Filomat 33:5 (2019), 1453–1461
https://doi.org/10.2298/FIL1905453K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Direct Integration of Systems of Linear Differential and Difference Equations

Syrgak Kydyraliev[a], Anarkul Urdaletova[b]

[a] *American University of Central Asia, Bishkek, Kyrgyzstan*
[b] *Kyrgyz–Turkish Manas University, Bishkek, Kyrgyzstan*

Abstract. Traditionally the Euler method is used for solving systems of linear differential equations. The method is based on the use of eigenvalues of a system's coefficients matrix. Another method to solve those systems is the D'Alembert integrable combination method. In this paper, we present a new method for solving systems of linear differential and difference equations. The main idea of the method is using the coefficients matrix eigenvalues to find integrable combinations of system variables. This method is particularly advantageous when nonhomogeneous systems are considered.

1. Introduction

Two great scientists of the XVIII century, Leonard Euler (1707–1783) and Jean le Rond D'Alembert (1717–1783), were men of great intellect and observation skills, made many discoveries, and wrote a lot of scientific works. Now students all over the world study these works. One day one of these scientists invented a method for solving systems of linear differential equations. Having heard this, the other one said "Whatever, I can invent a method at least as good." And he did. Since then poor students have to study both methods: the Euler method, based on the use of eigenvalues of a system's coefficients matrix, and the D'Alembert integrable combination method [1–5]. But life does not stop. Implacable scientific and technological progress keeps increasing the amount of information that must be learned. As a result, in the interest of time we often have to choose only one method. There are different ways to solve this dilemma, but, in our opinion, the best way out of this situation is to combine both methods. Fortunately, the potential embodied in the original components is so great that the synergy effect appears: the combined method is simpler than the original ones. We describe this method below.

2. Main Idea

The linear system of differential equations with constant coefficients is

$$\begin{cases} y' = ay + bz + f(x), \\ z' = cy + dz + g(x), \end{cases} \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 97M10; Secondary 39A05, 39A06, 39A60
Keywords. Systems of linear differential equations, difference equations
Received: 25 September 2018; Revised: 27 November 2018; Accepted: 28 November 2018
Communicated by Fahreddin Abdullayev
Email addresses: kydyraliev_s@auca.kg (Syrgak Kydyraliev), urdaletova.anarkul@gmail.com (Anarkul Urdaletova)

where a, b, c, d are constant coefficients, while f and g are determined functions.

A characteristic equation of system (1) is the equation

$$\Delta(\lambda) = 0, \tag{2}$$

where the function $\Delta(\lambda)$ is determined by the coefficients of system (1) as follows:

$$\Delta(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - c \cdot b.$$

The roots of equation (2) are called characteristic numbers of system (1).

Let p and q be the characteristic numbers of the system (1).

Then, in order to solve the system, at first, we subtract function py from the left and the right parts of the 1st equation of system (1) and pz from both parts of the 2nd equation:

$$\begin{cases} y' - py = (a-p)y + bz + f(x), \\ z' - pz = cy + (d-p)z + g(x). \end{cases} \tag{3}$$

As p is a characteristic number, the determinant $\begin{vmatrix} a-p & b \\ c & d-p \end{vmatrix}$ is equal to zero, which means that the rows of the determinant are proportional. Let k be the proportionality coefficient.

We factorize the left part of the equations; multiply the 2nd equation by k

$$\begin{cases} (ye^{-px})'e^{px} = (a-p)y + bz + f(x), \\ k(ze^{-px})'e^{px} = kcy + k(d-p)z + kg(x), \end{cases}$$

and subtract the 2nd equation from the 1st. Then

$$[(y-kz)e^{-px}]'e^{px} = f(x) - kg(x). \tag{4}$$

Integrating equation (4), we get

$$y - kz = \left(\int [f(x) - kg(x)]e^{-px}dx + C_1 \right) e^{px}. \tag{5}$$

At second, we repeat the procedure, taking q instead of p, and get

$$y - mz = \left(\int [f(x) - mg(x))]e^{-qx}dx + C_2 \right) e^{qx}. \tag{6}$$

We consider equalities (5) and (6) as an algebraic system of the unknowns y and z, and solving it we get the solution of system (1).

