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Best Approximation of Holomorphic Functions from Hardy Space in Terms of Taylor Coefficients

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Abstract. We describe the set of holomorphic functions from the Hardy space H^q, $1 \le q \le \infty$, for which the best polynomial approximation $E_n(f)_q$ is equal to $|f^{(n)}(0)|/n!$.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and let dm be a normalized Lebesgue measure on $\mathbb{T}$. The Hardy space H^q for $1 \le q \le \infty$ is the class of holomorphic in the $\mathbb{D}$ functions f satisfied $\|f\|_q < \infty$, where

$$\|f\|_q := \begin{cases} \sup\limits_{\rho\in(0,1)} \left(\displaystyle\int_{\mathbb{T}} |f(\rho t)|^q dm(t) \right)^{1/q}, & 1 \le q < \infty, \\ \sup\limits_{z\in\mathbb{D}} |f(z)|, & q = \infty. \end{cases}$$

The best approximation of $f \in H^q$ is the quantity

$$E_n(f)_q := \inf_{P_{n-1}\in\mathcal{P}_{n-1}} \|f - P_{n-1}\|_q, \quad n \in \mathbb{N},$$

where $\mathcal{P}_{n-1}$ is the set of all algebraic polynomials of degree at most $n-1$.

The polynomial P^*_{n-1} satisfied $\|f - P^*_{n-1}\|_q = E_n(f)_q$ is called a best approximation to f among the set $\mathcal{P}_{n-1}$ in the metric $\|\cdot\|_q$.

It is well known that for Taylor coefficients $\widehat{f_k} := \frac{f^{(k)}(0)}{k!}$ of any function $f \in H^q$, $1 \le q \le \infty$, the Cauchy's inequality holds:

$$\left|\widehat{f_k}\right| \le \|f\|_q \quad \forall\, k \in \mathbb{Z}_+.$$

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More precise inequalities are

$$\left|\widehat{f_k}\right| \le E_k(f)_q \quad \forall\, k \in \mathbb{Z}_+. \tag{1}$$

It is easy to see that

$$E_n(f)_2 = \left|\widehat{f_n}\right| > 0 \tag{2}$$

if and only if

$$f(z) = \sum_{k=0}^{n} \widehat{f_k} z^k, \quad \left|\widehat{f_n}\right| > 0. \tag{3}$$

Let us call a holomorphic function f *trivial polynomial* if f has the form (3) (or, equivalently, which satisfies (2)) for a given natural n.

It is clear that for any $q \ge 1$, given natural n and a trivial polynomial f we have equality

$$E_n(f)_q = \left|\widehat{f_n}\right|.$$

The following question arise naturally: Does there exist a non trivial polynomial $f \in H^q$, $1 \le q \le \infty$, such that for given natural n,

$$E_n(f)_q = \left|\widehat{f_n}\right| \ ?$$

As we will show later, a positive answer to this question is confirmed if and only if $q = 1$.

More precisely, the aim of the paper is to describe the set of functions $f \in H^q$, $1 \le q \le \infty$, for which

$$E_n(f)_q = \left|\widehat{f_n}\right|$$

for a given natural n.

2. Main Results

We begin with the simpler case, when $1 < q \le \infty$.

Theorem 2.1. *Suppose that $n, N \in \mathbb{N}$, $n \le N$, $f \in H^q$, $1 < q \le \infty$ and $\left|\widehat{f_N}\right| > 0$. The equality $E_k(f)_q = \left|\widehat{f_N}\right|$, $k = n, \dots, N$, holds true if and only if*

$$f(z) = \sum_{l=0}^{n-1} \widehat{f_l} z^l + \widehat{f_N} z^N.$$

Moreover, the polynomial

$$P^*_{n-1}(z) = \sum_{l=0}^{n-1} \widehat{f_l} z^l$$

is the unique best approximation to f among class $\mathcal{P}_{k-1}$ for each $k = n, \dots, N$, in the metric $\|\cdot\|_q$.

