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The Existence of a Solution of the Two-Dimensional Direct Problem of Propagation of the Action Potential Along Nerve Fibers

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Abstract. In this article, we consider a generalized parabolic two-dimensional direct problem of the process of propagation of the action potential along nerve fibers. The problem is reduced to a generalized hyperbolic problem using the Laplace transform. A generalized two-dimensional direct hyperbolic problem is reduced to a regular hyperbolic problem using methods for rectifying characteristics and isolating singularities. Using the piecewise-continuous function, the existence of the solution of the last problem is proved. From the equivalence of problems it follows that there exists a generalized solution of the parabolic problem.

1. Introduction

When the membrane of a nerve fiber is excited, electrical impulses appear in it. A single nerve impulse is called the action potential. The propagation of the nerve impulse along the nerve fiber (axon) occurs without damping at a constant rate. The propagation velocity of a stable pulse is inversely proportional to the square root of the fiber diameter.

The potential for actions is also called the electric impulse, caused by the change in the ionic permeability of the membrane and associated with the propagation along the nerve and muscle excitation waves.

The formation of the action potential is due to two ionic flows through the biomembrane, which are approximately equal in magnitude but shifted in time.

The membrane excitation is described by the Hodgkin-Huxley equation [1].

$$I_m = C_m \left(dV_{m/} / \partial t \right) + \sum I_i,$$

where I_m is the current through the membranes, C_m is the membrane capacity, $\sum I_i$ is the sum of ion currents across the membrane, V_m is the action potential. The propagation of the potential along a nerve fiber, in one-dimensional space, is described by the telegraph equation [1].

$$\frac{r_a(x)}{2\rho_a(x)} \frac{\partial^2 V_m(x,t)}{\partial x^2} - C_m(x) \frac{\partial V_m(x,t)}{\partial t} - \frac{V_m(x,t)}{\rho_m(x) \cdot l} = 0,$$

where $r_a(x)$ is the radius of nerve fiber (axons), $\rho_a(x)$ is a specific resistance of nerve fiber plasma, $C_m(x)$ is the capacitance per unit area of membrane, $\rho_m(x)$ is the specific resistance of the membrane material, l is the membrane thickness, $V_m(x,t)$ is the action potential.

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2. Statement of the problem

Statement of the parabolic problem. The mathematical model of the process of propagation of the action potential along nerve fibers, in two-dimensional space, is described by the following telegraph equation of parabolic type [1]:

$$C_m(x,y)\vartheta'_t(x,y,t) = \frac{r_a(x,y)}{2\rho_a(x,y)}\Delta\vartheta - \frac{\vartheta(x,y,t)}{\rho_m(x,y)\cdot l}, \quad (x,t)\in R^2_+, \quad y\in R, \quad (1'),$$

where $C_m(x,y)$ is capacitance per unit area of membrane, $r_a(x,y)$ - radius of nerve fiber, $\rho_m(x,y)$ - specific resistance of the membrane material, $\rho_a(x,y)$)- specific plasma resistance of nerve fibers, l - membrane thickness, $\vartheta(x,y,t)$ - intracellular action potential, indices a and m - means nerve fibers and membranes respectively, $\Delta\vartheta(x,y,t) = \vartheta''_{xx}(x,y,t) + \vartheta''_{yy}(x,y,t)$ -Laplace operator.

We take the initial and boundary conditions as follows:

$$\vartheta(x,y,t)|_{t<0} \equiv 0, \vartheta'_x(x,y,t)|_{x=0} = h(y)\theta(t) + r(y)\theta_1(t) + p(y)\theta_2(t), \quad t\in R_+, \quad (2'),$$

where $h(y)$, $r(y)$, $p(y)$ are given functions, $\theta(t)$ is the Heaviside function, $\theta_1(t) = t\theta(t)$, $\theta_2(t) = \frac{t^2}{2!}\theta(t)$.

A direct parabolic problem consists in determining the function $\vartheta(x,y,t)$ from problem (1′) - (2′) for known values of the coefficients $C_m(x,y)$, $r_a(x,y)$, $\rho_m(x,y)$, $\rho_a(x,y)$, l and also for known functions $h(y)$, $r(y)$, $p(y)$.

By the Laplace transform from the parabolic equation, we pass to the hyperbolic equation [2].

Equivalent direct hyperbolic problem. We assume that the coefficients of the equations $r_a(x,y)/\left(2\rho_a(x,y)\cdot C_m(x,y)\right)$, $1/\left(l\cdot\rho_m(x,y)\cdot C_m(x,y)\right)$ grow no faster than the functions $Ce^{\alpha t}$ for $t\to\infty$, where C, α are positive constants and $f_1(t,y) = \int\limits_0^\infty f(\tau,y)G(t,\tau)d\tau$, $G(t,\tau) = \frac{1}{\sqrt{\pi t}}e^{\frac{-z^2}{4t}}$.

Then we make the following calculations

$$\vartheta(x,y,t) = \int\limits_0^\infty V(x,y,\tau)G_t(t,\tau)d\tau = |G_t = G_{\tau\tau}| = \int\limits_0^\infty V(x,y,\tau)G_{\tau\tau}(t,\tau))d\tau =$$
$$= V(x,y,t)G_\tau(t,\tau)|_0^\infty - V(x,y,t)G(t,\tau)|_0^\infty + \int\limits_0^\infty V_{\tau\tau}(x,y,\tau)G(t,\tau)d\tau == \int\limits_0^\infty V_{\tau\tau}(x,y,\tau)G(t,\tau)d\tau.$$

$$\frac{r_a(x,y)}{\left(2\rho_a(x,y)C_m(x,y)\right)}\Delta\vartheta(x,y,t) - \frac{1}{l\cdot\rho_m(x,y)C_m(x,y)}\cdot\vartheta(x,y,t) =$$
$$= \int\limits_0^\infty\left[\frac{r_a(x,y)}{\left(2\rho_a(x,y)\cdot C_m(x,y)\right)}\Delta V(x,y,\tau) - \frac{1}{l\cdot\rho_m(x,y)C_m(x,y)}V(x,y,\tau)\right]G(t,\tau)d\tau =$$
$$= \int\limits_0^\infty \frac{r_a(x,y)}{\left(2\rho_a(x,y)C_m(x,y)\right)}\Delta V(x,y,\tau)\cdot G(t,\tau)d\tau - \int\limits_0^\infty \frac{1}{l\rho_m(x,y)C_m(x,y)}V(x,y,\tau)G(t,\tau)d\tau.$$

hence for $t\in R_+$ we obtain

$$\vartheta_t(x,y,t) - \frac{r_a(x,y)}{\left(2\rho_a(x,y)C_m(x,y)\right)}\Delta\vartheta(x,y,t) - \frac{1}{l\cdot\rho_m(x,y)C_m(x,y)}\vartheta(x,y,t) =$$
$$= \int\limits_0^\infty\left\{V_{\tau\tau}(x,y,\tau) - \frac{r_a(x,y)}{\left(2\rho_a(x,y)C_m(x,y)\right)}\Delta V(x,y,\tau) - \frac{V(x(,y,\tau)}{l\cdot\rho_m(x,y)C_m(x,y)}\right\}G(t,\tau)d\tau.$$

$$\vartheta(0,y,t) = \lim_{t\to 0}\int\limits_0^\infty V(0,y,\tau)G(t,\tau)d\tau = \lim_{t\to 0}\frac{2}{\sqrt{x}}\int\limits_0^\infty V(0,y,2\sqrt{t\tau})e^{-\tau}d\tau =$$
$$= V(0,y,\tau) = f_1(t,y).$$

Thus, from the parabolic equation one can obtain a hyperbolic equation.

Statement of the hyperbolic problem. Thus, in medical practice, the distribution of the action potential for nerve fiber (by axon) is described by the telegraph equation hyperbolic type:

$$C_m(x,y)\frac{\partial^2 V(x,y,t)}{\partial t^2} = \frac{r_a(x,y)}{2\rho_a(x,y)}\Delta V - \frac{1}{\rho_m(x,y)\cdot l}V(x,y,t), \quad x\in R_+, \quad t\in R_+, \quad y\in R. \quad (1)$$

To find the unique solution of equation (1), we give the initial and boundary conditions of the following form:

$V(x,y,t)|_{t<0} \equiv 0, \quad V'_x(x,y,t)|_{x=0} = h(y)\delta(t) + r(y)\theta(t) + p(y)\theta_1(t),$

$x \in (0, T/2), \quad y \in [-D, D], \quad t \in [0,T],$ (2) where $\delta(t)$ is the delta Dirac function.

The first condition (2) means that the nerve-fiber medium is in the depressed state before the time t ¡0, and the potential begins to start at time t = 0 and the propagation of the action potential along the nerve fiber begins.

The second condition (2) means that the force of the instantaneous cord and flat sources with the values h (y), r (y), p (y), respectively, will act on the boundary x = 0.

Find a generalized solution of problem (1) - (2),those, $V(x,y,t)$ for known values of the coefficients $C_m(x,y)$, $r_a(x,y)$, l, $\rho_a(x,y)$, $\rho_m(x,y)$ and for known values $h(y)$, $r(y)$, $p(y)$.