3. Distinct Eigenvalues

Example 3.1. Let solve the initial value problem

$$\begin{cases} y' = 3y - 5z + 2\sqrt{x}e^{0.5x}; \ \ y_0 = 10, \\ z' = 0.5y - 0.5z + \sqrt{x}e^{0.5x}; \ \ z_0 = 2. \end{cases} \tag{7}$$

We will get the characteristic numbers of the system $(0.5; 2)$ from the equation

$$\begin{vmatrix} 3-\lambda & -5 \\ 0.5 & -0.5-\lambda \end{vmatrix} = \lambda^2 - 2.5\lambda + 1 = 0.$$

1) Using the first of them, we rewrite the system as

$$\begin{cases} y' - 0.5y = 2.5y - 5z + 2\sqrt{x}e^{0.5x}; \\ z' - 0.5z = 0.5y - z + \sqrt{x}e^{0.5x}. \end{cases}$$

We factorize the left parts

$$\begin{cases} (ye^{-0.5x})'e^{0.5x} = 2.5y - 5z + 2\sqrt{x}e^{0.5x}; \\ (ze^{-0.5x})'e^{0.5x} = 0.5y - z + \sqrt{x}e^{0.5x}. \end{cases}$$

and subtracting the 2nd multiplied by 5 from the 1st, we get

$$[(y - 5z)e^{-0.5x}]'e^{0.5x} = -3x^{0.5}e^{0.5x}.$$

We integrate it and get

$$y - 5z = (-2x^{1.5} + C_1)e^{0.5x}.$$

As $y_0 - 5z_0 = 10 - 5 \cdot 2 = 0$, we can find $C_1 = 0$.

2) We repeat the procedure taking the second characteristic number:

$$\begin{cases} y' - 2y = y - 5z + 2\sqrt{x}e^{0.5x}; \\ z' - 2z = 0.5y - 2.5z + \sqrt{x}e^{0.5x}. \end{cases}$$

Then

$$\begin{cases} (ye^{-2x})'e^{2x} = y - 5z + 2\sqrt{x}e^{0.5x}; \\ (ze^{-2x})'e^{2x} = 0.5y - 2.5z + \sqrt{x}e^{0.5x}. \end{cases}$$

We subtract the doubled 2nd from the 1st:

$$[(y - 2z)e^{-2x}]'e^{2x} = 0.$$

Then $y - 2z = C_2e^{-2x}$, and using $y_0 - 2z_0 = 10 - 2 \cdot 2 = 6$, we can find $C_2 = 6$.

In order to find y and z we solve the system

$$\begin{cases} y - 5z = -2x^{1.5}e^{0.5x}, \\ y - 2z = 6e^{2x}, \end{cases}$$

and we get that the solution of systems (7) is a pair of functions

$$\begin{cases} y = \frac{4}{3}x^{1.5}e^{0.5x} + 10e^{2x}, \\ z = \frac{2}{3}x^{1.5}e^{0.5x} + 2e^{2x}. \end{cases}$$

4. Multiple Eigenvalue

In Example 3.1 we have managed to get two integrable combinations $y - 5z$ and $y - 2z$, because there were two different characteristic numbers.

And what to do in the case, where the characteristic equation has a multiple root?

Fortunately, our method gives possibility to find the solution of system (1) in this situation too. Let's prove it.

The characteristic equation of the system (1) $\Delta(\lambda) = 0$ has only one root, if $\Delta(\lambda) = (a - \lambda)(d - \lambda) - c \cdot b = (r - \lambda)^2$, where r is the value of the characteristic number. Opening the brackets and equating the coefficients with the same λ powers, we get that $r = (a + d)/2$.