Corollary 2.2. *There is no function $f \in H^q$, $1 < q \le \infty$, such that*

$$E_k(f)_q = \left|\widehat{f_k}\right| = \text{const} > 0, \quad k = n, \dots, N,$$

for some natural numbers n and N, $n < N$.

Proof. Sufficiency of this assertion is obvious. To see necessity, observe first of all, that for any $s \in [1, q]$, $E_k(f)_1 \le E_n(f)_s \le E_k(f)_q = \left|\widehat{f_N}\right|$, $k = n, \dots, N$. Hence, if we take into account the inequality (1), we obtain the equality

$$E_N(f)_s = \left|\widehat{f_N}\right| \quad \forall\, s \in [1, q]. \tag{4}$$

Denote by $P^*_{n-1}(z) = \sum_{l=0}^{n-1} c_{l,n,q} z^l$ the polynomial of best approximation to f in the metric $\|\cdot\|_q$ and suppose that $1 < q \le 2$. Then by Hausdorff–Young inequality we get

$$\sum_{l=0}^{n-1} \left|\widehat{f_l} - c_{l,n,q}\right|^{q'} + \sum_{l=n}^{\infty} \left|\widehat{f_l}\right|^{q'} \le \left(E_k(f)_q\right)^{q'} = \left|\widehat{f_N}\right|^{q'}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Consequently,

$$c_{k,n,q} = \widehat{f_k}, \quad k = 0, 1, \dots, n-1,$$

and

$$\widehat{f_k} = 0, \quad k = n, n+1 \dots, \; k \ne N.$$

The case when $2 < q \le \infty$ follows immediately from the above if we take into account (4).

The uniqueness of the polynomial of best approximation follows by general theorem (see, for example [2]). □

The situation in case $q = 1$ is much more complicated.

The following is the main result of the paper.

Theorem 2.3. *Suppose that $n \in \mathbb{N}$, $f \in H^1$ and $\left|\widehat{f_n}\right| > 0$. The equality $E_n(f)_1 = \left|\widehat{f_n}\right|$ holds true if and only if*

$$f(z) = \sum_{k=0}^{2n} \widehat{f_k} z^k \tag{5}$$

and

$$2\,\mathrm{Re} \sum_{k=0}^{n} \frac{\widehat{f_{n+k}}}{\widehat{f_n}} z^k \ge 1 \quad \forall\, z \in \mathbb{T}. \tag{6}$$

Moreover, the polynomial

$$P^*_{n-1}(z) = \sum_{k=0}^{n-1} \left(\widehat{f_k} - \overline{\widehat{f_{2n-k}}} e^{i2\arg \widehat{f_n}}\right) z^k \tag{7}$$

is the unique best approximation to f among the set $\mathcal{P}_{n-1}$ in the metric $\|\cdot\|_1$.

Proof. To our goal we need the following

Lemma 2.4. *Suppose that $n \in \mathbb{N}$, $f \in H^1$, and $\left|\widehat{f_n}\right| > 0$. The following assertions are equivalent:*

(i)

$$2\,\mathrm{Re} \sum_{k=n}^{\infty} \frac{\widehat{f_k}}{\widehat{f_n}} z^{k-n} \ge 1 \quad \forall\, z \in \mathbb{D};$$

(ii)

$$\inf_{g\in\mathcal{P}_{n-1}+\overline{H^1}} \|f-g\|_1 = \left|\widehat{f_n}\right|, \tag{8}$$

where

$$\mathcal{P}_{n-1}+\overline{H_0^1} := \left\{g = P + h : P \in \mathcal{P}_{n-1}, \overline{h} \in H_0^1\right\},\ H_0^1 := \left\{h \in H^1 : h(0) = 0\right\};$$

(iii) the function

$$g^*(z) = \sum_{k=0}^{n-1}\left(\widehat{f_k} - \overline{\widehat{f_{2n-k}}}e^{i2\arg\widehat{f_n}}\right)z^k - e^{i2\arg\widehat{f_n}}\overline{\left(\sum_{k=2n+1}^{\infty}\widehat{f_k}z^{k-2n}\right)} \tag{9}$$

is the unique best approximation to f among the set $\mathcal{P}_{n-1}+\overline{H_0^1}$ in the metric $\|\cdot\|_1$.

