$, $(q \neq 0)$ defined by* (5) *and* (6) *if and only if $f(xy, z)$ satisfies the q-partial differential equations*

$$\delta_z\{f(xy, z) - \gamma q^{-1} f(xy, zq)\} = \delta_x \delta_y \{f(xy, z) - (\beta + \eta) f(xy, qz) + \beta\eta f(xy, q^2 z)\}$$

and

$$\delta_z\{f(xy, z) - \gamma q^{-1} f(xy, zq)\} = -\delta_x \tau_{-1,y} \{f(xy, z) - (\beta + \eta) f(xy, qz) + \beta\eta f(xy, q^2 z)\},$$

respectively.

Remark 1.14. Corollary 1.13 is equivalent to the main theorem in [7] (Theorem 2) by using Definition 14. T hus Theorem 1.8 generalizes Theorem 2 of the paper[7].

2. The Proof of Theorem 1.8

Proof. Since $f(x, z)$ is analytic function at $(0, 0) \in \mathbb{C}^2$, according to Proposition 1.7, $f(x, z)$ can be expanded in an absolutely convergent series in a neighbourhood of $(0, 0)$, that is, there be series $\mu_{n,k}$ such that

$$f(x, z) = \sum_{n,k=0}^{\infty} \mu_{n,k} x^n z^k,$$

then function $f(xy, z)$ will be expanded as

$$f(xy, z) = \sum_{n,k=0}^{\infty} \mu_{n,k} x^n y^n z^k = \sum_{k=0}^{\infty} z^k \sum_{n=0}^{\infty} \mu_{n,k} x^n y^n. \tag{24}$$

If $h \neq 0$ in c_k, substituting (24) into equation (18) results in

$$\delta_z \Big\{ \sum_{n,k=0}^{\infty} (1 - \gamma h^{k-1}) \mu_{n,k} x^n y^n z^k \Big\} = -\omega \delta_x \tau_{r,y} \Big\{ \sum_{n,k=0}^{\infty} \lambda^k \left[1 - (\beta + \eta) d^k + \beta\eta d^{2k}\right] \mu_{n,k} x^n y^n z^k \Big\}.$$

That is

$$\sum_{k=1}^{\infty} (1 - \gamma h^{k-1})(1 - q^k) z^{k-1} \sum_{n=0}^{\infty} \mu_{n,k} x^n y^n = -\delta_x \tau_{r,y} \sum_{k=1}^{\infty} \omega \lambda^{k-1} (1 - \beta d^{k-1})(1 - \eta d^{k-1}) z^{k-1} \sum_{n=0}^{\infty} \mu_{n,k-1} x^n y^n. \tag{25}$$

If $h = 0$ in c_k, substituting (24) into equation (19) yields

$$\delta_z\Big\{\sum_{n,k=0}^{\infty}\mu_{n,k}x^ny^nz^k\Big\} = -\omega\delta_x\tau_{r,y}\Big\{\sum_{n,k=0}^{\infty}\lambda^k\left[1-(\beta+\eta)d^k+\beta\eta d^{2k}\right]\mu_{n,k}x^ny^nz^k\Big\}.$$

That is

$$\sum_{k=1}^{\infty}(1-q^k)z^{k-1}\sum_{n=0}^{\infty}\mu_{n,k}x^ny^n = -\delta_x\tau_{r,y}\sum_{k=1}^{\infty}\omega\lambda^{k-1}(1-\beta d^{k-1})(1-\eta d^{k-1})z^{k-1}\sum_{n=0}^{\infty}\mu_{n,k-1}x^ny^n. \tag{26}$$

Comparing the coefficients of z^{k-1} in (25) or (26), we always have

$$\sum_{n=0}^{\infty}\mu_{n,k}x^ny^n = -\frac{c_k}{(1-q^k)c_{k-1}}\delta_x\tau_{r,y}\sum_{n=0}^{\infty}\mu_{n,k-1}x^ny^n.$$

