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Almost Multiplicative Maps and $\varepsilon-$ Spectrum of an Element in Fréchet Q-Algebra

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Abstract. Let $(A, (p_k))$ be a Fréchet Q-algebra with unit e_A. The $\varepsilon-$ spectrum of an element x in A is defined by

$$\sigma_\varepsilon(x) = \{\lambda \in \mathbb{C} : p_{k_0}(\lambda e_A - x)p_{k_0}(\lambda e_A - x)^{-1} \geq \tfrac{1}{\varepsilon}\}$$

for $0 < \varepsilon < 1$. We show that there is a close relation between the $\varepsilon-$spectrum and almost multiplicative maps. It is also shown that

$$\{\varphi(x) : \varphi \in M^{\varepsilon}_{alm}(A), \varphi(e_A) = 1\} \subseteq \sigma_\varepsilon(x)$$

for every $x \in A$, where $M^{\varepsilon}_{alm}(A)$ is the set of all $\varepsilon-$ multiplicative maps from A to $\mathbb{C}$.

1. Introduction

Let A be an algebra over the complex field. A subset V of A is called *idempotent* if $VV \subseteq V$, and it is called *balanced* if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$. An algebra A is called a *Fréchet algebra* if it is a complete metrizable topological linear space and has a neighborhood basis (V_n) of zero such that V_n is *absolutely convex* (convex and balanced) and $V_nV_n \subseteq V_n$ for all $n \in \mathbb{N}$. The topology of a Fréchet algebra A can be generated by a sequence (p_k) of separating submultiplicative seminorms (p_k is called $m-seminorm$), i.e., $p_k(xy) \leq p_k(x)p_k(y)$ for all $n \in \mathbb{N}$ and $x, y \in A$, such that $p_k(x) \leq p_{k+1}(x)$, whenever $n \in \mathbb{N}$ and $x \in A$. If A is unital then p_k can be chosen such that $p_k(1) = 1$ for all $n \in \mathbb{N}$. The Fréchet algebra A with the above generating sequence of seminorms (p_k) is denoted by $(A, (p_k))$. A Fréchet algebra $(A, (p_k))$ is a uniform Fréchet algebra if $p_k(a^2) = (p_k(a))^2$, for all $k \in \mathbb{N}$ and $a \in A$. Note that a sequence (x_k) in the Fréchet algebra $(A, (p_k))$ converges to $x \in A$ if and only if $p_n(x_k - x) \to 0$ for each $n \in \mathbb{N}$, as $k \to \infty$. Banach algebras are important examples of Fréchet algebras.

Let A be an algebra. For $x, y \in A$, we define $x \diamond y = x + y - xy$. An element $x \in A$ is called *quasi-invertible* if there exists $y \in A$ such that

$$x \diamond y = y \diamond x = 0.$$

The set of all quasi-invertible elements of A is denoted by $q - InvA$. For a unital algebra A, the set of all invertible elements and don't invertible elements of A is denoted by $InvA$ and $SignA$ respectively . A

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topological algebra A is called a *Q-algebra* if $q - InvA$ is open, or equivalently, if $q - InvA$ has an interior point in A [16, Lemma E2]. For a unital algebra A, *the spectrum* $\sigma_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda 1 - x$ is not invertible in A. For a non-unital algebra A, the spectrum of $x \in A$ is defined by $\sigma_A(x) = \{0\} \cup \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \frac{x}{\lambda} \notin q - InvA\}$. *The spectral radius* $r_A(x)$ of an element $x \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$. In particular, when A is a Banach algebra, it is known that $r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$. The *(Jacobson) radical* of A, denoted by $radA$, is the intersection of all maximal left (right) ideals in A. It is known that for any algebra A

$$radA = \{x \in A : r_A(xy) = 0, \text{ for every } y \in A\}\cdot$$

The algebra A is called *semisimple* if $radA = \{0\}$. If A is a commutative Fréchet algebra, then $radA = \bigcap_{\varphi\in M(A)} ker\varphi$, where $M(A)$ is the continuous character space of A, i.e, the space of all continuous non-zero multiplicative linear functionals on A. See, for example, [4, Proposition 8.1.2].

A well-known class of Fréchet algebras is the class of Fréchet Q-algebras.