This lemma is a slight improvement of the main result of [4]. Namely, we decompose the statement "$(i) \Leftrightarrow ((ii) \wedge (iii))$", proved in [4, Theorem 3], for the equivalence "$(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$". The proof of Lemma 2.4 essentially not differ from original one in [4].

Returning to the proof of Theorem 2.3, firs we prove the direct implication.

It is obvious that

$$\left|\widehat{f_n}\right| \le \inf_{g\in\mathcal{P}_{n-1}+\overline{H^1}} \|f-g\|_1.$$

On the other side,

$$\inf_{g\in\mathcal{P}_{n-1}+\overline{H^1}} \|f-g\|_1 \le E_n(f)_1.$$

Thus

$$\inf_{g\in\mathcal{P}_{n-1}+\overline{H^1}} \|f-g\|_1 = E_n(f)_1 = \left|\widehat{f_n}\right|.$$

So, the function $g^* := P^*$, where P^* is a polynomial of best approximation to f among the set $\mathcal{P}_{n-1}$, is the best approximation to f among the set $\mathcal{P}_{n-1}+\overline{H^1}$. But by Lemma 2.4 such function g^* is unique and has the form (9). Hence the polynomial P^* is unique and has the form (7). This yields equalities $\widehat{f_k} = 0$ for $k = 2n+1, \ldots$, that is, f has the form (5) required by Theorem 2.3. By Lemma 2.4 all this facts are equivalent to (6).

Let us prove the inverse implication. Suppose that (5) and (6) are satisfied. Than by Lemma 2.4, the equality (8) hold, that is also equivalent to assertion (*iii*) of Lemma 2.4. Thus the function $P^* := g^*$ has the form (7) and

$$E_n(f)_1 \le \|f-P^*\|_1 = \left|\widehat{f_n}\right|.$$

The result follows by (1). □

Remark 2.5. If function f satisfy conditions (5) and (6), then

$$\left|\widehat{f_n}\right| > \left|\widehat{f_{n+1}}\right|, \ldots, \left|\widehat{f_{2n}}\right|.$$

Indeed,

$$\left|\frac{\widehat{f_{k+n}}}{\widehat{f_n}}\right| = \left|\int_0^{2\pi}\left(1 + 2\,\mathrm{Re}\sum_{l=1}^{n}\frac{\widehat{f_{l+n}}}{\widehat{f_n}}e^{ilx}\right)e^{-ikx}\frac{dx}{2\pi}\right| \le$$

$$\leq \int_0^{2\pi}\left(1+2\,\mathrm{Re}\sum_{l=1}^{n}\frac{\widehat{f_{l+n}}}{\widehat{f_n}}e^{ilx}\right)\frac{dx}{2\pi}=1,\quad k=1,2,\ldots,n.$$

If we suppose that $\left|\widehat{f_{l+n}}\right|=\left|\widehat{f_n}\right|$, $l=1,\ldots,n$, we would obtain that trigonometric polynomial

$$T_n(x)=1+2\,\mathrm{Re}\sum_{l=1}^{n}\frac{\widehat{f_{l+n}}}{\widehat{f_n}}e^{ilx}=1+2\sum_{l=1}^{n}\left(\cos\theta_l\cos(lx)+\sin\theta_l\sin(lx)\right),$$

where $\theta_l=\arg\widehat{f_{l+n}}-\arg\widehat{f_n}$, is positive. Thus by Egerváry–Szász's theorem [3],

$$\left((2\cos\theta_k)^2+(2\sin\theta_k)^2\right)^{1/2}=2\leq 2\cos\frac{\pi}{\left[\frac{n}{k}\right]+2},\quad k=1,2,\ldots,n,.$$

We have the contradiction.

