Filomat 33:5 (2019), 1463–1469
https://doi.org/10.2298/FIL1905463C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

$\mathbb{R}$-Compact Uniform Spaces in the Category $ZUnif$

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Abstract. A number of basic properties of $\mathbb{R}$-compact spaces in the category $Tych$ of Tychonoff spaces and their continuous mappings are extended to the category $ZUnif$ of uniform spaces with the special normal bases and their *coz*-mappings.

1. Introduction

E. Hewitt introduced the class of $\mathbb{R}$-compact (Tychonoff) spaces [19]. That class was independently defined by L. Nachbin [24] in terms of uniformities. The important topological and uniform properties of $\mathbb{R}$-compact spaces are established in works of T. Shirota [25] and S. Mrówka [22]. From a categorical point of view $\mathbb{R}$-compact spaces coincide with epi-reflective hull of the real line in the category $Tych$ of Tychonoff spaces and their continuous mappings [12, 18]. Various problems of the theory of $\mathbb{R}$-compact spaces are investigated in the books [4, 10, 15, 27]. Spectral Theorem for $\mathbb{R}$-compact spaces is given in [9].

$\mathbb{R}$-compact extensions over the special bases (separating nest-generated intersection ring (s.n.-g.i.r.) or strong delta normal base) have been investigated in [2, 3, 16, 26]. For any uniform space uX the set $\mathcal{Z}_u$ of zero-sets of all uniformly continuous functions forms s.n.-g.i.r. or strong delta normal base [4]. It is naturally arisen the category $ZUnif$, whose objects are uniform spaces uX with base $\mathcal{Z}_u$ and morphisms are *coz*-mappings (where a mapping $f : uX \to vY$ between uniform spaces uX and vY is *coz*-mapping, if $f^{-1}(\mathcal{Z}_v) \subset \mathcal{Z}_u$) [8, 14]. The Wallman-Shanin compactification $\beta_u X = \omega(X, \mathcal{Z}_u)$ and the Wallman-Shanin realcompactification $\upsilon_u X = \upsilon(X, \mathcal{Z}_u)$ both are defined over the base $\mathcal{Z}_u$ [8]. In the category $ZUnif$ a uniform space uX is $\mathbb{R}$-compact if $X = \upsilon_u X$. The category $Tych$ is a full subcategory of $ZUnif$.

In this work it is shown that a number of basic properties of $\mathbb{R}$-compact spaces in the category $Tych$ can be extended to the category $ZUnif$.

2. Preliminaries and Notations

Assume $\mathbb{R}$ is the real line with the ordinary metric $\rho(x, y) = |x - y|$ and the uniformity $u_{\mathbb{R}}$ generated by the metric ρ, $\mathbb{N}$ is the set of natural numbers, $I = [0, 1]$ is the unit segment with the metric and uniformity induced from $\mathbb{R}$. If $f : X \to Y$ is a mapping and $F \subset X$, then $f|_F : F \to Y$ is the restriction of f on F. If $Y = \mathbb{R}$, then a mapping $f : X \to \mathbb{R}$ is a function, where $Z(f) = f^{-1}(0)$ and $X \setminus Z(f) = f^{-1}(\mathbb{R} \setminus \{0\})$. $\mathbb{R}^X$ is the set of

2010 *Mathematics Subject Classification*. Primary 54E15; Secondary 54C20, 54C45, 54D30, 54D35, 54D60, 54H13
Keywords. $\mathbb{R}$-compactum, $\mathbb{R}$-compact space, *coz*-mapping, *coz*-function, z_u-ultrafilter, CIP z_u-ultrafilter
Received: 28 September 2018; Revised: 06 January 2019; Accepted: 21 January 2019
Communicated by Fahreddin Abdullayev
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all functions from X into $\mathbb{R}$. If $A \subset \mathbb{R}^X$ and $F \subset X$, then $\mathcal{Z}(A) = \{Z(f) : f \in A\}$ and $A|_F = \{f|_F : f \in A\}$. For a system $\mathcal{F} = \{F_s\}_{s\in S}$ of sets $\bigcup\mathcal{F} = \bigcup_{s\in S} F_s$ is the union and $\bigcap\mathcal{F} = \bigcap_{s\in S} F_s$ is the intersection of elements from $\mathcal{F}$. For systems $\mathcal{F}$ and $\mathcal{F}'$ their inner intersection is $\mathcal{F} \wedge \mathcal{F}' = \{F \cap F' : F \in \mathcal{F}, F' \in \mathcal{F}'\}$. If $\mathcal{F}' = \{X\}$, then $\mathcal{F} \wedge X = \{F \cap X : F \in \mathcal{F}\}$.