In 1985, K. Jarosz introduced the concept of *almost multiplicative maps* between Banach algebras [9]. For the Banach algebras A and B, a linear map $T : A \to B$ is called *almost multiplicative* if there exists $\varepsilon \geq 0$ such that

$$\|Tab - TaTb\| \leq \varepsilon \|a\| \|b\|,$$

for every $a, b \in A$. K. Jarosz investigated the problem of automatic continuity for almost multiplicative linear maps between Banach algebras. Since then, many authors investigated almost multiplicative maps between different classes of Banach algebras. See, for example, [10]. In 2014, Honary, Omidi and Sanatpour investigated the problem of automatic continuity for (weakly) almost multiplicative linear maps between Fréchet algebras [7] and [6].

In this paper, we prove some results about Fréchet Q-algebras, $\varepsilon-$ spectrum of an element in Fréchet Q-algebra and relationship between $\varepsilon-$ spectrum and almost multiplicative maps. Next we show that, $\varepsilon-$spectrum is a particular case of the generalized spectrum defined by Ransford [14]. Several results on spectrum are generalized to $\varepsilon-$spectrum. For example, it is proven that if $(A, (p_k))$ is a unital Fréchet Q-algebra and $T : A \to \mathbb{C}$ is an $\varepsilon-$ multiplicative map such that $T(e_A) = 1$, then for every $x \in A$, $T(x) \in \sigma_\varepsilon(x)$. Also in some situations, if $(A, (p_k))$ is a commutative Fréchet Q-algebra with unit e_A and $\varepsilon > 0$, then for every $x \in A$ and $\lambda \in \sigma_\varepsilon(x)$, there exists weakly almost multiplicative map $T : A \to \mathbb{C}$ such that $T(x) = \lambda$ and $T(e_A) = 1$. In [13], the authors have given an analogue of the Spectral Mapping Theorem for condition spectrum. The $\varepsilon-$spectrum is a useful tool in the numerical solution of operator equations. For example, Suppose that X is a Banach space, $T : X \longrightarrow X$ is a bounded linear map and $y \in X$. Consider the operator equation

$$Tx - \lambda x = y,$$

then

(i) $\lambda \notin \sigma(T)$ implies that this operator equation is solvable,

(ii) $\lambda \notin \sigma_\varepsilon(T)$implies that this operator equation has a stable solution.

It is clear that, if $\sigma_\varepsilon(a) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$, then $a = \lambda$. A very well known classical problem in operator theory known as $T = I$ problem, asks the following question:
Let T be an operator on a Banach space. Suppose that $\sigma(T) = \{1\}$. Under what additional conditions can we conclude $T = I$? From the above condition, it follows that if $\sigma_\varepsilon(T) = \{1\}$, then $T = I$. In other words if we replace spectrum by ε-spectrum in the $T = I$ problem, then no additional conditions are required.

We first bring some notions on topological algebras.

Definition 1.1. A subset V of a complex algebra A is *m-convex* (*multiplicatively convex*) if V is convex and *idempotent*, i.e., $VV \subseteq V$. A topological algebra is *locally convex* if there is a base of neighborhoods of zero consisting of convex sets. Since each locally convex algebra has a base of neighborhoods of zero consisting of absolutely convex sets, we may always assume that a locally convex algebra has a base of neighborhoods of zero consisting of absolutely convex sets. A topological algebra $(A, (p_\alpha)_\alpha)$, whose topology is defined by a family $(p_\alpha)_\alpha$ of m-seminorms is called *m-convex* or *lmc* algebra . In the other word a topological algebra A is a *locally multiplicatively convex algebra* (*lmc algebra*) if there is a base of neighborhoods of the origin consisting of sets which are absolutely convex and multiplicative (idempotent).

Definition 1.2. A copmlete m-convex algebra is said *Arens-Michael algebra*, while the term Fréchet algebra stands for a metrizable Arens-Michael algebra.

It is interesting to note that the Fréchet algebras defined in Section 1, are *lmc* algebras. Moreover, an *lmc* algebra A is a Q-algebra if and only if the spectral radius r_A is continuous at zero and it is uniformly continuous on A if A is also commutative. See, for example, [3, Theorem 6.18] or [15, III. Proposition 6.2].

2. Preliminaries

We first bring the following some results for easy reference.