Iterating this relation $k-1$ times, we obtain

$$\sum_{n=0}^{\infty}\mu_{n,k}x^ny^n = (-1)^k\frac{c_k}{(q;q)_k}\delta_x^k\tau_{r,y}^k\sum_{n=0}^{\infty}\mu_{n,0}x^ny^n.$$

By formula (24) we get

$$\begin{aligned}f(xy,z) &= \sum_{k=0}^{\infty}z^k\sum_{n=0}^{\infty}\mu_{n,k}x^ny^n = \sum_{k=0}^{\infty}(-1)^k\frac{c_k}{(q;q)_k}z^k\sum_{n=k}^{\infty}\mu_{n,0}q^{rnk-r\binom{k+1}{2}}\frac{(q;q)_n(aq;q)_n}{(q;q)_{n-k}(aq;q)_{n-k}}x^{n-k}y^{n-k}\\ &= \sum_{n=0}^{\infty}\mu_{n,0}\sum_{k=0}^{\infty}(-1)^kc_k\begin{bmatrix}n\\k\end{bmatrix}_q q^{rnk-r\binom{k+1}{2}}\frac{(aq;q)_n}{(aq;q)_{n-k}}x^{n-k}y^{n-k}z^k = \sum_{n=0}^{\infty}\mu_{n,0}U_n(x,y,z;q).\end{aligned}$$

On the other hand, we prove that if $f(xy,z)$ can be expanded in terms of $U_n(x,y,z;q)$, then $f(xy,z)$ satisfies Equation (18), the proof of case $k=0$ is omitted since it is similar to that of $k\neq 0$.

Assume that

$$\begin{aligned}f(xy,z) &= \sum_{n=0}^{\infty}\mu_nU_n(x,y,z;q)\\ &= \sum_{n=0}^{\infty}\mu_n\sum_{k=0}^{n}(-1)^kc_k\begin{bmatrix}n\\k\end{bmatrix}_q q^{rnk-r\binom{k+1}{2}}\frac{(aq;q)_n}{(aq;q)_{n-k}}x^{n-k}y^{n-k}z^k.\end{aligned}$$

The right hand side of (18)

$$\begin{aligned}&-\omega\delta_x\tau_{r,y}\{f(xy,\lambda z)-(\eta+\beta)f(xy,\lambda dz)+\eta\beta f(xy,\lambda d^2z)\}\\ &= -\omega\delta_x\tau_{r,y}\Big\{\sum_{n=0}^{\infty}\mu_n\sum_{k=0}^{n}\lambda^k\left[1-(\eta+\beta)d^k+\eta\beta d^{2k}\right](-1)^kc_k\begin{bmatrix}n\\k\end{bmatrix}_q q^{rnk-r\binom{k+1}{2}}\frac{(aq;q)_n}{(aq;q)_{n-k}}x^{n-k}y^{n-k}z^k\Big\}\\ &= \sum_{n=0}^{\infty}\mu_n\sum_{k=0}^{n-1}(-1)^{k+1}c_{k+1}(1-\gamma d^k)\begin{bmatrix}n\\k\end{bmatrix}_q(1-q^{n-k})q^{rn(k+1)-r\binom{k+1}{2}-rk}\frac{(aq;q)_n}{(aq;q)_{n-k-1}}x^{n-k-1}y^{n-k-1}z^k\\ &= \sum_{n=0}^{\infty}\mu_n\sum_{k=0}^{n-1}(-1)^{k+1}(1-q^{k+1})(1-\gamma d^k)c_{k+1}\begin{bmatrix}n\\k+1\end{bmatrix}_q q^{rn(k+1)-r\binom{k+2}{2}}\frac{(aq;q)_n}{(aq;q)_{n-k-1}}x^{n-k-1}y^{n-k-1}z^k\\ &= \sum_{n=0}^{\infty}\mu_n\sum_{k=1}^{n}(-1)^k(1-q^k)(1-\gamma d^{k-1})c_k\begin{bmatrix}n\\k\end{bmatrix}_q q^{rnk-r\binom{k+1}{2}}\frac{(aq;q)_n}{(aq;q)_{n-k}}x^{n-k}y^{n-k}z^{k-1}\\ &= \delta_z\{f(xy,z)-\gamma d^{-1}f(xy,dz)\}\end{aligned}$$

where Equation (17) is used in the third equality. We deduced that $f(xy,z)$ satisfies Equation (18), therefore we complete the proof of Theorem 1.8. □