Existence of a solution. Suppose that the following conditions are satisfied with respect to the coefficients of the equation:

$$\left.\begin{array}{l} C_m(x,y), r_a(x,y), \rho_a(x,y), \rho_m(x,y) \in \wedge_1, \\ h(y), r(y), p(y) \in \wedge_2, \quad l>0, \end{array}\right\} \quad (3)$$

where

$$\wedge_1 = \{C_m(x,y) \in C^2((0,d)\times(-D_1,D_1)), 0 < M_1 \le C_m(x,y) \le M_2,$$
$$supp\{C_m(x,y)\} \subset ((0,d)\times(-D_1,D_1)), d = ||C_m(x,y)||_C^2 \le M_2\}.$$

$$\wedge_2 = \{supp\{h(y)\} \in (-D,D), h(y) \in C(-D,D)\},$$
$$D = D_1 + T(M_2 + l), T = 2l/(M_1 - l), M_1, M_2, D = positive - constant \quad numbers.$$

Reduction of problem (1)-(2) to the regular problem. To rectify the characteristic, we introduce a new variable $\alpha(x,y)$, which is a solution of the Eikonal problem of the form:

$$\left.\begin{array}{l} \alpha_x^2(x,y) + \alpha_y^2(x,y) = \frac{2\rho_a(x,y)C_m(x,y)}{r_a(x,y)}, \\ \alpha(x,y)|_{x=0} = 0, \quad \alpha_x(x,y)|_{x=0} = \frac{2\rho_a(0,y)\cdot C_m(0,y)}{r_a(0,y)}, \\ \alpha_x(x,y) > 0, \quad \lim\limits_{x\to\infty}\alpha(x,y) = \infty, \end{array}\right\}.$$

We also introduce new functions $Cm(\alpha(z,y),y) = C_m(x,y)$, $\rho a(\alpha,y) = \rho_a(x,y)$, $\rho m(\alpha,y) = \rho_m(x,y)$, $ra(\alpha,y) = r_a(x,y)$, $v(\alpha(x,y),y,t) = V(x,y,t)$.

We represent the solution of the direct problem from the singular and regular parts of the solution of the following form [4]:

$v(\alpha,y,t) = \tilde{v}(\alpha,y,t) + S(t,y)\theta(t-|\alpha|) + R(t,y)\theta_1(t-|\alpha|) + P(t,y)\theta_2(t-|\alpha|),$

where $\tilde{v}(\alpha,y,t)$ is a continuous function.

Then we obtain a direct problem with the data on the characteristics:

$$\left.\begin{array}{l} \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial \alpha^2} + L_1 v(\alpha,y,t), |\alpha| < t < T, y \in (-D,D), \\ v(\alpha,y,t)|_{|\alpha|=t} = S(t,y), t\in[0,T], y\in(-D,D), \\ v(\alpha,y,t)|_{y=-D} = v(\alpha,y,t)|_{y=D} = 0, \end{array}\right\} \quad (4)$$

where

$$L_1(\alpha,y,t) = \frac{ra(\alpha,y)}{2\rho a(\alpha,y)Cm(\alpha,y)}\left[\frac{\partial^2 v}{\partial y^2} + \Delta\alpha\frac{\partial v}{\partial \alpha} + \alpha'_y\frac{\partial^2 v}{\partial\alpha\partial y}\right] - \frac{1}{Cm(\alpha,y)\rho_m(\alpha,y)\cdot l} * v(\alpha,y,t), \quad (5)$$

$$S(t,y)=\frac{h(y)\cdot 2\rho a(0,y)Cm(0,y)}{2ra(0,y)}+\frac{1}{2}\int_0^t\left\{\frac{ra(\tau,y)}{2\rho a(\tau,y)Cm(\tau,y)}[\alpha_y S_y(\tau,y)-\Delta\alpha S(\tau,y)]\right\}d\tau, \quad (6)$$

$$\begin{aligned} R(t,y)=&\tfrac{r(y)2\rho a(0,y)Cm(0,y)}{2ra(0,y)}+\tfrac{1}{2}\int_0^t\{\tfrac{ra(\tau,y)}{2\rho a(\tau,y)Cm(\tau,y)}[S_{yy}(\tau,y)-\alpha_y R_y(\tau,y)-\Delta\alpha R(\tau,y)]-\\ &-\tfrac{S(\tau,y)}{Cm(\tau,y)\rho m(\tau,y)l}-S_{\tau\tau}(\tau,y)\},\, t\in[0,T],\, y\in(-D,D),\end{aligned} \quad (7)$$

$$\begin{aligned} P(t,y)=&\tfrac{p(y)2\rho a(0,y)Cm(0,y)}{2ra(0,y)}+\tfrac{1}{2}\int_0^t\{\tfrac{ra(\tau,y)}{2\rho a(\tau,y)Cm(\tau,y)}\left[R_{yy}(\tau,y)-\Delta\alpha P(\tau,y)-\alpha_y P_y(\tau,y)\right]-\\ &-\tfrac{1}{Cm(\tau,y)\rho m(\tau,y)\cdot l}R(\tau,y)-R_{\tau\tau}(\tau,y)\}d\tau,\quad t\in[0,T],\quad y\in[-D,D].\end{aligned}$$

We calculate the following equalities, which are necessary later:

$$\begin{aligned}&\tfrac{\partial v}{\partial y}|_{|\alpha|=t}=S_y(t,y),\ \tfrac{\partial v}{\partial t}|_{|\alpha|=t}=S_t(t,y)+R(t,y),\\ &\tfrac{\partial v}{\partial t}|_{|\alpha|=t}=-sign(\alpha)R(t,y),\quad t\in[0,T],\quad y\in(-D,D).\end{aligned} \quad (8)$$

By a generalized solution of the direct problem we mean a function $v(\alpha,y,t)$ that satisfies the equalities:

$$\int_0^t\int_{|\alpha|}^t\int_{-D}^D\left[\frac{\partial v}{\partial\tau}\frac{\partial\varphi}{\partial\tau}+\frac{\partial v}{\partial\alpha}\frac{\partial\varphi}{\partial\alpha}+Lv(\alpha,y,\tau)\varphi(\alpha,y,\tau)\right]dyd\alpha d\tau=\int_{|\alpha|}^t\int_{-D}^D S(\tau,y)\varphi(\alpha,y,\tau)dyd\tau,\quad t\in(0,T). \quad (9)$$

where, $\varphi(\alpha,y,t)\in C^2(\Omega(T,D))$, $\Omega(T,D)=\{(\alpha,y,t):\alpha\in(-T,T),y\in(-D,D),|\alpha|<t<T\}$.

We note that the existence of solutions of wave processes in various settings was considered by A.J. Satybaev and his graduate students [4-8].

A finite-difference analogue of the problem. To compile the difference analogue, we introduce the grid region:

$$\begin{aligned}&\Omega_{ij}^K=\{x_i=ih_1,y_i=jh_2,t_k=k\tau,|ih_1|<2\tau k<T,\\ &h_1=T/N,\tau=T/2N,h_2=D/L,jh_2\in(-D,D)\}.\end{aligned}$$

We construct the difference analogue of the problem:

$$\left.\begin{aligned}&u_{t\bar t}=u_{\alpha\bar\alpha}+L_1u_{ij}^k,(ih_1,jh_2,\tau k)\in\Omega_{ij}^k,\\ &u_{\pm i,j}^{|2i|}=S_j^{|2i|},i=\overline{-N,N};j=\overline{|2i|,2N},\\ &u_{i,L}^k=u_{i,-L}^k=0,i=\overline{-N,N};k=\overline{|2i|,2N},\end{aligned}\right\} \quad (10)$$

where the difference analogues S_j^k, R_j^k,P_j^k, $L_1u_{ij}^k$ will be:

$$S_j^k=\frac{h_j\rho a_j^0Cm_j^0}{ra_j^0}+\frac{1}{2}\sum_{l=0}^{k-1}\left\{\frac{ra_j^l}{2\rho a_j^l\cdot Cm_j^l}\left[\alpha_{\bar y}(S_{\bar y})_j^l+(\Delta\alpha)_j^lS_j^l\right]\right\}\tau,\quad k=\overline{1,M},j=\overline{-L,L}, \quad (11)$$

$$\begin{aligned}R_j^k=&\tfrac{r_j\cdot\rho a_j^0\cdot Cm_j^0}{ra_j^0}+\tfrac{1}{2}\sum_{l=0}^{k-1}\{\tfrac{ra_j^l}{2\rho a_j^l\cdot Cm_j^l}\left[(S_{y\bar y})_j^l-\alpha_{\bar y}(R_y)_j^l-\Delta\alpha R_j^l\right]-\\ &-\tfrac{S_j^l}{Cm_j^l\rho m_j^l\cdot l}-(S_{\tau\bar\tau})_j^l\}\tau,\quad k=\overline{1,M};\quad j=\overline{-L,L}.\end{aligned} \quad (12)$$

$$\begin{aligned}P_j^k=&\tfrac{p_j\rho a_j^0Cm_j^0}{ra_j^0}+\tfrac{1}{2}\sum_{l=0}^{k-1}\{\tfrac{ra_j^l}{2\rho a_j^lCm_j^l}\left[(R_{y\bar y})_j^l-\alpha_{\bar y}(P_{\bar y})_j^l-\Delta\alpha P_j^l\right]-\\ &-\tfrac{R_j^l}{Cm_j^l\rho m_j^l l}-(R_{\tau\tau})_j^l\}\tau,\quad k=\overline{1,M},j=\overline{-L,L}.\end{aligned} \quad (13)$$

$$L_1 u_{ij}^k = \frac{ra_{ij}}{2\rho a_{ij} Cm_{ij}} \left[(u_{\bar{y}})_{ij}^k + \Delta\alpha_{ij} (u_{\bar{\alpha}})_{ij}^k + (\alpha_{\bar{y}})_{ij} \cdot (u_{\alpha\bar{y}})_{ij}^k \right] - \frac{u_{ij}^k}{Cm_{ij} \cdot \rho m_{ij} \cdot l}. \quad (14)$$