Theorem 2.1. ([3, Theorem 4.6]) *Let $(A, (p_\alpha)_\alpha)$ be an m-convex algebra, then the following statements hold:*

(i) $\sigma_A(x) \neq \emptyset$, *for every $x \in A$.*
If morever A is complete (ie., an Arens-Michael algebra) one has,

(ii) $r_A(x) = \sup_{p_\alpha} lim_{n\to\infty} p_\alpha(x^n)^{\frac{1}{n}}$, *for every $x \in A$.*

Corollary 2.2. *Let $(A, (p_k))$ be a Fréchet algebra, then*

$$r_A(x) = \sup_{k\in\mathbb{N}} lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}}$$

for every $x \in A$.

Theorem 2.3. ([3, Theorem 6.18]) *Let $(A, (p_k))$ be Fréchet algebra, then the following statements are equivalent:*

(i) $(A, (p_k))$ *is a Q-algebra.*

(ii) *There is $k_0 \in \mathbb{N}$ such that $r_A(x) \leq p_{k_0}(x)$, for every $x \in A$.*

(iii) $r_A(x) = lim_{n\to\infty} p_{k_0}(x^n)^{\frac{1}{n}}$, *for every $x \in A$ and p_{k_0} as in (ii).*

If k_0 is the smallest natural number such that p_{k_0} satisfies in the Theorem (2.3), we say that p_{k_0} is original seminorm for Fréchet $Q-$algebra $(A, (p_k))$. In the sequel, we use this fixed p_{k_0} as original seminorm for every Fréchet Q-algebra.

Theorem 2.4. ([15, III. Corollary 6.4]) *Let $(A, (p_k))$ be a unital commutative Fréchet $Q-$algebra, then*

$$\emptyset \neq \sigma_A(x) = \{\varphi(x) : \varphi \in M(A)\}$$

for every $x \in A$.

Definition 2.5. Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet algebras and ε be non-negative. A linear map $T : A \to B$ is multiplicative, if $Tab = TaTb$ for every $a, b \in A$, it is *ε-multiplicative*, with respect to (p_n) and (q_n), if

$$q_n(Tab - TaTb) \leq \varepsilon p_n(a)p_n(b),$$

for all $n \in \mathbb{N}$, $a, b \in A$, and it is weakly ε-multiplicative, with respect to (p_n) and (q_n), if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(Tab - TaTb) \leq \varepsilon p_{n(k)}(a)p_{n(k)}(b),$$

for every $a, b \in A$. A linear map $T : A \to B$ is almost multiplicative or weakly almost multiplicative, if it is ε-multiplicative, or weakly ε-multiplicative, respectively, for some $\varepsilon \geq 0$.

Remark 2.6. Note that the topology of a Fréchet algebra A, generated by the sequence (p_k), coincides with the topology of A, generated by the subsequence (p_{n_k}). Hence, in the definition of weakly ε-multiplicative, if we only consider the topology of a Fréchet algebra A, regardless of a particular sequence of seminorms (p_n) for A, then we may assume, without loose of generality, that the inequality $q_k(Tab - TaTb) \leq \varepsilon p_k(a)p_k(b)$, holds for all k. Therefore, the notion of weakly ε-multiplicative is, in fact, the same as ε-multiplicative in the above sense.

Example 2.7. We consider the Fréchet algebra $(C(\mathbb{R}), (p_n))$, with the sequence of seminorms $p_n(f) = sup\{|f(x)| : x \in [-n, n]\}$. Take $r = \frac{1+\sqrt{1+4\varepsilon}}{2}$ for $\varepsilon > 0$. For a fixed $x_0 \in \mathbb{R}$ we define a map $T : C(\mathbb{R}) \to \mathbb{C}$ by $T(f) = rf(x_0)$, which is obviously a continuous linear functional. Since $T(fg) = rf(x_0)g(x_0)$ and $T(f)T(g) = r^2 f(x_0)g(x_0)$, it follows that T is not multiplicative. However, there exists $m \in \mathbb{N}$ such that $x_0 \in [-m, m]$ and hence $x_0 \in [-n, n]$ for all $n \geq m$. Therefore,

$$\begin{aligned}
|T(fg) - T(f)T(g)| &= |(r - r^2)f(x_0)g(x_0)| \\
&= (r^2 - r)|f(x_0)g(x_0)| \\
&\leq \varepsilon ||f||_{[-m,m]}||g||_{[-m,m]} \\
&= \varepsilon p_m(f)p_m(g) \leq \varepsilon p_n(f)p_n(g) \cdot
\end{aligned} \tag{1}$$

It follows from (1) that T is weakly ε-multiplicative and hence, it is almost multiplicative by Remark 2.6.