3. Generating Functions for Some Polynomials

As application of Theorem 1.8, we give the generating function for $U_n(x,y,z;q)$, which includes the generating functions for several polynomials mentioned above as special cases.

Theorem 3.1. *For* $\max\{|z|, \left|xy\right|, 1+r\} \le 1$, $\lim_{n\to\infty}|c_{n+1}t/c_n| < 1$, *we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{-r\binom{n}{2}} U_n(x,y,z;q)t^n}{(q;q)_n(aq;q)_n} = \sum_{n=0}^{\infty} \frac{c_n(tz)^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^k q^{-r\binom{k}{2}}(xyt)^k}{(q;q)_k(aq;q)_k}, \quad |t|<1 \text{ when } r=0, \tag{27}$$

where $U_n(x,y,z;q)$ *is defined by* (7).

Proof. We use Theorem 1.8 to prove Equation (27). Let

$$f(x,z) = \sum_{n=0}^{\infty} \frac{c_n(tz)^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^k q^{-r\binom{k}{2}}(xt)^k}{(q;q)_k(aq;q)_k}, \tag{28}$$

we first verify $f(x,z)$ is analytic at $(0,0)$.

Use $|z|<1$ to get

$$\left|\frac{c_n(tz)^n}{(q;q)_n}\right| \le \left|\frac{c_n t^n}{(q;q)_n}\right|.$$

By ratio test, $\sum_{n=0}^{\infty} c_n t^n/(q;q)_n$ is converging since $\lim_{n\to\infty}|c_{n+1}t|/c_n < 1$, thus $\sum_{n=0}^{\infty}\left|c_n(tz)^n/(q;q)_n\right|$ converges uniformly respect to z and then is analytic.

On the other hand, we have

$$\left|\frac{(-1)^k q^{-r\binom{k}{2}}(xyt)^k}{(q;q)_k(aq;q)_k}\right| \le \left|\frac{q^{-r\binom{k}{2}}t^k}{(q;q)_k(aq;q)_k}\right|,$$

when $r<0$, by ratio test

$$\sum_{n=0}^{\infty}\left|\frac{q^{-r\binom{k}{2}}t^k}{(q;q)_k(aq;q)_k}\right|$$

is always converging.

When $r=0$,

$$\sum_{n=0}^{\infty}\left|\frac{t^k}{(q;q)_k(aq;q)_k}\right|$$

is converging since $|t|<1$.

We conclude that

$$\sum_{n=0}^{\infty}\left|\frac{(-1)^k q^{-r\binom{k}{2}}(xyt)^k}{(q;q)_k(aq;q)_k}\right|$$

converges uniformly respect to x when $r=0$ and $|t|<1$ or $r=0$ and thus is analytic. By Hartogs' theorem 1.6, function $f(x,z)$ is analytic at $(0,0)$.

Next we check that $f(xy,z)$ satisfies Equation (18), we just proof the case of $h \neq 0$ and omit the proof of case $h = 0$ since it's easy to verify (19).