Taken into account Remark 2.5, we conclude similarly to Corollary 2.2, that there is no polynomial f of degree $2n, n\in\mathbb{N}$, such that $\left|\widehat{f_n}\right|=\ldots=\left|\widehat{f_{2n}}\right|=c>0$ and $E_k(f)_1=c$ for $k=n,\ldots,2n$. But in the next assertion we shall establish that there exist polynomial f of degree $2n$, such that

$$E_k(f)_1=\left|\widehat{f_k}\right|,\quad k=n,\ldots,2n. \tag{10}$$

Corollary 2.6. *Suppose that $n\in\mathbb{N}$ and $f(z)=\sum_{l=0}^{2n}\widehat{f_l}z^l$, $\prod_{l=n}^{2n}\left|\widehat{f_l}\right|>0$. The equality (10) holds true if and only if*

$$2\,\mathrm{Re}\sum_{l=0}^{2n-k}\frac{\widehat{f_{k+l}}}{\widehat{f_k}}z^l\geq 1\quad\forall\, z\in\mathbb{T},\quad k=n,\ldots,2n-1.$$

Moreover, the polynomials

$$P_{k-1}^{*}(z)=\sum_{l=0}^{2(k-n)-1}\widehat{f_l}z^l+\sum_{l=2(k-n)}^{k-1}\left(\widehat{f_l}-\overline{\widehat{f_{2k-l}}}e^{i2\arg\widehat{f_k}}\right)z^l,\ k=n,\ldots,2n,$$

(the sum $\sum_{l=N}^{M}\cdots$ for $M<N$ is empty) are the unique best approximation to f among class $\mathcal{P}_{k-1}$ in the metric $\|\cdot\|_1$.

As an illustration to Corollary 2.6 we state following assertions that are also of some independent interest.

Example 2.7. Suppose that $n\in\mathbb{N}$, $\{a_l\}_{l=n}^{2n}$ be sequence of positive numbers and let

$$f(z)=P(z)+\sum_{l=n}^{2n}a_lz^l,\quad P\in\mathcal{P}_{n-1}.$$

If

$$a_{n+j}-2a_{n+j+1}+a_{n+j+2}\geq 0,\quad j=0,1,\ldots,n-2,\ n\geq 2,$$

and

$$a_{2n-1}-2a_{2n}\geq 0,$$

then

$$E_k(f)_1 = \begin{cases} a_k, & k = n, \dots, 2n, \\ 0, & k \geq 2n+1. \end{cases} \tag{11}$$

The polynomials

$$P^*_{k-1}(z) = \begin{cases} P(z) - \sum_{l=2(k-n)}^{k-1} a_{2k-l} z^l + \sum_{l=n}^{k-1} a_l z^l, & k = n, \dots, 2n, \\ \\ f(z), & k \geq 2n+1 \end{cases}$$

are the best approximations to f among class $\mathcal{P}_{k-1}$, $k = n, \dots, 2n$, in the norm $\|\cdot\|_1$.

Indeed, the trigonometric polynomials

$$T_k(\theta) := \frac{a_{n+k}}{2} + \sum_{l=1}^{n-k} a_{n+k+l} \cos(l\theta), \quad k = 0, 1, \dots, n,$$

where the sum $\sum_{l=1}^{0} \cdots$ is empty, are not negative for all $\theta \in [0, 2\pi]$ [4]. Therefore for each $k = n, \dots, 2n$,

$$1 + 2\,\mathrm{Re}\left(\sum_{l=1}^{2n-k} \frac{a_{k+l}}{a_k} z^l + \sum_{l=2n-k+1}^{k} \frac{0}{a_k} z^l \right) =$$

$$= \frac{1}{a_k} T_{k-n}(\theta) \geq 0 \quad \forall\, \theta \in [0, 2\pi],\ z = e^{i\theta}.$$

Example 2.8. Suppose that $n \in \mathbb{N}$, $0 \leq \rho \leq \frac{1}{2}$ and let

$$f(z) = \sum_{l=0}^{2n} \rho^l z^l.$$

Then

$$E_k(f)_1 = \begin{cases} \rho^k, & k = n, \dots, 2n, \\ 0, & k \geq 2n. \end{cases}$$

The polynomials

$$P^*_{k-1}(z) = \begin{cases} \sum_{l=0}^{2(k-n)-1} \rho^l z^l + \sum_{l=2(k-n)}^{k-1} \left(1 - \rho^{2(k-l)}\right) \rho^l z^l, & k = n, \dots, 2n, \\ \\ f(z), & k \geq 2n \end{cases}$$

are the best approximations to f among the class $\mathcal{P}_{k-1}$ in the metric $\|\cdot\|_1$.