The following theorem has proved in [8, Proposition 2.7].

Theorem 2.8. *Let $(A, (p_k))$ and $(B, (q_k))$ be Fréchet algebras and $T : A \to B$ be a linear map. Then T is continuous if and only if for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ and $c_k > 0$ such that*

$$q_k(T(a)) \leq c_k p_{n(k)}(a), \tag{2}$$

for each $a \in A$. Moreover, if $(B, (q_k))$ is a uniform Fréchet algebra and T is a continuous homomorphism, then we may choose $c_k = 1$ for all $k \in \mathbb{N}$.

3. Main Results

In this section we prove some results about Fréchet Q-algebras and $\varepsilon-$ spectrum of an element in Fréchet Q-algebra.

Lemma 3.1. *Let $(A, (p_k))$ be a Fréchet algebra and $x \in A$ such that $r_A(x) < 1$. Then*

$$lim_{n\to\infty} p_k(x^n) = 0,$$

for every $k \in \mathbb{N}$.

Proof. Since $r_A(x) = \sup_{k\in\mathbb{N}} lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}} < 1$, we have

$$lim_{n\to\infty} p_k(x^n)^{\frac{1}{n}} = l_k < 1,$$

for every $k \in \mathbb{N}$. For k fixed and $\varepsilon = \frac{1-l_k}{2}$, there exists $N_k \in \mathbb{N}$ such that

$$p_k(x^n)^{\frac{1}{n}} - l_k < \varepsilon = \frac{1-l_k}{2},$$

for every $n \geq N_k$. Therefore

$$0 \leq p_k(x^n) < (\tfrac{1-l_k}{2})^n,$$

for every $n \geq N_k$, and hence $lim_{n\to\infty} p_k(x^n) = 0$ $(0 < l_k < 1)$. □

We note that if $(A,(p_k))$ is a Fréchet Q-algebra and $x \in A$, then $r_A(x) \leq p_{k_0}(x)$. Now by applying Theorem 2.3, we obtain the following result.

Corollary 3.2. *Let $(A,(p_k))$ be a Fréchet Q-algebra and $x \in A$ such that $p_{k_0}(x) < 1$. Then $lim_{n\to\infty} p_k(x^n) = 0$, for every $k \in \mathbb{N}$.*

Lemma 3.3. *Let $(A,(p_k))$ be a Fréchet Q-algebra and $x \in A$ such that $p_{k_0}(x) < 1$ or $r_A(x) < 1$, then $e_A - x$ is invertible.*

Proof. We set $S_n = e_A + x + \cdots + x^n$, then $(e_A - x)S_n = e_A - x^{n+1}$. Since $r_A(x) < 1$, by Lemma 3.1, we have $lim_{n\to\infty} p_k(x^n) = 0$, for every $k \in \mathbb{N}$. This means that $x^n \to 0$, therefore $(e_A - x)\sum_{n=1}^{\infty} x^n = e_A$ and $(e_A - x)^{-1} = \sum_{n=1}^{\infty} x^n$. □

Corollary 3.4. *Let $(A,(p_k))$ be a Fréchet Q-algebra and $x \in A$ such that $p_{k_0}(x) \leq |\lambda|$. Then $\lambda e_A - x$ is invertible and $p_{k_0}(\lambda e_A - x)^{-1} < \dfrac{1}{|\lambda| - p_{k_0}(x)}$.*

Proof. The proof is similar to Lemma 3.3 and $(\lambda e_A - x)^{-1} = \frac{1}{\lambda}\sum_{n=1}^{\infty}(\frac{x}{\lambda})^n$. Hence

$$p_{k_0}(\lambda e_A - x)^{-1} \leq \frac{1}{|\lambda| - p_{k_0}(x)},$$

because p_{k_0} is continuous. □

Lemma 3.5. *Let $(A,(p_k))$ be Fréchet Q-algebra with unit e_A, $x \in InvA$ and $y \in A$ such that $p_{k_0}(x - y) < \dfrac{1}{p_{k_0}(x^{-1})}$. Then $y \in InvA$.*