$$\begin{aligned}
&-\omega\delta_x\tau_{r,y}\{f(xy,\lambda z)-(\beta+\eta)f(xy,\lambda dz)+\beta\eta f(xy,\lambda d^2z)\}\\
&=-\omega\delta_x\tau_{r,y}\left\{\sum_{n=0}^{\infty}\frac{c_n(tz)^n\lambda^n[1-(\beta+\eta)d^n+\beta\eta d^{2n}]}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(-1)^kq^{-r\binom{k}{2}}(xyt)^k}{(aq;q)_n(q;q)_n)}\right\}\\
&=-\sum_{n=0}^{\infty}\frac{\omega^{n+1}\lambda^{\binom{n}{2}+n}(\beta;d)_{n+1}(\eta;d)_{n+1}(tz)^n}{(q;q)_n}\sum_{k=1}^{\infty}\frac{(-1)^kq^{-r\binom{k}{2}+rk-r}(xy)^{k-1}t^k}{(aq;q)_{k-1}(q;q)_{k-1}}\\
&=\sum_{n=0}^{\infty}\frac{c_{n+1}(1-\gamma h^n)z^nt^{n+1}}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(-1)^kq^{-r\binom{k}{2}}(xyt)^k}{(aq;q)_k(q;q)_k}=\delta_z\{f(xy,z)-\gamma h^{-1}f(xy,hz)\}.
\end{aligned}$$

By Theorem 1.8, there exists a μ_n such that

$$f(xy,z)=\sum_{n=0}^{\infty}\mu_nU_n(x,y,z;q). \tag{29}$$

Taking $z = 0$ on both sides of (29) and noticing $U(x,y,0;q) = x^ny^n$ yield

$$\sum_{k=0}^{\infty}\frac{(-1)^kq^{-r\binom{k}{2}}(xyt)^k}{(aq;q)_k(q;q)_k}=\sum_{k=0}^{\infty}\mu_nx^ny^n.$$

Equating the coefficients of x^ny^n, we obtain

$$\mu_n=\frac{(-1)^nq^{-r\binom{n}{2}}t^n}{(aq;q)_n(q;q)_n}.$$

Substitute μ_n into Equation (29) to end the proof of Theorem 3.1. □

By letting $c_n = q^{\binom{n}{2}}$, $c_n = (-1)^n(b;q)_n$, $c_n = a^n(a^{-1};q)_n$, $c_n = (bq)^{-n}$ in Equation (3.1) respectively, we get

Corollary 3.2. *Let* $\max\{|z|,|xy|\} \leq 1$, *we have*

$$\sum_{n=0}^{\infty}\frac{\mathcal{P}_n(x,y,z;q)t^n}{(q;q)_n(aq;q)_n}=(zt;q)_{\infty}\,{}_2\phi_1\left(\begin{matrix}0,0\\aq\end{matrix};q,xyt\right),\quad |t|<1. \tag{30}$$

$$\sum_{n=0}^{\infty}\frac{\Phi_n^{(a,b)}(x,y,z;q)t^n}{(q;q)_n(aq;q)_n}=\frac{(bzt;q)_{\infty}}{(zt;q)_{\infty}}\,{}_2\phi_1\left(\begin{matrix}0,0\\aq\end{matrix};q,xyt\right),\quad |t|<1. \tag{31}$$

$$\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{\Psi_n^{(a,b)}(x,y,z;q)t^n}{(q;q)_n(aq;q)_n}=\frac{(zt;q)_{\infty}}{(azt;q)_{\infty}}\,{}_1\phi_1\left(\begin{matrix}0\\aq\end{matrix};q,xyt\right),\quad |at|<1. \tag{32}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{2\binom{n}{2}}\mathcal{L}_n^{(a,b)}(x,y,z;q)t^n}{(q;q)_n(aq;q)_n}=\frac{1}{(zt/(bq);q)_{\infty}}\,{}_0\phi_1\left(\begin{matrix}-\\aq\end{matrix};q,-xyt\right),\quad \left|tb^{-1}q^{-1}\right|<1. \tag{33}$$