3. Proof of the existence of a solution

We define a piecewise continuous function $\tilde{u}(\alpha, y, t)$ inside a parallelepiped
$\Pi = \{k\tau \le t \le (k+1)\tau, \quad ih_1 \le \alpha \le (i+1)h_1, \quad ih_2 \le y \le (j+1)h_2\}$

$$\begin{aligned} &\tilde{u}(\alpha, y, t) = u_{ij}^k (i + 1 - \tfrac{\alpha}{h_1})(j + 1 - \tfrac{y}{h_2})(k + 1 - \tfrac{t}{\tau}) + u_{ij}^{k+1} \times \\ &\times (i + 1 - \tfrac{\alpha}{h_2})(j + 1 - \tfrac{y}{h_2})(\tfrac{t}{\tau} - k) + u_{i+1j}^k (\tfrac{\alpha}{h_1} - i)(j + 1 - \tfrac{y}{h_2}) \times \\ &\times (k + 1 - \tfrac{t}{\tau}) + u_{ij+1}^k (i + 1 - \tfrac{\alpha}{h_1})(\tfrac{y}{h_2} - j)(k + 1 - \tfrac{t}{\tau}) + u_{i+1j+1}^k \times \\ &\times (\tfrac{\alpha}{h_1} - i)(\tfrac{y}{h_2} - j)(k + 1 - \tfrac{t}{\tau}) + u_{i+1j}^{k+1} (\tfrac{\alpha}{h_1} - i)(j + 1 - \tfrac{y}{h_2}) \times \\ &\times (\tfrac{t}{\tau} - k) + u_{ij+1}^{k+1} (i + 1 - \tfrac{\alpha}{h_1})(\tfrac{y}{h_2} - j)(\tfrac{t}{\tau} - k) + u_{i+1j+1}^{k+1} \times (\tfrac{\alpha}{h_1} - i)(\tfrac{y}{h_2} - j)(\tfrac{t}{\tau} - k). \end{aligned} \quad (15)$$

Let the following conditions be satisfied:

$$\left. \begin{aligned} &\max_{|i| \le k \le M} h_1 h_2 \sum_{j=-L}^{L} \sum_{i=-N}^{N} (u_{ij}^k)^2 \le A, \\ &\max_{|i| \le k \le M} h_1 h_2 \sum_{j=-L}^{L} \sum_{i=-N}^{N} \left(\frac{u_{ij}^{k+1} - u_{ij}^k}{\tau} \right)^2 \le B_1, \\ &\max_{|i| \le k \le M} h_1 h_2 \sum_{j=-L}^{L} \sum_{i=-N}^{N} \left(\frac{u_{i+1j}^k - u_{ij}^k}{h_1} \right)^2 \le B_2, \\ &\max_{|i| \le k \le M} h_1 h_2 \sum_{j=-L}^{L} \sum_{i=-N}^{N} \left(\frac{u_{ij+1}^k - u_{ij}^k}{h_2} \right)^2 \le B_3. \end{aligned} \right\} \quad (16)$$

We show that

$$\left. \begin{aligned} &\max_{|\alpha| \le t \le T} \int_{-t}^{t} \int_{-D}^{D} u^2(\alpha, y, t) d\alpha dy \le A, \\ &\max_{|\alpha| \le t \le T} \int_{-t}^{t} \int_{-D}^{D} u_t^2(\alpha, y, t) d\alpha dy \le B_1, \\ &\max_{|\alpha| \le t \le T} \int_{-t}^{t} \int_{-D}^{D} u_\alpha^2(\alpha, y, t) d\alpha dy \le B_2, \\ &\max_{|\alpha| \le t \le T} \int_{-t}^{t} \int_{-D}^{D} u_y^2(\alpha, y, t) d\alpha dy \le B_3. \end{aligned} \right\} \quad (17)$$

We calculate the following integral for $t = k\tau, ih_1 \le \alpha \le (i+1)h_1, jh_2 \le y \le (j+1)h_2$:

$$\begin{aligned} &\int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} \tilde{u}^2(\alpha, y, kt) d\alpha dy = \int_{ih_1}^{(i+1)h_1} \int_{jh_2}^{(j+1)h_2} [u_{ij}^k (i + 1 - \tfrac{\alpha}{h_1})(j + 1 - \tfrac{y}{h_2}) + \\ &+ u_{i+1j}^k (\tfrac{\alpha}{h_1} - i)(j + 1 - \tfrac{y}{h_2}) + u_{ij+1}^k (i + 1 - \tfrac{\alpha}{h_1})(\tfrac{y}{h_2} - j) + u_{i+1j+1}^k \times \\ &\times (\tfrac{\alpha}{h_1} - i)(\tfrac{y}{h_2} - j)]^2 d\alpha dy = |\tfrac{\alpha}{h_1} - i = \xi, \tfrac{y}{h_2} - j = \eta| = h_1 h_2 \int \int_0^1 \int_0^1 [u_{ij}^k (1 - \xi) \times \\ &\times (1 - \eta) + u_{i+1j}^k (\xi(1 - \eta)) + u_{ij+1}^k ((1 - \xi)\eta) + u_{i+1j+1}^k (\xi \cdot \eta)]^2 d\xi d\eta. \end{aligned}$$

We denote by $a = u_{ij}^k$, $b = u_{i+1j}^k$, $c = u_{ij+1}^k$, $d = u_{i+1j+1}^k$. Then we have

$$\int_0^1\int_0^1[a(1-\xi)(1-\eta)+b(1-\eta)\xi+c(1-\xi)\eta+d\xi\eta]^2d\xi d\eta=\int_0^1\int_0^1\{a^2(1-\xi)^2\times$$
$$\times(1-\eta)^2+b^2(1-\xi)^2\xi^2+c^2(1-\xi)^2\eta^2+d^2\xi^2\eta^2+2ab(1-\xi)(1-\eta)^2\xi+$$
$$+2ac(1-\xi)^2(1-\eta)\eta+2ad(1-\xi)(1-\eta)\xi\eta+2bc(1-\eta)(1-\xi)\xi\eta+$$
$$+2bd(1-\eta)\xi^2\eta+2cd(1-\xi)\eta^2\cdot\xi\}d\xi d\eta=\int_0^1\{\tfrac{1}{3}a^2(1-\eta)^2+\tfrac{1}{3}b^2(1-\eta)^2+$$
$$+\tfrac{1}{3}c^2\eta^2+\tfrac{1}{3}d^2\eta^2+2ab[\tfrac{1}{2}-\tfrac{1}{3}](1-\eta)^2+2\tfrac{1}{3}ac\eta(1-\eta)+2ad[\tfrac{1}{2}-\tfrac{1}{3}]\times$$
$$\times\eta(1-\eta)+2bc[\tfrac{1}{2}-\tfrac{1}{3}]\eta(1-\eta)+2bd\tfrac{1}{3}(1-\eta)\eta+2cd[\tfrac{1}{2}-\tfrac{1}{3}]\eta^2\}d\eta=$$
$$=\tfrac{a^2}{3}\int_0^1(1-\eta)^2d\eta+\tfrac{b^2}{3}\int_0^1(1-\eta)^2d\eta+\tfrac{c^2}{3}\int_0^1\eta^2d\eta+2ad[\tfrac{1}{2}-\tfrac{1}{3}]\times$$
$$\times\int_0^1(1-\eta)^2d\eta+\tfrac{2ac}{3}\int_0^1\eta(1-\eta)d\eta+2ad[\tfrac{1}{2}-\tfrac{1}{3}]\int(1-\eta)d\eta+2bc[\tfrac{1}{2}-\tfrac{1}{3}]\times$$
$$\times\int_0^1\eta(1-\eta)d\eta+2bd\tfrac{1}{3}\int_0^1(1-\eta)\eta d\eta+2cd[\tfrac{1}{2}-\tfrac{1}{3}]\int_0^1\eta^2d\eta=\tfrac{a^2}{9}+\tfrac{b^2}{9}+\tfrac{c^2}{9}+\tfrac{d^2}{9}+$$
$$+2ab[\tfrac{1}{2}-\tfrac{1}{3}]\tfrac{1}{3}+\tfrac{2ac}{3}[\tfrac{1}{2}-\tfrac{1}{3}]+2ad[\tfrac{1}{2}-\tfrac{1}{3}]^2+2bc[\tfrac{1}{2}-\tfrac{1}{3}]^2+\tfrac{2bd}{3}[\tfrac{1}{2}-\tfrac{1}{3}]+$$
$$+2cd\tfrac{1}{3}[\tfrac{1}{2}-\tfrac{1}{3}]=\tfrac{a^2}{9}+\tfrac{b^2}{9}+\tfrac{c^2}{9}+\tfrac{d^2}{9}+\tfrac{ab}{9}+\tfrac{ac}{9}+\tfrac{ad}{18}+\tfrac{bc}{18}+\tfrac{bd}{9}+\tfrac{cd}{9}\le\tfrac{1}{9}[a^2+b^2+$$
$$+c^2+d^2+a^b+ac+\tfrac{ad}{2}+\tfrac{bc}{2}+bd+cd]\le\tfrac{1}{9}[a^2+b^2+c^2+d^2+\tfrac{a^2}{4}+\tfrac{b^2}{4}+$$
$$+\tfrac{a^2}{4}+\tfrac{c^2}{4}+\tfrac{a^2}{2}+\tfrac{d^2}{2}+\tfrac{b^2}{2}+\tfrac{c^2}{2}+\tfrac{b^2}{2}+\tfrac{d^2}{2}+\tfrac{c^2}{2}+\tfrac{d^2}{2}]\le\tfrac{1}{4}[a^2+b^2+c^2+d^2]$$

$$\int_{ih_1}^{(i+1)h_1}\int_{ih_2}^{(j+1)h_2}\tilde{u}^2(\alpha,y,k\tau)d\alpha dy\le\frac{1}{4}h_1h_2[(u_{ij}^k)^2+(u_{ij+1}^k)^2+(u_{i+1j}^k)^2+(u_{i+1j+1}^k)^2]$$