Proof. By hypothesis $p_{k_0}((x-y)x^{-1}) \leq p_{k_0}(x-y)p_{k_0}(x^{-1}) < 1$, then yx^{-1} is invertible, by Lemma 3.3. Therefore $y \in InvA$. □

Corollary 3.6. *Let $(A,(p_k))$ be Fréchet Q-algebra with unit e_A, $x \in InvA$ and $y \in Sing(A)$. Then $p_{k_0}(x-y) \geq \dfrac{1}{p_{k_0}(x^{-1})}$.*

Lemma 3.7. *Let $(A,(p_k))$ be Fréchet Q-algebra with unit e_A, $x \in InvA$ and $y \in A$ such that $p_{k_0}(y)p_{k_0}(x^{-1}) < 1$. Then $x + y \in InvA$.*

Proof. By hypothesis $p_{k_0}(-yx^{-1}) \leq p_{k_0}(y)p_{k_0}(x^{-1}) < 1$, then $e_A + yx^{-1}$ is invertible, by Lemma 3.3. Therefore $x + y = (e_A + yx^{-1})x \in InvA$. □

Now, we introduce the concept of $\varepsilon-$spectrum, as follows.

Definition 3.8. Let $(A,(p_k))$ be a Fréchet Q-algebra with unit e_A. For $\varepsilon > 0$, the $\varepsilon-$ spectrum of an element x in A is defined by,

$$\sigma_\varepsilon(x) = \{\lambda \in \mathbb{C} : p_{k_0}(\lambda e_A - x)p_{k_0}(\lambda e_A - x)^{-1} \geq \tfrac{1}{\varepsilon}\}$$

with the convention $p_{k_0}(\lambda e_A - x)p_{k_0}(\lambda e_A - x)^{-1} = \infty$ when $\lambda e_A - x$ is not invertible. The $\varepsilon-$ spectral radius $r_\varepsilon(x)$ is define as

$$r_\varepsilon(x) = sup\{|\lambda| : \lambda \in \sigma_\varepsilon(x)\}\cdot$$

Note that $\sigma_A(x) \subset \sigma_\varepsilon(x)$ and therefore $r_A(x) \leq r_\varepsilon(x)$.
The following example is useful.

Example 3.9. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and $A = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$. Then for $\lambda \in \mathbb{C}$, we have

$$\|A - \lambda I\| = \max\{|\lambda - \lambda_1|, |\lambda - \lambda_2|\}$$

and

$$\|(A - \lambda I)^{-1}\| = \max\{\tfrac{1}{|\lambda-\lambda_1|}, \tfrac{1}{|\lambda-\lambda_2|}\}.$$

Hence for $\varepsilon > 0$,

$$\begin{aligned}\sigma_\varepsilon(A) =& \{\lambda \in \mathbb{C} : \|A - \lambda I\|\|(A - \lambda I)^{-1}\| \geq \frac{1}{\varepsilon}\}\\ =& \{\lambda \in \mathbb{C} : \frac{|\lambda - \lambda_1|}{|\lambda - \lambda_2|} \geq \frac{1}{\varepsilon}\} \cup \{\lambda \in \mathbb{C} : \frac{|\lambda - \lambda_2|}{|\lambda - \lambda_1|} \geq \frac{1}{\varepsilon}\}.\end{aligned}$$

In the next theorem, we give some properties of the spectral radius $r_\varepsilon(x)$.

Lemma 3.10. *Let $(A, (p_k))$ be a Fréchet Q-algebra, $x \in A$ and $0 < \varepsilon < 1$, then*

$$r_\varepsilon(x) \leq (\tfrac{1+\varepsilon}{1-\varepsilon})p_{k_0}(x).$$

Proof. Let $\lambda \in \sigma_\varepsilon(x)$. We know that $p_{k_0}(\lambda e_A - x)^{-1} \leq \dfrac{1}{|\lambda| - p_{k_0}(x)}$, hence

$$\tfrac{1}{\varepsilon} \leq p_{k_0}(\lambda e_A - x)p_{k_0}(\lambda e_A - x)^{-1} \leq \frac{|\lambda| + p_{k_0}(x)}{|\lambda| - p_{k_0}(x)}.$$

Now, it is easy to check that $|\lambda| \leq (\frac{1+\varepsilon}{1-\varepsilon})p_{k_0}(x)$. □

Next two theorems establish a relationship between $\varepsilon-$spectrum and spectrum of an element in a Fréchet Q-algebra.