Use (30) and (31) in Corollary 3.1 to get

Corollary 3.3. *For* $\max\{|z|, |xy|\} \le 1$,

$$\frac{1}{(bzt;q)_\infty}\sum_{n=0}^{\infty}\frac{\mathcal{P}_n(x,y,z;q)t^n}{(q;q)_n(aq;q)_n}\sum_{m=0}^{\infty}\frac{\Phi_m^{(a,b)}(x,y,z;q)t^m}{(q;q)_m(aq;q)_m}=\left[{}_2\phi_1\left(\begin{matrix}0,0\\aq\end{matrix};q,xyt\right)\right]^2,\quad |t|<1. \tag{34}$$

Another generating function for $\mathcal{L}_n^{(a,b)}(x,y,z;q)$ is

Theorem 3.4. *Let* $\max\left\{|xytbq|, |zt|\right\} < 1$, *we have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2} b^n (s;q)_n \mathcal{L}_n^{(a,b)}(x,y,z;q)t^n}{(q;q)_n(aq;q)_n}=\frac{(szt;q)_\infty}{(zt;q)_\infty}{}_1\phi_2\left(\begin{matrix}s\\aq,szt\end{matrix};q,-xytbq\right). \tag{35}$$

Proof. Let

$$f(x,z)=\frac{(szt;q)_\infty}{(zt;q)_\infty}{}_1\phi_2\left(\begin{matrix}s\\aq,szt\end{matrix};q,-xtbq\right). \tag{36}$$

It is easy to verify that $f(x,z)$ is analytic at $(0,0)$. We check that $f(xy,z)$ satisfies Equation (23):

$$\begin{aligned}
bq\delta_z f(xy,z) =& aq\delta_z\left\{\frac{(szt;q)_\infty}{(zt;q)_\infty}\right\}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2\binom{n}{2}}(s;q)_k(xytbq)^k}{(aq;q)_k(q;q)_k(sztq;q)_k}\\
&+bq\frac{(szt;q)_\infty}{(zt;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2\binom{n}{2}}(s;q)_k st(1-q^k)(xytbq)^k}{(aq;q)_k(q;q)_k(szt;q)_{k+1}}\\
=&\frac{(szt;q)_\infty}{(zt;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2\binom{n}{2}}(s;q)_k(xyt)^k(bq)^{k+1}t(1-sq^k)}{(aq;q)_k(q;q)_k(szt;q)_{k+1}}\\
=&\frac{(szt;q)_\infty}{(zt;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{2\binom{n}{2}}(s;q)_{k+1}(xy)^k(tbq)^{k+1}}{(aq;q)_k(q;q)_k(szt;q)_{k+1}}\\
=&-\delta_x\tau_{-2,y}\{f(xy,z)\},
\end{aligned}$$

where the formula

$$\delta_x\{u(x)v(x)\}=\delta_x\{u(x)\}v(qx)+u(x)\delta_x\{v(x)\}$$

for functions $u(x)$ and $v(x)$ is used in the first equation. By Theorem 1.12, there must be a μ_n such that

$$f(xy,z)=\sum_{n=0}^{\infty}\mu_n\mathcal{L}_n^{(a,b)}(x,y,z;q). \tag{37}$$

Setting $z=0$ in Equation (37), notice that

$$f(xy,0)=\sum_{k=0}^{\infty}\frac{(-1)^k q^{2\binom{k}{2}}(s;q)_k(xytbq)^k}{(aq;q)_k(q;q)_k},$$

by (36) and $\mathcal{L}_n^{(a,b)}(x,y,0;q)=x^n y^n$, we have

$$\mu_n=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2\binom{n}{2}}(s;q)_n(tbq)^n}{(aq;q)_n(q;q)_n(szt;q)_n}$$

by equating the coefficients of $x^n y^n$ on both sides of (37). Substituting μ_n into (37) yields (35). □

Remark 3.5. Taking $s=0$, $t\to t/(qb)$ in (35) yields (33).

4. A Generalized Andrews–Askey Integral Formula

Recall the definition of Jackson q-integral ([19], p. 23)

$$\int_a^b f(x)d_qx = (1-q)\sum_{n=0}^{\infty}[bf(bq^n) - af(aq^n)]q^n. \tag{38}$$

The Andrews-Askey integral formula states ([3], Theorem 1)

$$\int_u^v \frac{(qx/u, qx/v;q)_\infty}{(cx,dx;q)_\infty} d_qx = \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}, \tag{39}$$

based on which, the following integral formula was obtained in [35] by using the q-Leibniz rule.