All spaces are assumed to be Tychonoff and for any compactum we use its unique uniformity. Denote by $Tych$ the category of Tychonoff spaces and their continuous mappings. For a space $X \in Tych$ denote by $C(X)$ ($C^*(X)$) the set of all (bounded) continuous functions on X. We will assume $\mathcal{Z}(C(X)) = \mathcal{Z}(X)$. Elements of $\mathcal{Z}(X)$ are called *zero-sets* and elements of $C\mathcal{Z}(X) = \{X \setminus Z : Z \in \mathcal{Z}(X)\}$ are called *cozero-sets*. A uniform space uX is a Tychonoff space X with a uniformity u on it. Uniformities are given by uniform coverings [20]. If uX is a uniform space and $Y \subset X$, then $u|_Y$ is the restriction of the uniformity u on Y and $[Y]_X$ is the closure of Y in X. For a uniform space uX we denote by $U(uX)$ ($U^*(uX)$) the set of all (bounded) uniformly continuous functions on uX. Then $\mathcal{Z}(U(uX)) = \mathcal{Z}_u$ is the set of all *u-zero-sets* and the family $C\mathcal{Z}_u = \{X \setminus Z : Z \in \mathcal{Z}_u\}$ is the set of all *u-cozero-sets*. A covering consisting of cozero-sets (u-cozero-sets) is called *cozero-covering* (*u-cozero-covering*). The set $\mathcal{Z}_u$ forms on uX a base of closed sets of the uniform topology [5] and this base is a separating nest-generated intersection ring (s.n-g.i.r.) [6]. That base is defined in [26] and it is a normal base in the sense of [13]. The mapping $f : uX \to vY$ between uniform spaces uX and vY is called *coz-mapping*, if $f^{-1}(\mathcal{Z}_v) \subset \mathcal{Z}_u$ or $f^{-1}(C\mathcal{Z}_v) \subset C\mathcal{Z}_u$ [14]. All uniform spaces and *coz*-mappings form the category $ZUnif$ [14]. Objects uX and vY in $ZUnif$ are called *coz-homeomorphic*, if there exists a bijective *coz*-mapping $f : uX \to vY$ such that the inverse mapping $f^{-1} : vY \to uX$ is a *coz*-mapping. Every Tychonoff space X with the fine uniformity u_f is an element of $ZUnif$ and every continuous mapping $f : X \to Y$ is uniformly continuous $f : u_fX \to v_fY$ with respect to the fine uniformities u_f and v_f on X and Y, respectively. Since $\mathcal{Z}_{u_f} = \mathcal{Z}(X)$ and $\mathcal{Z}_{v_f} = \mathcal{Z}(Y)$, then f is a *coz*-mapping. Hence, the category $Tych$ is a full subcategory of $ZUnif$.

In the case $Y = \mathbb{R}$, the *coz*-mapping $f : uX \to \mathbb{R}$ is called *coz-function*. The set of all *coz*-functions on uX is denoted by $C(uX)$ and the set of all bounded *coz*-functions on uX is denoted by $C^*(uX)$. It is clear that $U(uX) \subset C(uX) \subset C(X)$ ($U^*(uX) \subset C^*(uX) \subset C^*(X)$). We note that $\mathcal{Z}_u = \mathcal{Z}(C(uX))$ [6].

A filter $\mathcal{F}$ over the base $\mathcal{Z}_u$ is called *z_u-filter*. A z_u-filter $\mathcal{F}$ is a *prime z_u-filter* if $Z \cup Z' \in \mathcal{F}$ implies either $Z \in \mathcal{F}$ or $Z' \in \mathcal{F}$, where Z and Z' are members of $\mathcal{Z}_u$. If $\mathcal{F}$ is a prime z_u-filter and if $x \in X$, then the point x is a cluster point of $\mathcal{F}$ if and only if z_u-filter $\mathcal{F}$ converges to x ($\equiv \bigcap\{Z : Z \in \mathcal{F}\} = \{x\}$) [4].

The Wallman-Shanin (WS-) compactification $\omega(X, \mathcal{Z}_u)$ of a uniform space uX is a β-like compactification [23] and is denoted by $\beta_uX = \omega(X, \mathcal{Z}_u)$. Points of β_uX are all maximal centered systems of elements of the base $\mathcal{Z}_u$ (further *z_u-ultrafilters*) and β_uX is endowed with the Wallman-Shanin (WS-)topology [1]. The compactification β_uX is an epi-reflective functor $\beta_u : uX \to \beta_uX$, that is *coz*-homeomorphic embedding. Compacta in the category $ZUnif$ are precisely elements of epi-reflective hull $\mathfrak{L}([0, 1])$ of the unit segment in $ZUnif$ [8].