Sum over the entire interval,then:

$$\sum_{j=-L}^{L}\sum_{i=-N}^{N}\int_{ih_2}^{(j+1)h_2}\int_{ih_1}^{(i+1)h_1}\tilde{u}^2(\alpha,y,k\tau)d\alpha dy\le h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}(u_{ij}^k)^2\le A. \quad (18)$$

For $t=(k+1)\tau$, $ih_1\le\alpha\le(i+1)h_1$, $jh_2\le y\le(j+1)h_2$ we can also show the inequality obtained above. Let us show the linearity of the functions $\tilde{u}(\alpha,y,t)$ with respect to t for $k\tau\le t\le(k+1)\tau$. Consider

$$\tilde{u}(\alpha,y,k\tau)=u_{ij}^k(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})(k+1-\tfrac{k\tau}{\tau})+u_{ij}^{k+1}(i+1-\tfrac{\alpha}{h_1})\times$$
$$\times(y+1-\tfrac{y}{h_2})(\tfrac{k\tau}{\tau}-k)+u_{i+1j}^k(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})(k+1-\tfrac{k\tau}{\tau})+u_{ij+1}^k\times$$
$$\times(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)(k+1-\tfrac{k\tau}{\tau})+u_{i+1j+1}^k(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j)(k+1-\tfrac{k\tau}{\tau})+$$
$$+u_{i+1j}^{k+1}(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})(\tfrac{k\tau}{\tau}-k)+u_{ij+1}^{k+1}(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)(\tfrac{k\tau}{\tau}-k)+$$
$$+u_{i+1j+1}^{k+1}(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j)(\tfrac{k\tau}{\tau}-k)=u_{ij}^k(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})+u_{i+1j}^k\times$$
$$\times(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})+u_{ij+1}^k(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)+u_{i+1j+1}^k(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j).$$
$$\tilde{u}(\alpha,y,(k+1)\tau)=u_{ij}^k(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})(k+1-\tfrac{(k+1)\tau}{\tau})+u_{ij}^{k+1}\times$$
$$\times(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})(\tfrac{(k+1)\tau}{\tau}-k)+u_{i+1j}^{k+1}(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})\times$$
$$\times(k+1-\tfrac{(k+1)\tau}{\tau})+u_{ij+1}^k(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)(k+1-\tfrac{(k+1)\tau}{\tau})+u_{i+1j+1}^k\times$$
$$\times(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{2h}-j)(k+1-\tfrac{(k+1)\tau}{\tau})+u_{i+1j}^{k+1}(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})\times$$
$$\times(\tfrac{(k+1)\tau}{\tau}-k)+u_{uj+1}^{k+1}(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)(\tfrac{(k+1)\tau}{\tau}-k)+u_{i+1j+1}^{k+1}(\tfrac{\alpha}{h_1}-i)\times$$
$$\times(\tfrac{y}{h_2}-j)(\tfrac{(k+1)\tau}{\tau}-k)=u_{ij}^{k+1}(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})+u_{i+1j}^{k+1}(\tfrac{\alpha}{h_1}-i)\times$$
$$\times(j+1-\tfrac{y}{h_2})+u_{ij+1}^{k+1}(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)+u_{i+1j+1}^{k+1}(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j).$$

It is obvious that

$$\begin{aligned}&\tilde{u}(\alpha,y,t)=(k+1-\tfrac{t}{\tau})\tilde{u}(\alpha,y,k\tau)+(\tfrac{t}{\tau}-k)\tilde{u}(\alpha,y,(k+1)\tau)=\\&=[(1-\tfrac{t-k\tau}{\tau})\tilde{u}(\alpha,y,k\tau)+(\tfrac{t-k\tau}{\tau})\tilde{u}(\alpha,y,(k+1)\tau)].\end{aligned}\tag{19}$$

This means the linearity of the function. From here

$$\begin{aligned}&\tilde{u}^2(\alpha,y,t)=(1-\tfrac{t-k\tau}{\tau})^2\tilde{u}^2(\alpha,y,k\tau)+2(1-\tfrac{t-k\tau}{\tau})(\tfrac{t-k\tau}{\tau})\tilde{u}(\alpha,y,k\tau)\times\tilde{u}(\alpha,y,(k+1)\tau)+\\&+(\tfrac{t-k\tau}{\tau})^2\tilde{u}^2(\alpha,y,(k+1)\tau)\le 2[(1-(\tfrac{t-k\tau}{\tau}))^2\times\tilde{u}^2(\alpha,y,k\tau)+(\tfrac{t-k\tau}{\tau})^2\tilde{u}^2(\alpha,y,(k+1)\tau)]\le\\&\le(1-\tfrac{t-k\tau}{\tau})\tilde{u}^2(\alpha,y,k\tau)+(\tfrac{t-k\tau}{\tau})\tilde{u}^2(\alpha,y,(k+1)\tau).\end{aligned}\tag{20}$$

Then it is obvious that for $\tau k\le t\le(k+1)\tau$ or $0\le\frac{t-\tau k}{\tau}\le 1$ the following inequality holds:

$$\max_{|\alpha|\le t\le T}\int_{-D}^{D}\int_{-t}^{t}\tilde{u}^2(\alpha,y,t)d\alpha dy\le\max_{|i|\le k\le M}\{\int_{-D}^{D}\int_{-t}^{t}\tilde{u}^2(\alpha,y,k\tau)d\alpha dy.$$

$$\int_{-D}^{D}\int_{-t}^{t}\tilde{u}^2(\alpha,y,(k+1)\tau)d\alpha dy\}\max_{|\alpha|\le k\le M}h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}(u_{ij}^k)^2\le A.\tag{21}$$

This means that the first inequality of (17) holds.

Let us prove that $\max\limits_{|\alpha|\le t\le T}\int\limits_{-D}^{D}\int\limits_{-t}^{t}\tilde{u}_t^2(\alpha,y,t)d\alpha dy\le B_1$, if $\max\limits_{|i|\le k\le M}h_1h_2\sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N}\left[\frac{u_{ij}^{k+1}-u_{ij}^k}{\tau}\right]^2\le B_1$.

For $t=k\tau, ih_1\le\alpha\le(i+1)h_1, jh_2\le y\le(j+1)h_2$, we consider

$$\begin{aligned}&\tilde{u}_t(\alpha,y,t)=u_{ij}^k(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})(-\tfrac{1}{\tau})+u_{ij}^{k+1}(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})\tfrac{1}{\tau}+\\&+u_{i+1j}^k(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})(-\tfrac{1}{\tau})+u_{ij+1}^k(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)(-\tfrac{1}{\tau})+u_{i+1j+1}^k\times\\&\times(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j)(-\tfrac{1}{\tau})+u_{i+1j}^{k+1}(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})\tfrac{1}{\tau}+u_{ij+1}^k(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)\tfrac{1}{\tau}+\\&+u_{i+1j+1}^{k+1}(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j)\tfrac{1}{\tau}=(i+1-\tfrac{\alpha}{h_1})(j+1-\tfrac{y}{h_2})\cdot\tfrac{u_{ij}^{k+1}-u_{ij}^k}{\tau}+(\tfrac{\alpha}{h_1}-i)\times\\&\times(j+1-\tfrac{y}{h_2})\tfrac{u_{i+1j}^{k+1}-u_{i+1j}^k}{\tau}+(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)\tfrac{u_{ij+1}^{k+1}-u_{ij+1}^k}{\tau}+(\tfrac{\alpha}{h_1}-i)\times(\tfrac{y}{h_2}-j)\tfrac{u_{i+1j+1}^{k+1}-u_{i+1j+1}^k}{\tau}.\end{aligned}$$