Theorem 3.11. *Let $(A, (p_k))$ be a Fréchet Q-algebra, $\varepsilon > 0$ and $x \in A$, such that x is not a scalar multiple of the e_A. Suppose that $y \in A$, $\lambda \in \sigma_A(x + y)$ and $p_{k_0}(y) \leq \varepsilon p_{k_0}(\lambda e_A - x)$, then $\lambda \in \sigma_\varepsilon(x)$.*

Proof. If $\lambda \in \sigma_A(x)$, then $\lambda \in \sigma_\varepsilon(x)$. If $\lambda \notin \sigma_A(x)$, then $\lambda e_A - x \in InvA$. Since $\lambda \in \sigma_A(x + y)$, $\lambda e_A - x - y$ is not invertible. Now, from Corollary(3.6), we have

$$p_{k_0}(y) = p_{k_0}((\lambda e_A - x) - (\lambda e_A - x - y)) \geq \frac{1}{p_{k_0}(\lambda e_A - x)^{-1}}.$$

On the other hand, $p_{k_0}(y) \leq \varepsilon p_{k_0}(\lambda e_A - x)$. Hence, we conclude

$$p_{k_0}(\lambda e_A - x)p_{k_0}(\lambda e_A - x)^{-1} \geq \tfrac{1}{\varepsilon}.$$

Therefore, $\lambda \in \sigma_\varepsilon(x)$. □

Definition 3.12. A Fréchet Q-algebra $(A, (p_k))$ is said *compatible* if for every $x \in InvA$, there exists $y \in SignA$ such that $p_{k_0}(x - y)p_{k_0}(x^{-1}) = 1$.

Lemma 3.13. *Let $(A, (p_k))$ be a compatible Fréchet Q-algebra and $x \in A$, such that x is not a scalar multiple of e_A. Then for every $\lambda \in \sigma_\varepsilon(x)$, there exists $y \in singA$ such that $p_{k_0}(y) \leq \varepsilon p_{k_0}(\lambda e_A - x)$ and $\lambda \in \sigma_A(x + y)$.*

Proof. If $\lambda \in \sigma_A(x)$, we set $y = 0$. If $\lambda \notin \sigma_A(x)$, then $\lambda e_A - x \in InvA$. Since A is compatible, by Definition 3.12, there exists $c \in singA$ such that

$$p_{k_0}(\lambda e_A - x - c) = \frac{1}{p_{k_0}(\lambda e_A - x)^{-1}} \leq \varepsilon p_{k_0}(\lambda e_A - x),$$

because $\lambda \in \sigma_\varepsilon(x)$. By taking $y = \lambda e_A - x - c$, we conclude that

$$p_{k_0}(y) \leq \varepsilon p_{k_0}(\lambda e_A - x).$$

Also $\lambda e_A - x - y = c \in SingA$, therefore $\lambda \in \sigma_A(x+y)\cdot$ □

Lemma 3.14. *Let $(A,(p_k))$ be a semisimple commutative Fréchet algebra and x_0 be a non zero element in A. Then for every $\lambda \in \mathbb{C}$, there exists a weakly almost multiplicative linear functional $T : (A,(p_k)) \to \mathbb{C}$ such that $T(x_0) = \lambda$.*

Proof. Since $\bigcap_{\varphi \in M(A)} ker\varphi = radA = \{0\}$ and $x_0 \neq 0$, there exists $\varphi \in M(A)$ such that $\varphi(x_0) \neq 0$. If $\varphi(x_0) = 1$, we define $T : (A,(p_k)) \to \mathbb{C}$ by $T(x) = \lambda\varphi(x)$ and if $\varphi(x_0) \neq 1$ we set $T(x) = \frac{\lambda}{\varphi(x_0)}\varphi(x)$, for every $x \in A$. It is clear that T is a weakly almost multiplicative linear functional and $T(x_0) = \lambda$ in both cases. □

We know that if A is a commutative Banach algebra and $M(A)$ is the set of all complex homomorphisms from A to $\mathbb{C}$, then $\lambda \in \sigma(x)$ if and only if $T(x) = \lambda$, for some $T \in M(A)$ [17, Theorem 11.5].

Now, the following Lemma establish a relationship between $\sigma_\varepsilon(x)$ and $M^{\varepsilon}_{alm}(A)$.