Therefore, we can rewrite system (1):

$$\begin{cases} y' - \frac{a+d}{2}y = (a - \frac{a+d}{2})y + bz + f(x), \\ z' - \frac{a+d}{2}z = cy + (d - \frac{a+d}{2})z + g(x). \end{cases}$$

and

$$\begin{cases} y' - \frac{a+d}{2}y = \frac{a-d}{2}y + bz + f(x), \\ z' - \frac{a+d}{2}z = cy - \frac{a-d}{2}z + g(x). \end{cases}$$

The function $\Delta(\lambda)$ with $\lambda = (a+d)/2$ has the form

$$\begin{vmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{vmatrix}.$$

Since $\Delta(r) = 0$, rows of the determinant are proportional, consequently $c = -\frac{(a-d)^2}{4b}$.

Then system (1) can be rewritten

$$\begin{cases} y' - \frac{a+d}{2}y = \frac{a-d}{2}y + bz + f(x), \\ z' - \frac{a+d}{2}z = -\frac{(a-d)^2}{4b}y - \frac{a-d}{2}z + g(x). \end{cases} \tag{8}$$

Multiplying the 1st equation of system (8) by $\frac{a-b}{2}$, and adding the product to the 2nd equation multiplied by b, we get

$$\left(\frac{a-d}{2}y + bz\right)' - \frac{a+d}{2}\left(\frac{a-d}{2}y + bz\right) = \frac{a-d}{2}f(x) + bg(x). \tag{9}$$

The solution of equation (9) is some function $\frac{a-d}{2}y + bz = F(x)$.

The expression $\frac{a-d}{2}y + bz$ has place in the right part of the 1st equation of system (8), and in the right part of the 2nd equation of system (8) with the coefficient $\left(-\frac{a-d}{2b}\right)$.

Example 4.1. Let solve the system of equations

$$\begin{cases} y' = 5y - 7z + e^{-2x}/x, \\ z' = 7y - 9z. \end{cases}$$

The characteristic equation of the system

$$\begin{vmatrix} 5-\lambda & -7 \\ 7 & -9-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = 0$$

has multiple eigenvalue -2.

Then $\begin{cases} y' - (-2)y = 7y - 7z + e^{-2x}/x, \\ z' - (-2)z = 7y - 7z. \end{cases}$

Subtracting the 2nd equation from the 1st, we get

$$[(y-z)e^{2x}]'e^{-2x} = e^{-2x}/x.$$

We integrate it and get $y - z = (\ln x + C_1)e^{-2x}$.

The function $y - z$ stays in the right part of each equation of system. Therefore we will substitute the value of the function $y - z$ into the second equation and receive linear differential equation of the first order:

$z' + 2z = 7(\ln x + C_1)e^{-2x}$,

$(ze^{2x})'e^{-2x} = 7(\ln x + C_1)e^{-2x}$,

$(ze^{2x})' = 7(\ln x + C_1)$,

$z = 7(x\ln x - 1 + C_1x + C_2)e^{-2x}$.

To determine value of y we use expressions $y - z = (lnx + C_1)e^{-2x}$ and $z = 7(x\ln x - 1 + C_1x + C_2)e^{-2x}$:

$y = z + (y - z) = 7(x\ln x - 1 + C_1x + C_2)e^{-2x} + (\ln x + C_1)e^{-2x} = 7(x\ln x - 1 + \ln x + C_1(x+1) + C_2)e^{-2x}$.