The following is a characterization of WS-β-like compactifications.

Theorem 2.1. *For every uniform space uX there exists exactly one (up to a homeomorphism) β-like compactification β_uX with equivalent properties:*

(I) *Every coz-mapping f from uX into a compactum K has a continuous extension β_uf from β_uX into K.*
(II) *uX is C^*_u-embedded into β_uX.*
(III) *β_uX is a completion of X with respect to the uniformity u^z_p.*
(IV) *For any finite family $\{Z_n\}^k_{n=1}$ of u-zero-sets if $\cap^k_{n=1}Z_n = \emptyset$, then $\cap^k_{n=1}[Z_n]_{\beta_uX} = \emptyset$.*
(V) *For any finite family $\{Z_n\}^k_{n=1}$ of u-zero-sets $[\cap^k_{n=1}Z_n]_{\beta_uX} = \cap^k_{n=1}[Z_n]_{\beta_uX}$.*
(VI) *Distinct z_u-ultrafilters on uX have distinct limits in β_uX.*

In the above formulated theorem a uniform space uX is *C^*_u-embedded* into a uniform space vY if X is topologically a subspace of Y and $C^*(vY)|_X = C^*(uX)$, i.e. each bounded *coz*-function on uX can be extended to a bounded *coz*-function on vY [7], the uniformity u^z_p on X has a base of all finite u-cozero-coverings [8].

Compact uniform spaces in the category $ZUnif$ have the next characterizations.

Corollary 2.2. *For a uniform space uX the following are equivalent:*

(1) uX *is a compactum in* $ZUnif$.
(2) X *is complete with respect to the uniformity* u_p^z.
(3) $X = \beta_u X$.
(4) uX *is coz-homeomorphic to the closed uniform subspace of a power of* I.

All z_u-ultrafilters with CIP (countable intersection property) are the part $\upsilon_u X$ of the compactification $\beta_u X$. The *Wallman-Shanin (WS-) realcompactification* is the set $\upsilon_u X$ with the topology induced from the compactum $\beta_u X$ topology [26]. Moreover, the WS-realcompactification $\upsilon_u X$ is an epi-reflective functor $\upsilon_u : uX \to \upsilon_u X$, that is *coz*-homeomorphic embedding [8]. Realcompacta in the category $ZUnif$ are precisely elements of the epi-reflective hull $\mathfrak{L}(\mathbb{R})$ of the real line $\mathbb{R}$ in $ZUnif$ [8]. The following characterizations of the WS-realcompactification take place.

Theorem 2.3. *For every uniform space* uX *there exists exactly one (up to a coz-homeomorphism) realcompact space* $\upsilon_u X$ *contained in the* β*-like compactification* $\beta_u X$ *with equivalent properties:*

(I) *Every coz-mapping* f *from* uX *into a* $\mathbb{R}$-z_v*-complete uniform space* vR *has an extension to a coz-mapping* $\hat{f}$ *from* $\upsilon_u X$ *into* vR.
(II) *Every coz-mapping* f *from* uX *into a separable metric uniform space* $u_\rho M$ *has an extension to a coz-mapping* $\hat{f}$ *from* $\upsilon_u X$ *into* $u_\rho M$.
(III) $\upsilon_u X$ *is a completion with respect to the uniformity* u_ω^z.
(IV) uX *is* C_u*-embedded into* $\upsilon_u X$.
(V) $\upsilon_u X$ *is a completion with respect to the uniformity* u_c^z.
(VI) *For any countable family* $\{Z_n\}_{n\in\mathbb{N}}$ *of u-zero-sets if* $\cap_{n\in\mathbb{N}} Z_n = \emptyset$, *then* $\cap_{n\in\mathbb{N}}[Z_n]_{\upsilon_u X} = \emptyset$.
(VII) *For any countable family* $\{Z_n\}_{n\in\mathbb{N}}$ *of u-zero-sets* $\cap_{n\in\mathbb{N}}[Z_n]_{\upsilon_u X} = [\cap_{n\in\mathbb{N}} Z_n]_{\upsilon_u X}$.
(VIII) *Every point of* $\upsilon_u X$ *is a limit of unique countably centered* z_u*-ultrafilter on* uX.

Remind, that a uniform space vR is called $\mathbb{R}$-z_v*-complete* if every CIP z_v-ultrafilter converges.