Lemma 3.15. *Let $(A,(p_k))$ be a unital Fréchet Q-algebra and $T : A \to \mathbb{C}$ be an $\varepsilon-$ multiplicative map such that $T(e_A) = 1$. Then for every $x \in A$, $T(x) \in \sigma_\varepsilon(x)$.*

Proof. Set $\lambda = T(x)$. If $\lambda e_A - x \notin InvA$, then $\lambda \in \sigma_A(x) \subset \sigma_\varepsilon(x)$. Let $\lambda e_A - x \in InvA$, then

$$\begin{aligned} 1 &= |T(\lambda e_A - x)(\lambda e_A - x)^{-1}| \\ &= |T(\lambda e_A - x)(\lambda e_A - x)^{-1} - T(\lambda e_A - x)T((\lambda e_A - x)^{-1})| \\ &\leq \varepsilon p_{k_0}(\lambda e_A - x)p_{k_0}((\lambda e_A - x)^{-1})\cdot \end{aligned}$$

Therefore, $\lambda \in \sigma_\varepsilon(x)$. □

Next we show that there is close relation between the $\varepsilon-$spectrum and almost multiplicative maps.

Theorem 3.16. *Let $(A,(p_k))$ be a commutative compatible Fréchet Q-algebra with unit e_A and $\varepsilon > 0$. Then for every $x \in A$ and $\lambda \in \sigma_\varepsilon(x)$, there exists weakly almost multiplicative map $T : A \to \mathbb{C}$ such that $T(x) = \lambda$ and $T(e_A) = 1$.*

Proof. If x is a scalar multiple of e_A, then $\sigma_\varepsilon(x) = \sigma_A(x) = \{\lambda\}$. Now by applying Theorem 2.4, there exists $\varphi \in M(A)$ such that $\varphi(x) = \lambda$ and $\varphi(e_A) = 1$. Next, suppose that x is not a scalar multiple of e_A. By Lemma 3.13, there exists $y \in SingA$ such that $p_{k_0}(y) \leq \varepsilon p_{k_0}(\lambda e_A - x)$ and $\lambda \in \sigma_A(x+y)$(because A is compatible). Therefore there exists $\varphi \in M(A)$ such that $\lambda = \varphi(x+y) = \varphi(x) + \varphi(y)$. We know that A is locally convex topological vector space, then for $M = \{\alpha e_A : \alpha \in \mathbb{C}\}$ and applying [17, Theorem 3.5], there exists $\psi \in A^*$ such that $\psi(x) = 1$ and $\psi(e_A) = 0$ ($\psi = 0$ on M). Now we define $T : A \to \mathbb{C}$ with $T(a) = \varphi(a) + \varphi(y)\psi(a)$ for every $a \in A$. It is clear that, $T(e_A) = 1$. We claim that T is a weakly almost multiplicative map. Since ψ , φ are continuous linear maps and $\mathbb{C}$ is uniform, using Theorem 2.8, there exists $m, n \in \mathbb{N}$ such that

$$|\varphi(a)| < p_m(a) \quad , \quad |\psi(a)| < p_n(a) \tag{3}$$

for every $a \in A$. Set $N = \max\{m,n\}$, then by using (3),

$$\begin{aligned} &|T(ab) - TaTb| \\ &\leq \varphi(y)(\psi(ab) + \varphi(a)\psi(b) + \varphi(b)\psi(a) + \varphi(y)\psi(a)\psi(b)) \\ &\leq p_N(y)(p_N(a)p_N(b) + p_N(a)p_N(b) + p_N(a)p_N(b) + p_N(y)p_N(a)p_N(b)) \\ &= p_N(y)(3 + p_N(y))p_N(a)p_N(b) \\ &= \xi p_N(a)p_N(b), \end{aligned}$$

for every $a, b \in A$, where $\xi = p_N(y)(3 + p_N(y))$. Hence T is a weakly ξ- multiplicative map. □

4. Conclusion

At first glance, the $\varepsilon-$spectrum appears to differ from usual spectrum, but some properties of them are similar. Also the $\varepsilon-$spectrum has many properties that are different from the properties of the usual spectrum. In our opinion, comparing $M_{alm}(A)$ and $M(A)$ is interesting, especially finding the connection between them and $\sigma_\varepsilon(x)$, $\sigma(x)$.

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