Summing over the entire interval and denoting $a_1=\frac{u_{ij}^{k+1}-u_{ij}^k}{\tau}$, $b_1=\frac{u_{i+1j}^{k+1}-u_{i+1j}^k}{\tau}$, $c_1=\frac{u_{ij+1}^{k+1}-u_{ij+1}^k}{\tau}$, $d_1=\frac{u_{i+1j+1}^{k+1}-u_{i+1j+1}^k}{\tau}$, we obtain:

$$\begin{aligned}&h_1h_2\int\limits_{jh_2}^{(j+1)h_2}\int\limits_{ih_1}^{(i+1)h_1}\tilde{u}^2(\alpha,y,t)d\alpha dy=h_1h_2\int\limits_{jh_2}^{(j+1)h_2}\int\limits_{ih_1}^{(i+1)h_1}[(i+1-\tfrac{\alpha}{h_1})\times(j+1-\tfrac{y}{h_2})\tfrac{u_{ij}^{k+1}-u_{ij}^k}{\tau}+\\&(\tfrac{\alpha}{h_1}-i)(j+1-\tfrac{y}{h_2})\tfrac{u_{i+1j}^{k+1}-u_{i+1j}^k}{\tau}+(i+1-\tfrac{\alpha}{h_1})(\tfrac{y}{h_2}-j)\tfrac{u_{ij+1}^{k+1}-u_{ij+1}^k}{\tau}+(\tfrac{\alpha}{h_1}-i)(\tfrac{y}{h_2}-j)\times\\&\times\tfrac{u_{i+1j+1}^{k+1}-u_{i+1j+1}^k}{\tau}]^2d\alpha dy=|\tfrac{\alpha}{h_1}-i=\xi,\ \tfrac{y}{h_2}-j=\eta|=\\&=h_1^{h_2}\int\limits_0^1\int\limits_0^1[(1-\xi)(1-\eta)a_1+\xi(1-\eta)b_1+(1-\xi)\eta c_1+\xi\eta d_1]^2d\xi d\eta=\\&=h_1h_2\tfrac{1}{4}[a_1^2+b_1^2+c_1^2+d_1^2].\end{aligned}\tag{22}$$

From the latter it follows

$$\max_{|\alpha|\le t\le T}\int\limits_{-D}^{D}\int\limits_{-t}^{t}\tilde{u}^2(\alpha,y,t)d\alpha dy\le\max_{|i|\le k\le M}h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}\left[\frac{u_{ij}^{k+1}-u_{ij}^k}{\tau}\right]^2\le B_1.$$

We can show the linearity of the function $\tilde{u}_t(\alpha,y,t)$.

Let us prove that $\max\limits_{|\alpha|\leq t\leq T}\int\limits_{-D}^{D}\int\limits_{-t}^{t}\tilde{u}^2_\alpha(\alpha,y,t)d\alpha dy\leq B_2$, if $\max\limits_{|i|\leq k\leq M}h_1h_2\sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N}\left[\frac{u^k_{i+1j}-u^k_{ij}}{h_1}\right]^2\leq B_2$.
We compute in the rectangle $t=k\tau, ih_1\leq\alpha\leq(i+1)h_1, jh_2\leq y\leq(j+1)h_2$ the following:

$$\begin{aligned}&\tilde{u}_\alpha(\alpha,y,t)=(j+1-\tfrac{y}{h_2})(k+1-\tfrac{t}{h_2})\tfrac{u^k_{i+1j}-u^k_{ij}}{h_1}+(j+1-\tfrac{y}{h_2})(\tfrac{t}{\tau}-k)\times\\&\times\tfrac{u^{k+1}_{i+1j}-u^{k+1}_{ij}}{h_1}+(\tfrac{y}{h_2}-j)(k+1-\tfrac{t}{\tau})\tfrac{u^k_{i+1j+1}-u^k_{ij+1}}{h_1}+(\tfrac{y}{h_2}-j)(\tfrac{t}{\tau}-k)\tfrac{u^{k+1}_{i+1j+1}-u^{k+1}_{ij+1}}{h_1};\\&\tilde{u}_\alpha(\alpha,y,k\tau)=(j+1-\tfrac{y}{h_2})\cdot\tfrac{u^k_{i+1j}-u^k_{ij}}{h_1}+(\tfrac{y}{h_2}-j)\tfrac{u^k_{i+1j+1}-u^k_{ij+1}}{h_2}.\end{aligned}$$

Integrate $\tilde{u}^2(\alpha,y,k\tau)$

$$\begin{aligned}&\int\limits_{jh_2}^{(j+1)h_2}\int\limits_{ih_1}^{(i+1)h_1}\tilde{u}^2(\alpha,y,k\tau)d\alpha dy=\int\limits_{jh_2}^{(j+1)h_2}\int\limits_{ih_1}^{(i+1)h_1}[(i+1-\tfrac{y}{h_2})\cdot\tfrac{u^k_{i+1j}-u^k_{ij}}{h_1}+(\tfrac{y}{h_2}-j)\times\\&\times\tfrac{u^k_{i+1j+1}-u^k_{ij+1}}{h_2}]^2d\alpha dy=|\eta=\tfrac{y}{h_2}-j,\xi=\tfrac{\alpha}{h_1}-i,dy=h_2d\eta,d\alpha=h_1d\xi|=\\&=h_1h_2\int\limits_0^1[(1-\eta)\tfrac{u^k_{i+1j}-u^k_{ij}}{h_1}+\eta\tfrac{u_{i+1j+1}-u^k_{ij+1}}{h_1}]^2d\eta=\\&=|denoting,a_2=(u^k_{i+1j}-u^k_{ij})/h_1,b_2=(u^k_{i+1j+1}-u^k_{ij+1})/h_1|=\\&=h_1h_2\int\limits_0^1[(1-\eta)^2\cdot\alpha_2^2+2(1-\eta)\cdot\eta a_2b_2+\eta^2b_2^2]d\eta=\\&=h_1h_2[\tfrac13a_2^2+2a_2b_2[\tfrac12-\tfrac13]+\tfrac13b_2^2]\leq h_1h_2[\tfrac13a_2^2+\tfrac12a_2^2+\tfrac12b_2^2-\tfrac13a_2^2-\tfrac13b_2^2+\tfrac13b_2^2]=\\&=\tfrac12h_1h_2[a_2^2+b_2^2]=\tfrac{h_1h_2}{2}\left[(\tfrac{u^k_{i+1j}-u^k_{ij}}{h_1})^2+(\tfrac{u^k_{i+1j+1}-u^k_{ij+1}}{h_1})^2\right].\end{aligned}$$

Summing the last for $i=\overline{-N,N}, j=\overline{-L,L}$

$$\max_{|\alpha|\leq t\leq T}\int\limits_{-D}^{D}\int\limits_{-t}^{t}\tilde{u}^2_\alpha(\alpha,y,k\tau)\leq\max_{|i|\leq k\leq M}\frac{h_1h_2}{2}\sum_{j=-L}^{L}\sum_{i=-N}^{N}[(\frac{u^k_{i+1j}-u^k_{ij}}{h_1})^2++(\frac{u^k_{i+1j+1}-u^k_{ij+1}}{h_1})^2]\leq B_2.\quad(23)$$

In the same way, we can show that

$$\max_{|\alpha|\leq t\leq T}\int\limits_{-D}^{D}\int\limits_{-t}^{t}\tilde{u}^2_\alpha(\alpha,y,(k+1)\tau)d\alpha dy\leq\max_{|i|\leq k\leq M}\frac{h_1h_2}{2}\sum_{j=-L}^{L}\sum_{i=-N}^{N}[\frac{u^{k+1}_{i+1j}-u^{k+1}_{ij}}{h_1}+\frac{u^{k+1}_{i+1j+1}-u^{k+1}_{ij+1}}{h_1}]\leq B_2.$$

Let us show the linearity of the function $\tilde{u}_\alpha(\alpha,y,t)$. Let $t: k\tau<t<(k+1)\tau$, $ih_1<\alpha<(i+1)h_1$, $ih_2<y<(j+1)h_2$.

$$\tilde{u}_\alpha(\alpha,y,(k+1)\tau)=-u^{k+1}_{ij}\frac{1}{h_1}(j+1-\frac{y}{h_2})+u^{k+1}_{i+1j}\frac{1}{h_1}(j+1-\frac{y}{h_2})-u^{k+1}_{ij+1}\frac{1}{h_1}(\frac{y}{h_2}-j)+u^{k+1}_{i+1j+1}\frac{1}{h_1}(\frac{y}{h_2}-j);$$

$$\tilde{u}_\alpha(\alpha,y,k\tau)=-u^k_{ij}\frac{1}{h_1}(j+1-\frac{y}{h_2})+u^k_{i+1j}\frac{1}{h_1}(j+1-\frac{y}{h_2})-u^k_{ij+1}\frac{1}{h_1}(\frac{y}{h_2}-j)+u^k_{i+1j+1}\frac{1}{h_1}(\frac{y}{h_2}-j);$$

this implies:

$$\begin{aligned}&\tilde{u}_\alpha(\alpha,y,t)=(k+1-\tfrac{1}{\tau})\tilde{u}_\alpha(\alpha,y,k\tau)+(\tfrac{t}{\tau}-k)\tilde{u}_\alpha(\alpha,y,(k+1)\tau)=\\&=(1-\tfrac{t-\tau k}{\tau})\tilde{u}_\alpha(\alpha,y,k\tau)-(\tfrac{t}{\tau}-k)\tilde{u}_\alpha(\alpha,y,(k+1)\tau).\end{aligned}\quad(24)$$

Hence it follows that the function $\tilde{u}(\alpha,y,t)$ is a linear function.

$$\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_\alpha^2(\alpha,y,t)d\alpha dy \le (1-(\tfrac{t}{\tau}-k))\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_\alpha^2(\alpha,y,k\tau)d\alpha dy + (\tfrac{t}{\tau}-k)\times$$
$$\times\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_\alpha^2(\alpha,y,(k+1)\tau)d\alpha dy \le (1-(\tfrac{t}{\tau}-k))\max_{|i|\le k\le M}\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_\alpha^2(\alpha,y,k\tau)d\alpha dy+$$
$$+(\tfrac{t}{\tau}-k)\max_{|i|\le k\le M}\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_\alpha^2(\alpha,y,(k+1)\tau)d\alpha dy \le \max_{|i|\le k\le M}\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_\alpha^2(\alpha,y,k\tau)d\alpha dy \le$$
$$\le \max_{|i|\le k\le M}\frac{h_1h_2}{2}\sum_{j=-L}^{L}\sum_{i=-N}^{N}\left[\frac{u_{i+1j}^{k+1}-u_{ij}^{k+1}}{h_1}+\frac{u_{i+1j+1}^{k+1}-u_{ij+1}^{k+1}}{h_1}\right] \le B_2.$$

which was to be proved.