In the above theorem a uniform space uX is C_u*-embedded* into a uniform space vY if X is topologically a subspace of Y and $C(vY)|_X = C(uX)$, i.e. each *coz*-function on uX can be extended to a *coz*-function on vY [7], the uniformity u_ω^z on X has a base of all countable u-cozero-coverings and the uniformity u_c^z is weak generated by $C(uX)$ [8].

A uniform space uX is called $\mathbb{R}$*-compactum in* $ZUnif$, if $X = \upsilon_u X$. It follows immediately from Theorem 2.3 that $X = \upsilon_u X$ if and only if uX is *coz*-homeomorphic to some closed uniform subspace of $\mathbb{R}^{C(uX)}$, that is C_u-embedded into $\mathbb{R}^{C(uX)}$.

The following corollaries are immediate consequences of Theorem 2.3.

Corollary 2.4. *The WS-realcompactification* $\upsilon_u X$ *of a uniform space* uX *is the largest subspace of the* β*-like compactification* $\beta_u X$ *such that* uX *is* C_u*-embedded into it and* $\upsilon_u X$ *is the smallest* $\mathbb{R}$*-compactum between* X *and* $\beta_u X$.

Corollary 2.5. *For a uniform space* uX *the following are equivalent:*

(1) uX *is* $\mathbb{R}$*-compactum in* $ZUnif$.
(2) X *is complete with respect to the uniformity* u_ω^z.
(3) X *is complete with respect to the uniformity* u_c^z.
(4) $X = \upsilon_u X$.
(5) uX *is coz-homeomorphic to a closed uniform subspace of a power of* $\mathbb{R}$.

Further, for simplicity, $\mathbb{R}$-compact in the category $ZUnif$ of uniform spaces will be called $\mathbb{R}$*-compactum*, and $\mathbb{R}$-compact in the category $Tych$ Tychonoff spaces will be called $\mathbb{R}$*-compact space*.

As $\mathbb{R}$-compacta are elements of $\mathfrak{L}(\mathbb{R})$ in $ZUnif$, then from [12, 18] it follows:

Proposition 2.6. ([8]) *A closed subspace of an* $\mathbb{R}$*-compactum, product of any family of* $\mathbb{R}$*-compacta, intersection of any family of* $\mathbb{R}$*-compacta is an* $\mathbb{R}$*-compactum.*

We note that the intersection $\mathfrak{L}(\mathbb{R}) \cap Tych$ in $ZUnif$ coincides with the class of $\mathbb{R}$-compact Tychonoff spaces.

In this paper the most of properties of $\mathbb{R}$-compact spaces in the category $Tych$ are extended on the category $ZUnif$.

3. Properties of $\mathbb{R}$-Compacta

Proposition 3.1. *A metrizable space with a countable base is $\mathbb{R}$-compactum with respect to the metric uniformity.*

Proof. Let X be a metrizable space with a countable base and d be a metric which generates the topology on X. Denote by u the uniformity induced by d. Then $\mathcal{Z}_u = \mathcal{Z}(X)$. Since (X, d) is a paracompactum with a countable base, then the fine uniformity u_f of the space X consists of all countable cozero-coverings. Then any CIP z_u-ultrafilter p is a Cauchy filter with respect to the fine uniformity u_f. It is therefore evident that $u \subset u_f$, hence p is a Cauchy filter with respect to the metric uniformity u. Any metric space is weakly complete ($\equiv$ any CIP Cauchy filter converges [21]). Hence, $\cap p \neq \emptyset$. Thus, X is an $\mathbb{R}$-compactum with respect to the metric uniformity. □

Corollary 3.2. *Any metrizable space with a countable base is an $\mathbb{R}$-compact space.*

Proof. A metrizable space with a countable base is weakly complete with respect to the metric uniformity. Hence the fine uniformity is weakly complete, therefore every CIP z-ultrafilter has nonempty intersection. □

Remind that a *Polish* space is a complete metrizable space with a countable base [9].

Corollary 3.3. *Any Polish space is an $\mathbb{R}$-compactum with respect to the metric uniformity.*

Theorem 3.4. *Let uX be an $\mathbb{R}$-compactum and $f : uX \to vY$ be a coz-mapping between uniform spaces uX and vY. If $F \subset Y$ and F is an $\mathbb{R}$-compactum with respect to the uniformity $v|_F$, then $f^{-1}(F) = N$ is an $\mathbb{R}$-compactum with respect to the uniformity $u|_N$.*

Proof. Assume $u' = u|_N$, $v' = v|_F$, and $g = f|_N : N \to F$ (we note that g is a *coz*-mapping). Let p be an arbitrary $z_{u'}$-ultrafilter on N over the base $\mathcal{Z}_{u'}$. Then the families $\xi = \{Z \in \mathcal{Z}_u : Z \cap N \in p\}$ and $g^\sharp(p) = \{Z \in \mathcal{Z}_{v'} : g^{-1}(Z) \in p\}$ are prime CIP z_u- and $z_{v'}$-filters on X and F, respectively [27]. So there exist $x \in \cap\xi$ and $y \in \cap g^\sharp(p)$. We show that $x \in N$ and $x \in \cap p$.