5. Third Order System

The same result has place in the third order systems too:

Example 5.1. Let us solve the system of equations with initial conditions

$$\begin{cases} u' = 3u + v + w + 3e^{2x}x^{-0.5}, \quad u(0) = 1, \\ v' = -2u - 2w + 2019 \cdot 2020e^{2x}x^{2018}, \quad v(0) = 2, \\ w' = u + v + 3w + e^{3x}, \quad w(0) = 3. \end{cases} \tag{10}$$

First, construct and solve characteristic equation:

$$\begin{vmatrix} 3-k & 1 & 1 \\ -2 & -k & -2 \\ 1 & 1 & 3-k \end{vmatrix} = 0 \Rightarrow -k^3 + 6k^2 - 12k + 8 = 0 \Rightarrow k_1 = k_2 = k_3 = 2.$$

The root gives us the possibility to rewrite system (10):

$$\begin{cases} u' - 2u = u + v + w + 3e^{2x}x^{-0,5}, \\ v' - 2v = -2u - 2v - 2w + 2019 \cdot 2020e^{2x}x^{2018}, \\ w' - 2w = u + v + w + e^{3x}. \end{cases} \tag{11}$$

The sum of equations in (11) is a linear difference equation of the first order

$$(u + v + w)' - 2\,(u + v + w) = 3e^{2x}x^{-0.5} + 2019 \cdot 2020e^{2x}x^{2018} + e^{3x}.$$

We can easily find the solution of this equation by means of rewriting it in the following type

$$\left\{(u + v + w)\,e^{-2x}\right\}' e^{2x} = 3e^{2x}x^{-0.5} + 2019 \cdot 2020e^{2x}x^{2018} + e^{3x}.$$

We integrate it and, using the initial condition $u(0) + v(0) + w(0) = 6$, get $u + v + w = 6x^{0.5}e^{2x} + 2020x^{2019}e^{2x} + e^{3x} + 5e^{2x}$.

The function $u+v+w$ stays in the right part of each equation of system (11). Therefore we will substitute the value of the function and receive 3 linear differential equations of the first order:

$$\begin{cases} u' - 2u = 6e^{2x}x^{0.5} + 2020e^{2x}x^{2019} + e^{3x} + 5e^{2x} + 3e^{2x}x^{-0.5}, \\ v' - 2v = -2\left(6e^{2x}x^{0.5} + 2020e^{2x}x^{2019} + e^{3x} + 5e^{2x}\right) + 2019 \cdot 2020e^{2x}x^{2018}, \\ w' - 2w = 6e^{2x}x^{0.5} + 2020e^{2x}x^{2019} + e^{3x} + 5e^{2x} + e^{3x}. \end{cases}$$

Integrating them and using initial conditions we find the solution of problem (10):

$$\begin{cases} u = [4x^{1.5} + x^{2020} + e^{x} + 5x + 6x^{0.5}]e^{2x}, \\ v = [-8x^{1.5} - 2x^{2020} - 2e^{x} - 10x + 2010x^{2019} + 4]e^{2x}, \\ w = [4x^{1.5} + x^{2020} + 2e^{x} + 5x + 1]e^{2x}. \end{cases}$$

6. Systems with Variable Coefficients

The foregoing method can be also used in solving systems with variable coefficients. An example of this kind is stated below.

Example 6.1. Consider an initial value problem

$$\begin{cases} y' = 15y + 5xz + 15xe^{x^2/2+5x}, \quad y(0) = 21, \\ z' = 2y + (x + 5)z, \qquad z(0) = 1.2 \end{cases} \tag{12}$$

The characteristic equation of the system

$$\begin{vmatrix} 15-\lambda & -0.4 \\ 0.4 & x+5-\lambda \end{vmatrix} = \lambda^2 - (x+20)\lambda + 5x + 75 = 0$$

has the roots 5 and $x + 15$.