Let us prove that $\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_y^2(\alpha,y,t)d\alpha dy \le B_3$, if $\max_{|i|\le k\le M} h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}\left[\frac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2}\right]^2 \le B_3$.

$$\tilde{u}_y(\alpha,y,t) = (i+1-\tfrac{\alpha}{h_1})(k+1-\tfrac{t}{\tau})\frac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2} + (i+1-\tfrac{\alpha}{h_1})(\tfrac{t}{\tau}-k)\frac{u_{ij+1}^{k+1}-u_{ij}^{k+1}}{h_2}+$$
$$+(\tfrac{\alpha}{h_1}-i)(k+1-\tfrac{t}{\tau})\frac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2} + (\tfrac{\alpha}{h_1}-i)(\tfrac{t}{\tau}-k)\frac{u_{i+1j+1}^{k+1}-u_{i+1j}^{k+1}}{h_2}:$$

$$\tilde{u}_y(\alpha,y,k\tau) = (i+1-\frac{\alpha}{h_1})\frac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2} + (\frac{\alpha}{h_1}-i)\frac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2}:$$

$$\int_{jh_2}^{(j+1)h_2}\int_{ih_1}^{(i+1)h_1}\tilde{u}_y^2(\alpha,y,k\tau)d\alpha dy = \int_{jh_2}^{(j+1)h_2}\int_{ih_1}^{(i+1)h_1}[(i+1-\tfrac{\alpha}{h_1})\tfrac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2}+(\tfrac{\alpha}{h_1}-i)\times$$
$$\times\tfrac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2}]^2 d\alpha dy = |\eta = \tfrac{y}{h_2}-j, h_2d\eta = dy.\xi = \tfrac{\alpha}{h_1}-i, h_1d\xi = d\alpha| =$$
$$= h_1h_2\int_0^1[(1-\xi)\tfrac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2}+\xi\tfrac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2}]^2d\xi = |c_2\tfrac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2}, d_2 = \tfrac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2}| =$$
$$= h_1h_2[\tfrac{1}{3}c_2^2+2c_2d_2[\tfrac{1}{2}-\tfrac{1}{3}]+\tfrac{1}{3}d_2^2] \le \tfrac{h_1h_2}{2}(c_2^2+d_2^2) =$$
$$= \tfrac{h_1h_2}{2}[(\tfrac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2})^2+(\tfrac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2})^2].$$

Summing the latter for $i=\overline{-N,N}$; $j=\overline{-L,L}$, we obtain

$$\max_{|i|\le k\le M}\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_y^2(\alpha,y,k\tau)d\alpha dy \le \max_{|i|\le k\le M}\frac{h_1h_2}{2}\sum_{j=-L}^{L}\sum_{i=-N}^{N}[(\frac{u_{ij+1}^{k}-u_{ij}^{k}}{h_2})^2+(\frac{u_{i+1j+1}^{k}-u_{i+1j}^{k}}{h_2})^2] \le B_3. \quad (25)$$

The same inequality can be established for

$$\max_{|i|\le k\le M}\int_{-D}^{D}\int_{-t}^{t}\tilde{u}_y^2(\alpha,y,(k+1)\tau)d\alpha dy \le B_3.$$

Consider the following function in the parallelepiped $k\tau < t < (k+1)\tau$, $ih_1 < \alpha < (i+1)h_1$, $jh_2 < y < (j+1)h_2$:

$$\tilde{u}_y(\alpha,y,(k+1)\tau) = (i+1-\frac{\alpha}{h_1})\frac{u_{ij+1}^{k+1}-u_{ij}^{k+1}}{h_2} + (\frac{\alpha}{h_1}-i)\frac{u_{i+1j+1}^{k+1}-u_{i+1j}^{k+1}}{h_2}.$$

From here

$$\tilde{u}_y(\alpha,y,t) = (k+1-\frac{t}{\tau})\tilde{u}_y(\alpha,y,k\tau) + (\frac{t}{\tau}-k)\tilde{u}_y(\alpha,y,(k+1)\tau) \quad (26)$$

which shows the linearity of the function $\tilde{u}_y(\alpha,y,t)$.

Thus, the boundedness and linearity of piecewise-continuous functions $\tilde{u}(\alpha,y,t)$, $\tilde{u}_t(\alpha,y,t)$, $\tilde{u}_\alpha(\alpha,y,t)$, $\tilde{u}_y(\alpha,y,t)$.

We now show the existence of the following terms: $\frac{\partial\tilde{u}(\alpha,y,t)}{\partial\alpha}$, $\frac{\partial\tilde{u}(\alpha,y,t)}{\partial t}$, $\frac{\partial\tilde{u}(\alpha,y,t)}{\partial y}$.

One can choose sequence of functions $\left\{U_{ij}^k\right\}$, $\left\{W_{ij}^k\right\}$, $\left\{V_{ij}^k\right\}$, that converge to the above members. We denote $v_{ij}^k = \frac{u_{i+1j}^k - u_{ij}^k}{h_1}$.

Suppose that the conditions (16) are satisfied for (v_{ij}^k), that is,

$$\left.\begin{array}{l} \max h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N}(v_{ij}^k)^2 \le A, \\ \max h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N}\left(\frac{v_{ij}^{k+1}-v_{ij}^k}{\tau}\right)^2 \le B_1, \\ \max h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N}\left(\frac{v_{i+1j}^k-v_{ij}^k}{h_1}\right)^2 \le B_2, \\ \max h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N}\left(\frac{v_{ij+1}^k-v_{ij}^k}{h_2}\right)^2 \le B_3. \end{array}\right\} \quad (27)$$

Let us prove that $v_\alpha(\alpha, y, t) = \frac{\partial\tilde{u}(\alpha,y,t)}{\partial\alpha}$. Suppose that $\alpha_2 > \alpha_1$.

$$\begin{array}{l} \tilde{u}(\alpha_2, y, t) - \tilde{u}(\alpha_1, y, t) = \tilde{u}\left(\left[\frac{\alpha_2}{h_1}\right]h_1, \left[\frac{y}{h_2}\right]h_2, \left[\frac{t}{\tau}\right]\tau\right) - \tilde{u}\left(\left[\frac{\alpha_1}{h_1}\right]h_1, \left[\frac{y}{h_2}\right]h_2, \left[\frac{t}{\tau}\right]\tau\right) + \\ +O(\sqrt{h_1} + \sqrt{h_2} + \sqrt{\tau}) = \sum\limits_{i=[\frac{\alpha_1}{h_1}]}^{[\frac{\alpha_2}{h_1}]-1} \frac{\tilde{u}_{i+1j}^k - u_{ij}^k}{h_1} h_1 + O(\sqrt{h_1} + \sqrt{h_2} + \sqrt{\tau}) = \\ = \sum\limits_{i=[\frac{\alpha_1}{h_1}]}^{[\frac{\alpha_2}{h_1}]-1} v_{ij}^k h_1 + O(\sqrt{h_1} + \sqrt{h_2} + \sqrt{\tau}) = \sum\limits_{i=[\frac{\alpha_1}{h_1}]}^{[\frac{\alpha_2}{h_1}]-1} \int\limits_{ih_1}^{(i+1)h_1} v(\alpha, y, k\tau)d\alpha + \\ +O((\alpha_2-\alpha_1)\sqrt{h_1}) = \int\limits_{\alpha_1}^{\alpha_2} v(\alpha, y, k\tau)d\alpha + O\left((\alpha_2-\alpha_1)\sqrt{|t-k\tau|}\right) = \\ = \int\limits_{\alpha_1}^{\alpha_2} v(\alpha, y, t)d\alpha + O(\sqrt{h_1} + \sqrt{h_2} + \sqrt{\tau}). \end{array} \quad (28)$$

Hence for $h_1 \to 0$, $h_2 \to 0$, $\tau \to 0$, we have:

$$\tilde{u}(\alpha_2, y, t) - \tilde{u}(\alpha_1, y, t) = \int\limits_{\alpha_1}^{\alpha_2} v(\alpha, y, t)d\alpha. \quad (29)$$

Differentiating the last formula, we obtain $v(\alpha, y, t) = \frac{\partial\tilde{u}(\alpha,y,t)}{\partial\alpha}$.