Suppose that $x \notin N$. Then $f(x) \notin F$. Hence, $y \neq f(x)$. Therefore in the base $\mathcal{Z}_v$ there exist zero-set neighborhoods $f(x) \in Z$ and $y \in Z'$ such that $Z \cap Z' = \emptyset$. Since $y \in \cap g^\sharp(p)$, then $Z' \cap F \in g^\sharp(p)$, i.e. $g^{-1}(Z' \cap F) = g^{-1}(Z') \cap N \in p$. The preimage $f^{-1}(Z)$ is a zero-set neighborhood of x, hence $f^{-1}(Z) \in \xi$, i.e. $f^{-1}(Z) \cap N \in p$. Since $g^{-1}(Z') \cap N = f^{-1}(Z') \cap N$, then $(f^{-1}(Z) \cap N) \cap (f^{-1}(Z') \cap N) \in p$. Hence, from $f^{-1}(Z) \cap f^{-1}(Z') \neq \emptyset$, we have a contradiction, $Z \cap Z' \neq \emptyset$. Thus, $x \in N$.

Now suppose that $x \notin \cap p$. Then there exists $Z \in p$ such that $x \notin Z$. Since $[Z]_N = [Z]_X \cap N$ and $x \in N$, then $x \notin [Z]_X$. Hence there is a zero-set neighborhood $Z' \in \mathcal{Z}_u$ such that $x \in Z'$ and $Z' \cap [Z]_X = \emptyset$. Moreover, $Z \cap Z' = \emptyset$. Further, since $x \in Z' \cap N$, then $Z' \in \xi$. Then $Z' \cap N \in p$. Hence, $Z \cap (Z' \cap N) \neq \emptyset$, i.e. $Z \cap Z' \neq \emptyset$, which is a contradiction. Thus, $x \in \cap p$. □

Corollary 3.5. *Let uX be an $\mathbb{R}$-compactum and G be a u-cozero-set in X. Then G is an $\mathbb{R}$-compactum with respect to the uniformity $u|_G$.*

Proof. Since G is a u-cozero-set in uX, there exists a function $f \in U(uX)$, $f : uX \to \mathbb{R}$ such that $G = f^{-1}(\mathbb{R} \setminus \{0\})$. According to Proposition 3.1, $\mathbb{R} \setminus \{0\}$ is an $\mathbb{R}$-compactum because it is a metric space with a countable base. Hence, by Theorem 3.4, G is an $\mathbb{R}$-compactum with respect to the uniformity $u|_G$. □

Corollary 3.6. ([9]) *Let X be an $\mathbb{R}$-compact space and G be a cozero-set in X. Then G is an $\mathbb{R}$-compact space.*

Proof. Since G is a cozero-set in X, then there exists a function $f \in C(X)$, $f : X \to \mathbb{R}$ such that $G = f^{-1}(\mathbb{R} \setminus \{0\})$. According to Proposition 3.1 $\mathbb{R} \setminus \{0\}$ is an $\mathbb{R}$-compact space because it is a metric space with a countable base. Hence, by Theorem 3.4, G is an $\mathbb{R}$-compactum with respect to the uniformity $u_f|_G$. Then moreover G is an $\mathbb{R}$-compact space. □

The next theorem (and its corollary) is a generalization of results from [15, 8.10(a)].

Theorem 3.7. *If uX is C_u-embedded into vY, then $[X]_{\upsilon_v Y} = \upsilon_u X$.*

Proof. If uX is C_u-embedded into vY, then uX is C_u-embedded into $\upsilon_v Y$ [7]. Since $[X]_{\upsilon_v Y}$ is a subspace of $\upsilon_v Y$, then uX is C_u-embedded into $[X]_{\upsilon_v Y}$. Because $[X]_{\upsilon_v Y}$ is an $\mathbb{R}$-compactum as closed subspace of the $\mathbb{R}$-compactum $\upsilon_u Y$, we have $[X]_{\upsilon_v Y} = \upsilon_u X$ (see Proposition 2.6, Corollary 2.4). □