Using the first one, we rewrite the system as

$$\begin{cases} y' - 5y = 15y - 5y + 5xz + 15xe^{x^2/2+5x}, \\ z' - 5z = 2y + (x+5)z - 5z, \end{cases} \Rightarrow \begin{cases} y' - 5y = 10y + 5xz + 15xe^{x^2/2+5x}, \\ z' - 5z = 2y + xz, \end{cases}$$

Then we multiply 2nd equation by 5 and subtract the product from the 1st equation:

$$(y-5z)' - 5(y-5z) = 15xe^{x^2/2+5x}.$$

We integrate it and get

$$y - 5z = 15e^{x^2/2+5x} + C_1e^{5x}. \tag{13}$$

The initial conditions allow to find the value of C_1:

$$y(0) - 5z(0) = 15 + C_1 \Rightarrow 21 - 6 = 15 + C_1 \Rightarrow C_1 = 0.$$

To finish the process of solving, we will substitute the function

$y = 5z + 15e^{x^2/2+5x}$ in the right part of the 2nd equation of system (12) and solve it : $z' = 10z + 30e^{x^2/2+5x} + (x+5)z \Rightarrow$

$$\Rightarrow z' - (x+15)z = 30e^{x^2/2+5x} \Rightarrow z = -3e^{x^2/2+5x} + C_2e^{x^2/2+15x}.$$

Using initial value $z(0) = 1.2$ we find $C_2 : 1.2 = -3 + C_2 \Rightarrow C_2 = 4.2$.

Therefore,

$$z = -3e^{x^2/2+5x} + 4.2e^{x^2/2+15x},$$

and (from (13)) $y = 5z + 15e^{x^2/2+5x} \Rightarrow y = 21e^{x^2/2+15x}$.

7. Systems of Difference Equations

We have the same results for systems of linear difference equations. To illustrate this statement consider following examples.

Example 7.1. Let us solve the system of linear difference equations

$$\begin{cases} x_{n+1} = 0.7x_n + 0.25y_n + 0.1 \cdot 0.9^n\,, \\ y_{n+1} = -0.04x_n + 0.9y_n - 2\,, \end{cases} \tag{14}$$

with initial values $x_0 = 3$ and $y_0 = 2$.

The characteristic equation of system (14)

$$\begin{vmatrix} 0.7-k & 0,25 \\ -0.04 & 0.9-k \end{vmatrix} = 0 \Rightarrow (0.7-k)(0.9-k) - (-0.04)\cdot(0.25) = 0 \Rightarrow k^2 - 1.6k + 0.64 = 0$$

has the roots $k_1 = k_2 = 0.8$.

Using the eigenvalue 0.8, we rewrite system (14) as

$$\begin{cases} x_{n+1} - 0.8x_n = -0.1x_n + 0.25y_n + 0.1 \cdot 0.9^n\,, \\ y_{n+1} - 0.8y_n = -0.04x_n + 0.1y_n - 2\,, \end{cases} \tag{15}$$

Multiply the 2nd equation of system (15) by 2.5 and subtract from the 1st:

$$(x_{n+1} - 2.5y_{n+1}) = 0.8(x_n - 2.5y_n) + 0.1 \cdot 0.9^n + 5.$$

The result is the linear difference equation with the solution:

$$x_n - 2.5y_n = -2\,(0.8)^n + 5\frac{1-(0.8)^n}{1-(0.8)} + 0.1\frac{(0.9)^n - (0.8)^n}{0.9 - 0.8}.$$

We obtain the coefficient (-2) from initial values:

$$x_0 - 2.5y_0 = 3 - 2.5 \cdot 2 = -2.$$

After collecting homogenous terms we get

$$x_n - 2.5y_n = 25 - 28 \cdot (0.8)^n + 0.9^n. \tag{16}$$

Because the characteristic equation of system (14) has equal roots, we know that expression $x_n - 2.5y_n$ has place in the right parts of equations of system (15). It is easily seen in the right part of the 1st equation with coefficient (-0.1).