Indeed, the coefficients $a_l := \rho^l$, $l = n, \dots, 2n$, satisfies the conditions of Example 2.7:

$$a_{n+j} - 2a_{n+j+1} + a_{n+j+2} = \rho^{n+j} (1-\rho)^2 \geq 0, \quad j = 0, 1, \dots, n-2,$$

$$a_{2n-1} - 2a_{2n} = \rho^{2n-1}(1-2\rho) \geq 0 \quad \forall\, \rho \in \left[0, \frac{1}{2}\right].$$

Let us remark that bound $\rho \leq \frac{1}{2}$ in Example 2.8 can not be improved in general case. Namely, if $n = 1$ the criteria (6) take the following form

$$1 + 2\,\mathrm{Re}\sum_{k=1}^{n} \frac{a_{k+1}}{a_n} z^k = 1 + 2\rho\cos\theta \geq 0\ \forall\ \theta \in [0, 2\pi] \Leftrightarrow 0 \leq \rho \leq \frac{1}{2}. \tag{12}$$

In the following assertion we give a generalization of criteria (12) for arbitrary natural n.

Theorem 2.9. *Suppose that $n \in \mathbb{N}$, $0 < \rho < 1$ and let*

$$f(z) = \sum_{k=0}^{2n} \rho^k z^k.$$

Then the following assertions are equivalent:

(i) $\quad E_n(f)_1 = \rho^n$;

$$(ii) \quad 1 - \rho^2 - 2\rho^{n+1}\left(\cos\left((n+1)t\right) - \rho\cos(nt)\right) \geq 0 \quad \forall\ t \in [0, 2\pi]. \tag{13}$$

Proof. First of all we establish the following

Lemma 2.10. *Let the function $g(z) = \sum_{k=0}^{\infty} c_k z^k$ be holomorphic in the disk $\mathbb{D}$ and suppose that $|c_0| > 0$. Then*

$$2\,\mathrm{Re}\sum_{k=0}^{\infty} \frac{c_k}{c_0} z^k \geq 1 \quad \forall\ z \in \mathbb{D} \tag{14}$$

iff

$$|g(z)|^2 \geq |g(z) - c_0|^2 \quad \forall\ z \in \mathbb{D}.$$

This follows immediately from obvious identity

$$2\,\mathrm{Re}\,\zeta - 1 = |\zeta|^2 - |\zeta - 1|^2 \quad \forall\ \zeta \in \mathbb{C}.$$

Returning to the proof of Theorem 2.9 we note that by Theorem 2.3,

$$E_n(f)_1 = \rho^n \Longleftrightarrow 2\,\mathrm{Re}\sum_{k=0}^{n} \rho^k z^k \geq 1 \quad \forall\ z \in \mathbb{T}.$$

Further, by Lemma 2.10 we have the equivalence

$$2\,\mathrm{Re}\sum_{k=0}^{n} \rho^k z^k \geq 1 \quad \forall\ z \in \mathbb{T} \Longleftrightarrow$$

$$\Longleftrightarrow \left|\sum_{k=0}^{n} \rho^k z^k\right|^2 \geq \left|\sum_{k=1}^{n} \rho^k z^k\right|^2 \quad \forall\ z \in \mathbb{T}$$

It is easy to see that last inequality is equivalent to

$$\left|1 - \rho^{n+1} z^{n+1}\right|^2 \geq \rho^2 \left|1 - \rho^n z^n\right|^2 \quad \forall\ z \in \mathbb{T},$$

that after simplifying is equivalent to such one

$$1 - 2\rho^{n+1}\cos\left((n+1)t\right) + \rho^{2(n+1)} \geq \rho^2\left(1 - 2\rho^n\cos(nt) + \rho^{2n}\right) \quad \forall\ z = e^{it} \in \mathbb{T}.$$

□

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