Proposition 4.1. *If there are no zero factors in the denominator of the q-integral, then we have*

$$\int_u^v \frac{x^n(qx/u, qx/v;q)_\infty}{(cx,dx;q)_\infty} d_qx = \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty}\phi_n^{(\zeta,\xi,\rho)}(u,v;q)$$

where

$$\phi_n^{(\zeta,\xi,\rho)}(u,v;q) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{(\zeta,\xi;q)_i}{(\rho;q)_i} u^i v^{n-i}$$

is defined by (5) *and* $\zeta = cv$, $\xi = dv$, $\rho = cduv$.

In this section we introduce a generalized Andrews-Askey integral formula with $U_n(x,y,z;q)$ involved. The proof of this formula can be given by using Theorem 18.

Theorem 4.2. *If there are no zero factors in the denominator of the q-integral, let* $\max_{u\le x\le v}\{|x|\} = M$, $\max\{|w|, |st|, 1+r\} \le 1$, $\lim_{n\to\infty}|c_{n+1}M/c_n| < 1$ *and* $|M| < 1$ *when* $r = 0$, *then we have*

$$\int_u^v \frac{(qx/u, qx/v;q)_\infty G(s,w)}{(cx,dx;q)_\infty} d_qx = F(c,d,u,v)\sum_{n=0}^{\infty}\frac{(-1)^n q^{-r\binom{n}{2}}\phi_n^{(\zeta,\xi,\rho)}(u,v;q)U_n(s,t,w;q)}{(q,aq;q)_n}, \tag{40}$$

where $U_n(s,t,w;q)$ *is defined by* (7) *and*

$$G(s,w) = \sum_{n=0}^{\infty}\frac{c_n(xw)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-r\binom{k}{2}}(stx)^k}{(q;q)_k(aq;q)_k}, \quad F(c,d,u,v) = \frac{(1-q)v(q,u/v,qv/u,cduv;q)_\infty}{(cu,cv,du,dv;q)_\infty},$$

polynomial $\phi_n^{(\zeta,\xi,\rho)}(u,v;q)$ *is defined by* (5) *and* $\zeta = cv$, $\xi = dv$, $\rho = cduv$.

Proof. Let

$$f(s,w) = \int_u^v \frac{(qx/u, qx/v;q)_\infty G(s,w)}{(cx,dx;q)_\infty} d_qx,$$

we can verify that $f(s,w)$ is analytic at $(0,0)$, and $f(st,w)$ satisfies equation

$$\begin{aligned}
&-\omega\delta_w\tau_{r,t}\{f(st,\lambda w) - (\beta+\eta)f(st,\lambda dw) + \beta\eta f(st,\lambda d^2w)\\
&= \int_u^v \frac{(qx/u, qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{c_n x^{n+1}\omega^{n+1}\lambda^n}{(q;q)_n}(1-\beta d^n)(1-\eta d^n)\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\gamma\binom{k}{2}})(stx)^k}{(q;q)_k(aq;q)_k}d_qx\\
&= \int_u^v \frac{x(qx/u, qx/v;q)_\infty}{(cx,dx;q)_\infty}\sum_{n=0}^{\infty}\frac{c_{n+1}x^n w^n(1-\gamma h^n)}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\gamma\binom{k}{2}}(stx)^k}{(q;q)_k(aq;q)_k}d_qx\\
&=\delta_w\{f(st,w) - \gamma h^{-1}f(st,hw)\}.
\end{aligned}$$