Using this fact we get an equation

$$x_{n+1} - 0.8x_n = -0.1[25 - 28(0.8)^n + 0.9^n] + 0.1 \cdot 0.9^n \Rightarrow\; x_{n+1} - 0.8x_n = -2.5 + 2.8 \cdot (0.8)^n. \tag{17}$$

The solution of (17) is

$$x_n = 3\,(0.8)^n - 2.5\frac{1-(0.8)^n}{1-(0.8)} + 2.8n \cdot 0.8^{n-1} = 3\,(0.8)^n - 12.5 + 12.5 \cdot (0.8)^n + 3.5n\,(0.8)^n = [15.5 + 3.5n]\,(0.8)^n - 12.5.$$

To find the y_n we use (16):

$$\begin{aligned}
&x_n - 2.5y_n = 25 - 28 \cdot (0.8)^n + 0.9^n \;\Rightarrow\; 2.5y_n = x_n - [25 - 28 \cdot (0.8)^n + 0.9^n] \Rightarrow\\
&\Rightarrow 2.5y_n = [15.5 + 3.5n](0.8)^n - 12.5 - 25 + 28 \cdot (0.8)^n - 0.9^n \;\Rightarrow\; 2.5y_n = [43.5 + 3.5n](0.8)^n - 37.5 - 0.9^n\\
&\Rightarrow y_n = [17.4 + 1.4n](0.8)^n - 15 - 0.4(0.9)^n.
\end{aligned}$$

8. Samuelson's Model

Example 8.1. Let Y_t be a national income, C_t is consumption, I_t is investment.

Consider a classic version of the Samuelson's model

$$Y_t = C_t + I_t, \tag{18}$$

$$C_{t+1} = aY_t + b, \tag{19}$$

$$I_{t+1} = c(C_{t+1} - C_t), \tag{20}$$

where a and c are constant.

A standard approach to solving system (18) – (20) is to reduce it to a second order difference equation

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b.\ (t = 0, 1, \ldots)$$

In contrast to this approach, we will use systems. The use of systems can cover a wider class of problems.

We reduce the system of equations (18) –(20) to the system of two linear difference equations of the first order.

To do this we substitute the value Y_t from equation (18) in equation (19) and obtain an equation

$$C_{t+1} = a_t + aI_t + b. \tag{21}$$

Then we substitute the value C_{t+1} from equation (21) in equation (20):

$$I_{t+1} = c(a_t + aI_t + b - C_t).$$

Collecting similar terms, we get a system of equations concerning C_t and I_t:

$$\begin{cases} C_{t+1} = aC_t + aI_t + b, \\ I_{t+1} = c(a-1)C_t + caI_t + cb. \end{cases} \tag{22}$$

Assume that the coefficient a, which expresses the marginal propensity to consume is 0.9, *and the value of investments at any moment of time exceeds the increase in consumption two times:* $c = 2$.

Let's also assume that the value of autonomous consumption of b is 10, *consumption in the initial time* (C_0) *is* 70, *the value of investments* (I_0) *is* 10.

Then, according to (22) changes in consumption and investment are described by system

$$\begin{cases} C_{t+1} = 0.9C_t + 0.9I_t + 10, \\ I_{t+1} = 2(0.9-1)C_t + 0.9 \cdot 2I_t + 2 \cdot 10. \end{cases} \tag{23}$$

We will begin the solution of system (23) with the formulation and solution of the characteristic equation:

$$\begin{vmatrix} 0.9-k & 0.9 \\ -0.2 & 1.8-k \end{vmatrix} = 0 \Rightarrow (0.9-k)(1.8-k) - 0.9 \cdot (-0.2) = 0 \Rightarrow k^2 - 2.7k + 1.8 = 0 \Rightarrow k_1 = 1.5; k_2 = 1.2.$$

We use the root of the characteristic equation $k_1 = 1.5$ to rewrite system (23) in the form

$$\begin{cases} C_{t+1} - 1.5C_t = 0.9C_t - 1.5C_t + 0.9I_t + 10, \\ I_{t+1} - 1.5I_t = -0.2C_t + 1.8I_t - 1.5I_t + 20. \end{cases}$$

