Exact Solutions of Fractional Partial Differential Equation Systems with Conformable Derivative

Ozan Özkan[a], Ali Kurt[b]

[a]Department of Mathematics, Selçuk University, Konya, Turkey
[b]Department of Mathematics, Pamukkale University, Denizli, Turkey

Abstract. Main goal of this paper is to have the new exact solutions of some fractional partial differential equation systems (FPDES) in conformable sense. The definition of conformable fractional derivative (CFD) is similar to the limit based definition of known derivative. This derivative obeys both rules which other popular derivatives do not satisfy such as derivative of the quotient of two functions, the derivative product of two functions, chain rule and etc. By using conformable derivative it is seen that the solution procedure for (PDES) is simpler and more efficient.

1. Introduction

The application of differential equations which arise in the field of medicine, engineering, social sciences, physics, and different branches of applied sciences is one of the interesting and most important area. Although there are many problems including differential equations, there are not any prevalent techniques for the solution of such problems. Huge amount of researchers use the integral transforms which is one of the greatest known scheme used for the solution of ordinary and partial differential equations. After the use of integral transform methods the differential, partial differential, integral, integro differential equations turn into an algebraic equation. So the solution procedure becomes simpler.

Fractional calculus, which has been aroused great interest with respect to its extensive area of applications in nearly all disciplines of applied sciences and engineering became a favorite subject in the last decades [11–13, 16]. Fractional derivatives were not used in physics, engineering and other disciplines although they have a long mathematical history. One of the reason of this event could be that there are multiple nonequivalent definitions of fractional derivatives and integrals. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character [17]. Any other reason is that Riemann-Liouville and Caputo fractional derivative include integral forms in their definitions and these integral forms make calculations complicated. In addition to this scientists determined many deficiencies of these definitions. For instance [10]

1. The Riemann-Liouville derivative of a constant do not equal to zero (Caputo derivative satisfies).

2. The formula of the derivative of the product of two functions do not satisfied by both Riemann-Liouville and Caputo definitions.

$$D_a^\alpha(fg) = gD_a^\alpha(f) + fD_a^\alpha(g).$$

3. the known formula of the derivative of the quotient of two functions do not satisfied by both Riemann-Liouville and Caputo definitions.

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{fD_a^\alpha(f) - gD_a^\alpha(g)}{g^2}.$$

4. The chain rule do not satisfied by both Riemann-Liouville and Caputo definitions.

$$D_a^\alpha(f \circ g)(t) = f^\alpha(g(t))g^\alpha(t).$$

5. All fractional derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general.

6. In the Caputo definition it is assumed that the function f is differentiable.

For these reasons scientists decided to express efficient, applicable, limpid and simple definition of arbitrary order derivation and integration. In 2014, a new, well behaved arbitrary order derivative and integral definition that satisfies basic properties of Newtonian concept derivative and integral are expressed by Khalil et. al. [10].

Definition 1.1. $f : [0, \infty) \to \mathbb{R}$ be a function. The αth order *"CFD"* of f is stated by,

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0, 1)$.

Definition 1.2. If f is α-differentiable in some $(0, a), a > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function f starting from $a \geq 0$ is stated as:

$$I_\alpha^a(f)(t) = \int_a^t f(x)d_\alpha x = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Neither Riemann-Liouville definition nor Caputo definition have physical and geometrical descriptions. But CFD has the physical and geometrical explanations [25] and satisfies the following basic properties and theorems referred in [1, 10]

1. $D_\alpha(cf + dg) = cD_\alpha(f) + cD_\alpha(g)$ for all $a, b \in \mathbb{R}$.

2. $D_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.

3. $D_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$.

4. $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$.

5. $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$.

6. If f is differentiable, then $D_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

In addition to these advantages CFD is type of local fractional derivatives are important in themselves being used, due to their well-behaved, in generalizing and describing certain models depending on ordinary or partial differential equations [6, 7, 14, 18–24]. Exact solutions can be obtained. Besides that CFD can be used to generalize certain inequalities to arbitrary order [2, 4]. Moreover, CFD and their modified Laplace are defined in [1] and investigated in [8] below are so important in generating and dealing with generalized types of nonlocal fractional derivatives [3, 4, 9]. Also Al-Refai and Abdeljawad [5] suggested a fractional generalization of the well-known Sturm-Liouville eigenvalue problems in conformable sense.

The rest of the article is organized as follows. In Section 2 the definition of newly defined conformable double Laplace transform (CDLT) and some theorems over basic properties of this definition are given.In Section 3 the exact solutions for system of fractional partial differential equations are expressed as implementation of conformable double Laplace transform.

2. Basics of Double Laplace Transform of Conformable Type

Definition 2.1. Let $u(x,t)$ be an exponential order and continuous on the interval $[0, \infty)$ and for some $a, b \in \mathbb{R}$ $\sup_{x>0,t>0} \frac{|u(x,t)|}{e^{\frac{ax^\beta}{\beta} + \frac{bt^\alpha}{\alpha}}} < \infty$ satisfied. The CDLT can be stated as [15]

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [u(x,t)] = U(p,s) = \int_0^\infty \int_0^\infty e^{-p\frac{x^\beta}{\beta} - s\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t d_\beta x, \tag{1}$$

where $p, s \in \mathbb{C}$, $0 < \alpha, \beta \le 1$ and the integrals are in the sense of conformable fractional integral.