Corollary 3.8. ([15]) *Let uX and vY be $\mathbb{R}$-compacta and uX is C_u-embedded into vY. Then X is closed in Y.*

Proof. By the condition $X = \upsilon_u X$ and $Y = \upsilon_v Y$. Then $[X]_Y = [X]_{\upsilon_v Y} = \upsilon_u X = X$. □

Definition 3.9. ([22]) A subset F of a space X is said to be *G_δ-closed*, if for each $x \notin F$ there exists a G_δ-subset G such that $x \in G$ and $G \cap F = \emptyset$. *G_δ-closure* of F is the set of all $x \in X$ which satisfy the condition: whenever G is G_δ-set containing x, then $G \cap F \neq \emptyset$ and the G_δ-closure of F is denoted by $G_\delta - cl_X F$. A subspace F is said to be *G_δ-dense* in X, if $X = G_\delta - cl_X F$, i.e. if each G_δ-set in X meets F.

Theorem 3.10. *Every G_δ-closed subset F of an $\mathbb{R}$-compactum uX is an $\mathbb{R}$-compactum with respect to any uniformity v on F such that $\mathcal{Z}_u \wedge F \subseteq \mathcal{Z}_v$.*

Proof. Let F be a G_δ-closed subset in X and v be a uniformity on F such that $\mathcal{Z}_u \wedge F \subseteq \mathcal{Z}_v$. Let p be an arbitrary z_v-ultrafilter over the base $\mathcal{Z}_v$. Let $\xi = \{Z \in \mathcal{Z}_u : Z \cap F \in p\}$. It is easy to check that ξ is a prime CIP z_u-filter on uX. Hence, ξ is contained in the unique CIP z_u-ultrafilter q on uX [27] and since uX is an $\mathbb{R}$-compactum, $\{x\} = \cap q \subseteq \cap \xi$. We show that $x \in F$ and $x \in \cap p$.

Assume, to the contrary, $x \notin F$. Since F is G_δ-closed in X, then there is a G_δ-set $G = \cap_{i\in\mathbb{N}} O_i$ such that $x \in G$ and $G \cap F = \emptyset$. By properties of the base $\mathcal{Z}_u$ [7] there are zero-set neighborhoods $Z_i \in \mathcal{Z}_u$ $(i \in \mathbb{N})$ such that $x \in Z_i \subset O_i$. Since the prime z_u-filter ξ converges to x, then $Z_i \in \xi$ for all $i \in \mathbb{N}$. If we suppose that $Z_i \cap F \neq \emptyset$ for all $i \in \mathbb{N}$, then $Z_i \cap F \in p$ $(i \in \mathbb{N})$. Hence we have a contradiction $\bigcap_{i\in\mathbb{N}}(Z_i \cap F) = (\bigcap_{i\in\mathbb{N}} Z_i) \cap F \neq \emptyset$, since p is a prime CIP z_v-ultrafilter on F. On the other hand, $\bigcap_{i\in\mathbb{N}} Z_i \subset G$ and $G \cap F = \emptyset$. Thus, there exists an index $k \in \mathbb{N}$ such that $x \in Z_k$ and $Z_k \cap F = \emptyset$. But $Z_k \in \xi$, hence $Z_k \cap F \in p$, which is impossible. Thus, $x \in F$.

Suppose that $x \notin \cap p$. Then there exists $Z \in p$ such that $x \notin Z$. Since $[Z]_F = [Z]_X \cap F$ and $x \in F$, then $x \notin [Z]_X$. Then there is a zero-set neighborhood $Z' \in \mathcal{Z}_u$ such that $x \in \mathcal{Z}'_u$ and $Z' \cap Z = \emptyset$. It is evident that $Z' \in \xi$. Therefore $Z' \cap F \in p$ implies $(Z' \cap F) \cap Z \in p$ and $Z' \cap Z \neq \emptyset$, which is a contradiction. Thus, $x \in \cap p$ and F is an $\mathbb{R}$-compactum with respect to the uniformity v. □

Corollary 3.11. ([22]) *Every G_δ-closed subspace of an $\mathbb{R}$-compact space is also an $\mathbb{R}$-compact space.*

Proof. It follows, as we noted above, from the fact that an $\mathbb{R}$-compact space X is an $\mathbb{R}$-compactum with respect to the fine uniformity u_f or over the base $\mathcal{Z}(X)$. □

Theorem 3.12. *The following are equivalent:*

(I) *uX is an $\mathbb{R}$-compactum.*

(II) *For any $y \in \beta_u X \setminus X$ there exists a continuous function $h : \beta_u X \to I$ such that $h(y) = 0$ and $h(x) > 0$ for all $x \in X$.*