We denote by $W_{ij}^k = \frac{\tilde{u}_{ij}^{k+1} - \tilde{u}_{ij}^k}{\tau}$. Suppose also that an inequality of the form (27) holds for W_{ij}^k. We show the equality $W(\alpha, y, t) = \frac{\partial\tilde{u}(\alpha,y,t)}{\partial t}$.

$$\begin{array}{l} \tilde{u}(\alpha, y, t_2) - \tilde{u}(\alpha, y, t_1) = \tilde{u}\left(\left[\frac{\alpha}{h_1}\right]h_1, \left[\frac{y}{h_2}\right]h_2, \left[\frac{t_2}{\tau}\right]\tau\right) - \tilde{u}\left(\left[\frac{\alpha}{h_1}\right]h_1, \left[\frac{y}{h_2}\right]h_2, \left[\frac{t_1}{\tau}\right]\tau\right) + \\ +O(\sqrt{h_1} + \sqrt{h_2} + \sqrt{\tau}) = \sum\limits_{k=[\frac{t_1}{\tau}]}^{[\frac{t_2}{\tau}]-1} \frac{\tilde{u}_{ij}^{k+1} - u_{ij}^k}{\tau}\tau + O(\sqrt{h_1} + \sqrt{h_2} + \sqrt{\tau}). \end{array} \quad (30)$$

As well as above, reasoning

$$\tilde{u}(\alpha, y, t_2) - \tilde{u}(\alpha, y, t_1) = \int\limits_{t_1}^{t_2} W(\alpha, y, t)dt. \quad (31)$$

Differentiating (31), we obtain $W(\alpha,y,t)=\frac{\partial\tilde{u}(\alpha,y,t)}{\partial t}$.
We denote by $V_{ij}^k=\frac{\tilde{u}_{ij+1}^k-\tilde{u}_{ij}^k}{h_2}$.
Suppose also that an inequality of the form (27) holds for V_{ij}^k.
We show that equality $V(\alpha,y,t)=\frac{\partial\tilde{u}(\alpha,y,t)}{\partial y}$.

$$\begin{aligned}&\tilde{u}(\alpha,y_2,t)-\tilde{u}(\alpha,y_1,t)=\tilde{u}\left(\left[\tfrac{\alpha}{h_1}\right]h_1,\left[\tfrac{y_2}{h_2}\right]h_2,\left[\tfrac{t}{\tau}\right]\tau\right)-\tilde{u}\left(\left[\tfrac{\alpha}{h_1}\right]h_1,\left[\tfrac{y_1}{h_2}\right]h_2,\left[\tfrac{t}{\tau}\right]\tau\right)+\\&+O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau})=\sum_{y=[\frac{y_1}{h_2}]}^{[\frac{y_2}{h_2}]}\frac{\tilde{u}_{ij+1}^k-u_{ij}^k}{h_2}h_2+O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau}).\end{aligned}\tag{32}$$

Consequently,

$$V(\alpha,y,t)=\frac{\partial\tilde{u}(\alpha,y,t)}{\partial y}.\tag{33}$$

Thus, we can choose converging to a sequence of grid functions $\{u_{ij}^k\},\{U_{ij}^k\},\{W_{ij}^k\},\{V_{ij}^k\}$ that converge to functions u,U,W,V, hence to functions $u,\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial\tau},\frac{\partial u}{\partial y}$. We now show the existence of the following derivatives $\frac{\partial^2u}{\partial t^2},\frac{\partial^2u}{\partial\alpha^2},\frac{\partial^2u}{\partial y^2},\frac{\partial^2u}{\partial\alpha dy}$.
The existence of the derivatives $\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial y},\frac{\partial u}{\partial t}$ is proved.

We denote by $v_{2ij}^k=\frac{v_{i+1j}^k-v_{ij}^k}{h_1}$. Suppose that

$$\left.\begin{aligned}&\max_{|i|\le k\le M}h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}(v_{2ij}^k)^2\le A,\\&\max_{|i|\le k\le M}h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}\left(\frac{v_{2ij}^{k+1}-v_{2ij}^k}{\tau}\right)^2\le B_1,\\&\max_{|i|\le k\le M}h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}\left(\frac{v_{2i+1j}^k-v_{2ij}^k}{h_1}\right)^2\le B_2,\\&\max_{|i|\le k\le M}h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}\left(\frac{v_{2ij+1}^k-v_{2ij}^k}{h_2}\right)^2\le B_3.\end{aligned}\right\}\tag{34}$$

$$\begin{aligned}&\tilde{v}(\alpha_2,y,t)-\tilde{v}(\alpha_1,y,t)=\tilde{v}\left(\left[\tfrac{\alpha_2}{h_1}\right]h_1,\left[\tfrac{y}{h_2}\right]h_2,\left[\tfrac{t}{\tau}\right]\tau\right)-\tilde{v}\left(\left[\tfrac{\alpha_1}{h_1}\right]h_1,\left[\tfrac{y}{h_2}\right]h_2,\left[\tfrac{t}{\tau}\right]\tau\right)+O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau})=\\&=\sum_{i=[\frac{\alpha_1}{h_1}]}^{[\frac{\alpha_2}{h_1}]-1}\frac{\tilde{v}_{i+1j}^k-v_{ij}^k}{h_1}+O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau})=\sum_{i=[\frac{\alpha_1}{h_1}]}^{[\frac{\alpha_2}{h_1}]-1}v_{2ij}^kh_1+O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau}).\end{aligned}$$

We integrate from ih_1 to $(i+1)h_1$, then

$$\begin{aligned}&\sum_{i=[\frac{\alpha_1}{h_1}]}^{[\frac{\alpha_2}{h_1}]-1}\int_{ih_1}^{(i+1)h_1}v_2(\alpha,y,k\tau)d\alpha+O(|\alpha_2-\alpha_1|\sqrt{h_1})=\int_{\alpha_1}^{\alpha^2}v(\alpha,y,k\tau)d\alpha+\\&+O(|\alpha_2-\alpha_1|)+O(\sqrt{h_2})+O(|\alpha_2-\alpha_1|)\sqrt{(t-k\tau)}.\end{aligned}$$

Consequently $\tilde{v}(\alpha_2,y,t)-\tilde{v}(\alpha_1,y,t)=\int_{\alpha_1}^{\alpha_2}v_2(\alpha,y,t)+O(\sqrt{h_1,h_2,\tau})$.

Differentiating the latter, we obtain $v_2(\alpha,y,t)=\frac{\partial v(\alpha,y,t)}{\partial\alpha}=\frac{\partial^2u(\alpha,y,t)}{\partial\alpha^2}$. In the same way, it can be shown that $W_2(\alpha,y,t)=\frac{\partial W(\alpha,y,t)}{\partial t}=\frac{\partial^2u(\alpha,y,t)}{\partial t^2}$; $V_2(\alpha,y,t)=\frac{\partial V(\alpha,y,t)}{\partial y}=\frac{\partial^2u(\alpha,y,t)}{\partial y^2}$.

Let us show the existence of a derivative $\frac{\partial^2 u}{\partial\alpha dy}$. We denote by $P_{2ij}^k = \frac{U_{ij+1}^k - U_{ij}^k}{h_2}$. Suppose that

$$\left.\begin{array}{l} \max\limits_{|i|\le k\le M} h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N} (P_{2ij}^k)^2 \le A, \\ \max\limits_{|i|\le k\le M} h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N} \left(\frac{P_{2ij}^{k+1}-P_{2ij}^k}{\tau}\right)^2 \le B_1, \\ \max\limits_{|i|\le k\le M} h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N} \left(\frac{P_{2i+1j}^{k}-P_{2ij}^k}{h_1}\right)^2 \le B_2, \\ \max\limits_{|i|\le k\le M} h_1h_2 \sum\limits_{j=-L}^{L}\sum\limits_{i=-N}^{N} \left(\frac{P_{2ij+1}^{k}-P_{2ij}^k}{h_2}\right)^2 \le B_3. \end{array}\right\} \quad (35)$$

$$\tilde{v}(\alpha,y_2,t)-\tilde{v}(\alpha,y_1,t) = \tilde{v}\left(\left[\frac{\alpha}{h_1}\right]h_1,\left[\frac{y_2}{h_2}\right]h_2,\left[\frac{t}{\tau}\right]\tau\right) - \tilde{v}\left(\left[\frac{\alpha}{h_1}\right]h_1,\left[\frac{y_1}{h_2}\right]h_2,\left[\frac{t}{\tau}\right]\tau\right) +$$
$$+O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau}) = \sum_{i=[\frac{y_1}{h_2}]}^{[\frac{y_2}{h_2}]-1} \frac{\tilde{v}_{ij+1}^k - v_{ij}^k}{h_2} + O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau}) =$$
$$= \sum_{j=[\frac{y_1}{h_2}]}^{[\frac{y_2}{h_2}]-1} P_{2ij}^k h_2 + O(\sqrt{h_1}+\sqrt{h_2}+\sqrt{\tau}).$$

We integrate from jh_2 to $(j+1)h_2$, then

$$\sum_{j=[\frac{y_1}{h_2}]}^{[\frac{y_2}{h_2}]-1} \int_{jh_2}^{(j+1)h_2} P_2(\alpha,y,k\tau)dy + O(|y_2-y_1|\sqrt{h_2}) = \int_{y_1}^{y^2} P_2(\alpha,y,k\tau)dy +$$
$$+O(|y_2-y_1|) + O(\sqrt{h_2}) + O(|y_2-y_1|)\xi\sqrt{(t-k\tau)}.$$

Consequently $\tilde{v}(\alpha,y_2,t)-\tilde{v}(\alpha,y_1,t) = \int\limits_{y_1}^{y_2} P_2(\alpha,y,t) + O([|y_2-y_1|]\sqrt{t-k\tau})$.