By Theorem 3.2, there exists a sequence μ_n such that

$$f(st,w)=\sum_{n=0}^{\infty}\mu_n U_n(s,t,w;q). \tag{41}$$

Set $w=0$ and use the fact of $U_n(s,t,0;q)=s^nt^n$ to get

$$\sum_{n=0}^{\infty}\int_u^v \frac{x^n(qx/u,qx/v;q)_\infty}{(cx,dx;q)_\infty}d_qx\cdot\frac{(-1)^nq^{-r\binom{n}{2}}(st)^n}{(q;q)_n(aq;q)_n}=\sum_{n=0}^{\infty}\mu_n s^nt^n \tag{42}$$

Equating the coefficients of s^nt^n on both sides of Equation (42) and using the Proposition 4.1, we get

$$\mu_n=F(c,d,u,v)\frac{(-1)^nq^{-r\binom{n}{2}}\phi_n^{(\zeta,\xi,\rho)}(u,v;q)}{(q,aq;q)_n}.$$

Substitute μ_n into (41) to obtain (40). We complete the proof of Theorem 4.2. □

Setting $c_n=q^{\binom{n}{2}}$, $c_n=(-1)^n(b;q)_n$, $c_n=a^n(a^{-1};q)_n$, $c_n=(bq)^{-n}$ in Theorem 4.2 respectively yields

Corollary 4.3. *If there are no zero factors in the denominator of the q-integral, let* $\max_{u\le x\le v}\{|x|\}=M$, $F(c,d,u,v)$ *and* $\phi_n^{(\zeta,\xi,\rho)}(u,v;q)$ *are defined as in Theorem 4.2, If* $\max\{|w|,|st|\}\le 1$,*then we have*

$$\int_u^v\frac{(qx/u,qx/v,wx;q)_\infty G_1(s,t,x)}{(cx,dx;q)_\infty}d_qx=F(c,d,u,v)\sum_{n=0}^{\infty}\frac{\phi_n^{(\zeta,\xi,\rho)}(u,v;q)\mathcal{P}_n(s,t,w;q)}{(q,aq;q)_n},\ M<1, \tag{43}$$

$$\int_u^v\frac{(qx/u,qx/v,bwx;q)_\infty G_1(s,t,x)}{(cx,dx,wx;q)_\infty}d_qx=F(c,d,u,v)\sum_{n=0}^{\infty}\frac{\phi_n^{(\zeta,\xi,\rho)}(u,v;q)\Phi_n^{(a,b)}(s,t,w;q)}{(q,aq;q)_n},\ M<1, \tag{44}$$

$$\int_u^v\frac{(qx/u,qx/v,wx;q)_\infty G_2(s,t,x)}{(cx,dx,awx;q)_\infty}d_qx=F(c,d,u,v)\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}\phi_n^{(\zeta,\xi,\rho)}(u,v;q)\Psi_n^{(a,b)}(s,t,w;q)}{(q,aq;q)_n},\ |aM|<1, \tag{45}$$

$$\int_u^v\frac{(qx/u,qx/v;q)_\infty G_3(s,t,x)}{(cx,dx,wx/(bq);q)_\infty}d_qx=F(c,d,u,v)\sum_{n=0}^{\infty}\frac{(-1)^nq^{2\binom{n}{2}}\phi_n^{(\zeta,\xi,\rho)}(u,v;q)\mathcal{L}_n^{(a,b)}(s,t,w;q)}{(q,aq;q)_n},\ \left|b^{-1}q^{-1}M\right|<1, \tag{46}$$

where

$$G_1(s,t,x)={}_2\phi_1\left(\begin{matrix}0,0\\ aq\end{matrix};q,stx\right),\ G_2(s,t,x)={}_1\phi_1\left(\begin{matrix}0\\ aq\end{matrix};q,stx\right),\ G_3(s,t,x)={}_0\phi_1\left(\begin{matrix}-\\ aq\end{matrix};q,-stx\right).$$

Acknowledgements

The author would like to thank Professor G.E. Andrews for his valuable suggestions. This paper was completed during the author's visit to the Department of Mathematics, Pennsylvania State University under the support of ECNU found for International Education.

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