Proof. *(I)* ⇒ *(II)*. Since $X = \upsilon_u X$, then there exists a unique z_u-ultrafilter p_y without CIP, that converges to $y \in \beta_u X \setminus X$. Then there exists a sequence $\{Z_i\}_{i\in\mathbb{N}} \subset p_y$ such that $\bigcap_{i\in\mathbb{N}} Z_i = \emptyset$. We can assume that $Z_i = Z(g_i)$, where $g_i : uX \to I$ is a *coz*-function $(i \in \mathbb{N})$. Therefore, by the properties of $C(uX)$ [6], it follows that $g = \Sigma_{i\in\mathbb{N}}(g_i/2^i) : uX \to \mathbb{R}$ is a *coz*-function and $Z(g) = \bigcap_{i\in\mathbb{N}} Z_i$, i.e. $Z(g) = \emptyset$. For each $i \in \mathbb{N}$, $g_i(x) > 0$ for all $x \in X$, hence $g(x) \neq 0$ for all $x \in X$ and g cannot be *coz*-extendable to $Y = X \cup \{y\}$ with respect to the uniformity induced from the compactum $\beta_u X$. Suppose g has a *coz*-extension $\tilde{g} : Y \to \mathbb{R}$. For each $i \in \mathbb{N}$ it takes place $[Z_i]_Y = Z_i \cup \{y\}$. Because $\tilde{g}$ is continuous, then we have $\tilde{g}(y) = \tilde{g}(\bigcap_{i\in\mathbb{N}}[Z_i]_Y) \subseteq [\tilde{g}(\bigcap_{i\in\mathbb{N}} Z_i)]_{\mathbb{R}} = [g(Z(g))]_{\mathbb{R}} = \emptyset$. It is a contradiction. It can be supposed that $g(x) \geq 1$ for all $x \in X$.

Since $C(uX)$ is inversion-closed, then $f = 1/g$ is a *coz*-function [6] and $f : uX \to I$. Then the function f can be extended to the function $\beta_u f : \beta_u X \to I$ [17, 26]. If $\beta_u f(y) \neq 0$, then $\tilde{g} = 1/\beta_u f$ is a *coz*-extension of g to Y. So, $\beta_u f(y) = 0$ and $f(x) > 0$ for all $x \in X$. Assume $h = \beta_u f$.

(II) ⇒ *(I)*. It follows from Corollary 3.5 and Proposition 2.6, because uX is an $\mathbb{R}$-compactum as the intersection of cozero subspaces of $\beta_u X$, which are $\mathbb{R}$-compacta. □

Corollary 3.13. ([22]) *The following are equivalent:*

(I) *X is an $\mathbb{R}$-compact space.*

(II) *For any $y \in \beta X \setminus X$ there exists continuous function $h : \beta X \to I$ such that $h(y) = 0$ and $h(x) > 0$ for all $x \in X$.*

Proof. If X is endowed by the fine uniformity u_f, then X is an $\mathbb{R}$-compactum with respect to the uniformity u_f and $C(u_f X) = C(X)$. □

Definition 3.14. A uniform space uX is said to be *strongly C^*_u-embedded* into a uniform space vY if X is a topological subspace of Y and for any bounded *coz*-function $g \in C^*(uX)$ there exists a bounded *coz*-function $h \in C^*(vY)$ such that $h|_X = g$ and $sup|h| = sup|g|$, where $sup|h| = sup\{h(y) : y \in Y\}$ and $sup|g| = sup\{g(x) : x \in X\}$. If $u = u_f$, $v = v_f$ are the fine uniformities, then the space X is *strongly C^*-embedded* into the space Y.

Theorem 3.15. *Let uX be a uniform space such that $X = \bigcup_{n\in\mathbb{N}} X_n$. For each $n \in \mathbb{N}$ let u_n be a uniformity on X_n such that X_n is strongly $C^*_{u_n}$-embedded in uX and X_n is an $\mathbb{R}$-compactum with respect to the uniformity u_n. Then uX is an $\mathbb{R}$-compactum.*

Proof. Let $y \in \beta_u X \setminus X$. If $y \notin \bigcup_{n\in\mathbb{N}} [X_n]_{\beta_u X}$ for all $n \in \mathbb{N}$, and since $\beta_u X$ is a Tychonoff space, then there exists a continuous function $f_n : \beta_u X \to I$ such that $f_n(y) = 0$ and $f_n(x) = 2^{-n}$ for all $x \in [X_n]_{\beta_u X}$. Then $g_n = f_n|_X$ is a *coz*-function and $\beta_u g_n = f_n$ [6]. The series $g = \Sigma_{n\in\mathbb{N}} g_n$ is uniformly converging, so $g \in C(uX)$ and $g : uX \to I$ [6]. It is therefore evident that $\beta_u g(y) = 0$ and $g(x) > 0$ for all $x \in X$.