For $h_1\to 0, h_2\to 0, \tau\to 0$, we differentiate the last equality in y. $P_2(\alpha,y,t) = \frac{\partial v(\alpha,y,t)}{\partial y} = \frac{\partial^2 u(\alpha,y,t)}{\partial\alpha\partial y}$. etc.

Let the steps τ, h_1, h_2 through t, α, y run through some numerical sequences $\{\tau_s\}, \{h_{1s}\}, \{h_{2s}\}$, where $(\tau_s, h_{1s}, h_{2s} > 0)$ and $\lim\limits_{s\to\infty}(\tau_s, h_{1s}, h_{2s}) \to 0$.

Suppose that for each s, finite-difference solutions of problem (6) are constructed. Then, considering that all these solutions outside the characteristic angle are equal to zero, then there exists a sequence $\{(u_{ij}^k)^s\}$, for some u_{ij}^k that weakly converges in the norm $W_2^1(\Omega(T,D))$ and strongly converges in the norm $L_2(\Omega(T,D))$ to the function $u(\alpha,y,t)$, i.e.

$$||u_{ij}^k - u(\alpha,y,t)||_{W_2(\Omega(T,D))} \to 0, \quad ||u_{ij}^k - u(\alpha,y,t)||_{L_2(\Omega(T,D))} \to 0. \quad (36)$$

We show that the function $u(\alpha,y,t)$ is a generalized solution of problem (6), that is, the validity of equality (13). For u_{ij}^k, we have the equality

$$\tau h_1 h_2 \sum_{j=-L}^{L}\sum_{i=-N}^{N}\sum_{k=|i|}^{M} \left\{\left[(u_{i,j}^k)_{t\bar{t}}^s - (u_{i,j}^k)_{\alpha\bar{\alpha}}^s - L(u_{i,j}^k)^s\right]\cdot\phi_{ij}^k\right\} = 0. \quad (37)$$

Using the formula "summation by parts" and "differentiation" of products, we transform each term of the last equality (for brevity, the index s is omitted)

$$\sum_{k=|i|}^{M}(u_{i,j}^k)_{t\bar{t}}\phi_{i,j}^k = -\sum_{k=|i|}^{M}(u_{i,j}^k)_t(\phi_{i,j}^k)_{\bar{t}}(k) + (u_{i,j}^k)_t(M)(\phi_{i,j}^k)(M) - (u_{i,j}^k)_t \times (|i|)(\phi_{i,j}^k)|i| :$$

$$\sum_{k=|i|}^{M}(u^k_{i,j})_{\alpha\bar{\alpha}}\phi^k_{i,j}=-\sum_{k=|i|}^{M}[((u^k_{i,j})_{\bar{\alpha}}\phi^k_{i,j})_\alpha-(u^k_{i,j})_\alpha(\phi^k_{i,j})_\alpha]=-\sum_{k=|i|}^{M}\{[(u^k_{i,j})_\alpha\times(\phi^k_{i,j})_{\bar{\alpha}}+$$
$$+(u^k_{i,j})_\alpha(M)\phi^k_{i,j}(M)-(u^k_{i,j})(|i|)\phi^k_{i,j}(|i|)]_\alpha-(u^k_{i,j})_\alpha(\phi^k_{i,j})_\alpha\}=$$
$$=-\sum_{k=|i|}^{M}[(u^k_{i,j})_\alpha(\phi^k_{i,j})_{\bar{\alpha}}+(u^k_{i,j})_\alpha(\phi^k_{i,j})(i+1)(k)]+(u^k_{i,j})_\alpha(M)\phi^k_{i,j}(M)-(u^k_{i,j})(|i|)(\phi^k_{i,j})_\alpha(|i|);$$

$$\sum_{k=|i|}^{M}\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}u_{y\bar{y}}(\phi^k_{i,j})=-\sum_{k=|i|}^{M}[\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})_{\bar{y}}(\phi^k_{i,j})_y-\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})\bar{y}(\phi^k_{i,j})_y-$$
$$-\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})_{\bar{y}}(\phi^k_{i,j})](k)=-\sum_{k=|i|}^{M}[\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}u^k_{i,j}\phi^k_{i,j}-\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(D)u^k_{i,j}(D)(\phi^k_{i,j})(D)- \quad (38)$$
$$-\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(-D)(u^k_{i,j})(-D)\phi^k_{i,j}(-D)]_y+\frac{ra_{ij}}{2\rho a_{ijcm_{ij}}}(D)(u^k_{i,j})_y(\phi^k_{i,j})_y+$$
$$+(\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}})_y(u^k_{i,j})_{\bar{y}}\phi^k_{i,j}=-\sum_{k=|i|}^{M}\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})_y(\phi^k_{i,j})_y+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})_y\phi^k_{i,j}(i+1);$$

Then formula (37) will be

$$\tau h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}\sum_{k=|i|}^{M}[(u^k_{i,j})_t(\phi^k_{i,j})_t+(u^k_{i,j})_\alpha(\phi^k_{i,j})_\alpha+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})_y(\phi^k_{i,j})_y+$$
$$+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}\alpha_y(u^k_{i,j})_\alpha(\phi^k_{i,j})_y+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}\Delta\alpha(u^k_{i,j})_\alpha\cdot(\phi^k_{i,j})_\alpha]+(u^k_{i,j})_\alpha(\phi^k_{i,j})(i+1)+$$
$$+(\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}})_y(u^k_{i,j})_y(\phi^k_{i,j})(i+1)+(\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}\alpha_y)_y(u^k_{i,j})_\alpha(\phi^k_{i,j})(j+1)+$$
$$+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}\Delta\alpha_{i,j}(u^k_{i,j})_\alpha(\phi^k_{i,j})(j+1)+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(u^k_{i,j})_\alpha\phi^k_{i,j}= \quad (39)$$
$$=h_1h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}[(u^k_{i,j})_t(M)\phi^k_{i,j}(M)-(u^k_{i,j})_t(|i|)\phi^k_{i,j}(|i|)+(u^k_{i,j})_\alpha(M)\phi^k_{i,j}(M)-$$
$$-(u^k_{i,j})_\alpha(|i|)\phi^k_{i,j}(|i|)+L(u^k_{i,j})(M)\phi^k_{i,j}(M)-L(u^k_{i,j})(|i|)\phi^k_{i,j}(|i|)]=$$
$$=\tau h_2\sum_{j=-L}^{L}\sum_{i=-N}^{N}S_{i,j}\phi^k_{i,j},\quad k=\overline{|i|,M}.$$

Passing to the limit, for $\tau\to 0$, $h_1\to 0$, $h_2\to 0$, we obtain

$$\int_0^t\int_{-D}^{D}\int_{|\alpha|}^{t}[(\tilde{u}^k_{ij})_\tau(\tilde{\phi}^k_{ij})_\tau+(\tilde{u}^k_{ij})_\alpha(\tilde{\phi}^k_{ij})_\alpha+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}(\tilde{u}^k_{ij})_y(\tilde{\phi}^k_{ij})_y+$$
$$+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}\alpha_y(\tilde{u}^k_{ij})_\alpha(\tilde{\phi}^k_{ij})+\frac{ra_{ij}}{2\rho a_{ij}Cm_{ij}}\Delta\alpha_{ij}(\tilde{u}^k_{ij})_\alpha\tilde{\phi}^k_{ij}-\frac{1}{Cm_{ij}\rho m_{ij}\cdot l}*u^k_{ij}*\phi^k_{ij}= \quad (40)$$
$$=\int_{|\alpha|}^{t}\int_{-D}^{D}S_{ij}\phi^k_{ij}d\tau dy,\quad t\in[0,T].$$

Piecewise continuous functions are denoted by a wavy dash at the top, which coincide with the corresponding function at the grid nodes.Since, all these piecewise-continuous functions converge to the corresponding function, and also taking into account that $(\tilde{u}^k_{ij}), (\tilde{u}^k_{ij})_t, (\tilde{u}^k_{ij})_\alpha, (\tilde{u}^k_{ij})_y$ converge weakly to the functions u, $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial y}$ respectively. Then passing to the limit, we obtain the generalized solution (13). Thus, the theorem is proved:

Theorem 1. *Suppose that conditions (3), (16), (27), (34), (35) are satisfied and the function $u(\alpha, y, t)$ is continuous and has continuous partial derivatives of the first order in $\Omega(T, D)$ and let $S(t, y) \in L_2(\Omega(T, D))$. Then there exists a generalized solution of the problem (6) in the space $W^1_2(\Omega(T, D))$.*

From the equivalence of the direct problems (6) and (1) - (2) it follows that there exists a generalized solution of the direct problem (1) - (2).

In connection with the fact that the hyperbolic problem is obtained from the parabolic problem (1 ′) - (2′) by the equivalent Laplace transform, the existence of a generalized solution of the parabolic problem (1 ′) - (2′) also follows.

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