Then

$$\begin{cases} C_{t+1} - 1.5C_t = -0.6C_t + 0.9I_t + 10, \\ I_{t+1} - 1.5I_t = -0.2C_t + 0.3I_t + 20. \end{cases} \tag{24}$$

Multiply the 2nd equation of system (24) by 3 and subtract from the 1st:

$$(C_{t+1} - 3I_{t+1}) - 1.5(C_t - 3I_t) = -50.$$

This equation is a linear difference equation of the first order and its solution is

$$C_t - 3I_t = 40\,(1.5)^t - 50\frac{1-(1.5)^t}{1-(1.5)} = 100 - 60(1.5)^t. \tag{25}$$

Integer 40 is the initial condition $C_0 - 3I_0$: $\quad 70 - 3 \cdot 10 = 40$.

We repeat the procedure by taking 2nd root of the characteristic equation $k_2 = 1.2$:

Rewrite system (23) in the form

$$\begin{cases} C_{t+1} - 1.2C_t = -0.3C_t + 0.9I_t + 10, \\ I_{t+1} - 1.2I_t = -0.2C_t + 0.6I_t + 20. \end{cases} \tag{26}$$

Multiply the 2nd equation of system (26) by 1.5 and subtract from the 1st:

$$(C_{t+1} - 1.2I_{t+1}) - 1.5(C_t - 3I_t) = -20.$$

The solution of this equation with initial condition

$C_0 - 1.5I_0 = 70 - 1.5 \cdot 10 = 55$ has the form

$$C_t - 1.5I_t = 55\,(1.2)^t - 20\frac{1-(1.2)^t}{1-(1.2)} = 100 - 45(1.2)^t. \tag{27}$$

The equations (25) and (27) constitute a system, from which we can find C_t and I_t :

$$\begin{cases} C_t - 1.5I_t = 100 - 45\,(1.2)^t\,, \\ C_t - 3I_t = 100 - 60\,(1.5)^t\,. \end{cases} \tag{28}$$

We subtract the 2nd equation from the 1st in system (28):

$$1.5I_t = -45(1.2)^t + 60(1.5)^t,$$

and dividing by 1.5, we find that

$$I_t = -30(1.2)^t + 40(1.5)^t.$$

Substituting the value of I_t in (25), we obtain

$$C_t = 100 - 60(1.5)^t + 3[40(1.5)^t - 30(1.2)^t] = 100 + 60(1.5)^t - 90(1.2)^t.$$

Referring to (18) we obtain the function describing the change in GDP:

$$Y_t = C_t + I_t = [100 + 60(1.5)^t - 90(1.2)^t] + [-30(1.2)^t + 40(1.5)^t] = 100 + 100(1.5)^t - 120(1.2)^t.$$

Function Y_t is growing, and it is not difficult to understand that the critical issue is the value of the coefficient $c = 2$.

9. Conclusion

Traditionally, when studying differential equations, at first, the general theory of linear equations is constructed. Then, on the basis of this theory, methods for solving equations with constant coefficients are studied. The method that we propose allows us to proceed in a more natural way, starting with equations with constant coefficients. An additional advantage: our method makes it possible to simplify the process of integrating some systems of differential equations. To verify this, try to solve the systems (7) and (10) by traditional methods.

References

[1] W.E. Boyce, R.C. DiPrima, Elementary Differential Equations and Boundary Value Problems, Wiley, New York, 1986.

[2] M.M. Guterman, Z.H. Nitecki, Differential Equations. A Fist Course, Saunders College Publishing, Philadelphia, 1988.

[3] S.K. Kydyraliev, A.B. Urdaletova, Solving linear differential equations by operator factorization, The College Math. J. 27 (1996) 199–203.

[4] S.K. Kydyraliev, A.B. Urdaletova, Introduction to Linear Difference and Differential Equations, Economics University, Bishkek, 2000.

[5] N.M. Matveev, Methods of Integrating Ordinary Differential Equations, Higher School, Minsk, 1974.