Now suppose that $y \in [X_k]_{\beta_u X}$ for some $k \in \mathbb{N}$. Because $u_k X_k$ is strongly $C^*_{u_k}$-embedded in uX, and uX is C^*_u-embedded in $\beta_u X$, then $u_k X_k$ is $C^*_{u_k}$-embedded in $\beta_u X$ [7]. Hence $[X_k]_{\beta_u X} = \beta_{u_k} X_k$ [7]. Since X_k is an $\mathbb{R}$-compactum with respect to the uniformity u_k, there exists a continuous function $g : \beta_{u_k} X_k \to I$ such that $g(y) = 0$ and $g(x) > 0$ for all $x \in X_k$ (by Theorem 3.12). By strongly $C^*_{u_k}$-embeddedness of $u_k X_k$ into uX there exists a bounded *coz*-function $h \in C^*(uX)$ such that $h|_{X_k} = g|_{X_k}$ and $sup|h| = sup|g|$. Hence $h(x) > 0$ for all $x \in X$.

Let $\beta_u h : uX \to [-\infty, +\infty]$ be a continuous extension of h [6]. Then $\beta_u h|_{X_k} = h|_{X_k}$ implies $\beta_u h(y) = 0$ and $h(x) > 0$ for all $x \in X$. Thus, uX is an $\mathbb{R}$-compactum. □

Corollary 3.16. *Let X be a Tychonoff space such that $X = \bigcup_{n\in\mathbb{N}} X_n$ and every X_n is $\mathbb{R}$-compact and strongly C^*-embedded subspace of X ($n \in \mathbb{N}$). Then X is an $\mathbb{R}$-compact space.*

Proof. If X is endowed with the fine uniformity u_f and all X_n are endowed by the fine uniformities $(u_n)_f$, we get the result. □

Remark 3.17. It is known that every closed subspace of a normal space is strongly C^*-embedded (by the Brouwer–Tietze–Uryshon Theorem [11, Theorem 2.21]). So, we obtain the next corollary.

Corollary 3.18. ([22]) *Let X be a normal space such that $X = \bigcup_{n\in\mathbb{N}} X_n$, where any X_n is a closed $\mathbb{R}$-compact subspace of X. Then X is an $\mathbb{R}$-compact space.*

Definition 3.19. Let uX, vY be uniform spaces and $X \subset Y$. A uniform space uX is said to be *z_u-embedded* in vY, if $\mathcal{Z}_v \wedge X = \mathcal{Z}_u$.

It is clear, that C_u-embeddedness implies z_u-embeddedness. Simple examples demonstrate that z_u-embeddedness need not imply C_u-embeddedness. We note, if X is z-embedded in Y, then X is z_{u_f}-embedded in Y, i.e. $\mathcal{Z}_{v_f} \wedge X = \mathcal{Z}_{u_f}$, where u_f, v_f are fine uniformities on X and Y, respectively.

Below we formulate some problems.

(I) *Let $Z \in \mathcal{Z}_u$ for a uniform space uX. Which of the following statements are equivalent:*
  (1) *a set Z is $C_{u|_Z}$-embedded in X;*
  (2) *a set Z is $C^*_{u|_Z}$-embedded in X;*
  (3) *a set Z is $z_{u|_Z}$-embedded in X.*

(II) *Let $S \in C\mathcal{Z}_u$ for a uniform space uX. Is S $z_{u|_S}$-embedded in uX?*

(III) *Let uX be a uniform space such that $X = \bigcup_{n\in\mathbb{N}} X_n$. For each $n \in \mathbb{N}$ let u_n be a uniformity on X_n such that X_n is z_{u_n}-embedded in uX and X_n is an $\mathbb{R}$-compactum with respect to the uniformity u_n. Is uX an$\mathbb{R}$-compactum?*

Remark 3.20. In the category *Tych*, A. Chigogidze proved the Spectral Theorem for $\mathbb{R}$-compact spaces [9]. So, the following problem naturally arises: *Prove the Spectral Theorem for $\mathbb{R}$-compacta in the category $ZUnif$.*

Acknowledgement

The authors express their deep gratitude to the referee for corrections in the original draft of the paper and for many helpful and constructive remarks.

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