Definition 2.2. The conformable Laplace transform (CLT) with respect to x is denoted as [15]

$$\mathscr{L}_x^\beta [u(x,t)] = U(p,t) = \int_0^\infty e^{-p\frac{x^\beta}{\beta}} u(x,t) d_\beta x, \tag{2}$$

where the the considered integral is in conformable sense with respect to x.

The symbol $\mathscr{L}_x^\beta [u(x,t)]$ indicates the conformable integral of (2), we consider the variable which the single CLT applied by the help of the subscript x on $\mathscr{L}$. Alike the CLT with respect to variable t is stated as

$$\mathscr{L}_t^\alpha [u(x,t)] = U(x,s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t. \tag{3}$$

Definition 2.3. Let $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [u(x,t)] = U(p,s)$. Then the double inverse conformable Laplace transform can be defined as

$$\mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} [U(p,s)] = u(x,t) = \frac{1}{4\pi^2} \lim_{\varrho,\omega\to\infty} \int_{(\alpha(c-i\varrho))^{\frac{1}{\alpha}}}^{(\alpha(c+i\varrho))^{\frac{1}{\alpha}}} \int_{(\beta(d-i\omega))^{\frac{1}{\beta}}}^{(\beta(d+i\omega))^{\frac{1}{\beta}}} e^{\frac{(st)^\alpha}{\alpha} + \frac{(px)^\beta}{\beta}} U_{\beta,\alpha}(p,s) d_\beta p d_\alpha s,$$

where $U_{\beta,\alpha}(p,s) = U\left(\frac{p^\beta}{\beta}, \frac{s^\alpha}{\alpha}\right)$.

One can easily see that the double inverse CLT satisfies the following properties .

1. Double inverse CLT is linear. Namely let $a, b \in \mathbb{R}$, $\mathscr{L}_\beta \mathscr{L}_\alpha [u(x,t)] = U(p,s)$, $\mathscr{L}_\beta \mathscr{L}_\alpha [v(x,t)] = V(p,s)$ then

$$\mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} [aU_{\beta,\alpha}(p,s) + bV_{\beta,\alpha}(p,s)] = a\mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} [U_{\beta,\alpha}(p,s)] + b\mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} [V_{\beta,\alpha}(p,s)].$$

2. Let $\mathscr{L}_\beta \mathscr{L}_\alpha [uf(x,t)] = U(p,s)$ and $c, d \in \mathbb{R}$ then

$$\mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} [U_{\beta,\alpha}(p+d, s+c)] = e^{\frac{-ct^\alpha}{\alpha} - \frac{dx^\beta}{\beta}} \mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} [U_{\beta,\alpha}(p,s)].$$

3. Let $\mathscr{L}_\beta \mathscr{L}_\alpha [u(x,t)] = U(p,s)$ and $\sigma, \gamma \in \mathbb{R}$ afterwards

$$\mathscr{L}_\beta^{-1} \mathscr{L}_\alpha^{-1} \left[U_{\beta,\alpha} \left(\frac{p}{\gamma^\beta}, \frac{s}{\sigma^\alpha} \right) \right] = \gamma^\beta \sigma^\alpha u(\gamma x, \sigma t).$$

2.1. Properties of Conformable Double Laplace Transform

In this part we expressed some properties of CDLT.

Theorem 2.4. ([15]) *Let $u(x,t)$, $w(x,t)$ have the CDLT. Thus,*

i. $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [c_1 u(x,t) + c_2 w(x,t)] = c_1 \mathscr{L}_t^\alpha \mathscr{L}_x^\beta [u(x,t)] + c_2 \mathscr{L}_t^\alpha \mathscr{L}_x^\beta [w(x,t)]$ *where c_1 and c_2 are real constants.*

ii. $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [e^{-d\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}} u(x,t)] = U(p+d, s+c).$

iii. $\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [f(\gamma x, \sigma t)] = \frac{1}{r} U\left(\frac{p}{\gamma^\beta}, \frac{s}{\sigma^\alpha}\right),$ *where $r = \gamma^\beta \sigma^\alpha$.*

iv. $(-1)^{m+n} \mathscr{L}_t^\alpha \mathscr{L}_x^\beta \left[\frac{x^{m\beta}}{\beta^m} \frac{t^{n\alpha}}{\alpha^n} u(x,t) \right] = \frac{\partial^{m+n} U(p,s)}{\partial p^m \partial s^n}.$

Proof.   i. By using the definition of CDLT the proof of (*i*) can be shown easily.

ii.

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta \left[e^{-d\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}} u(x,t) \right] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\beta}{\beta} - s\frac{t^\alpha}{\alpha}} e^{-d\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t d_\beta x$$

$$= \int_0^\infty e^{-p\frac{x^\beta}{\beta} - d\frac{x^\beta}{\beta}} \left(\int_0^\infty e^{-s\frac{t^\alpha}{\alpha} - c\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t \right) d_\beta x. \tag{4}$$

With the aid of CLT definition

$$\int_0^\infty e^{-(s+c)\frac{t^\alpha}{\alpha}} u(x,t) d_\alpha t = U(x, s+c). \tag{5}$$

Now subrogating the Eqn. 5 into Eqn. 4 yields

$$\int_0^\infty e^{-(p+d)\frac{x^\beta}{\beta}} U(x, s+c) d_\beta x = U(p+d, s+c).$$

iii. Let $\tau = \gamma x$ and $\chi = \sigma t$, so the proof can be expressed as follows

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [u(\gamma x, \sigma t)] = \int_0^\infty \int_0^\infty e^{-p\frac{x^\beta}{\beta} - s\frac{t^\alpha}{\alpha}} u(\gamma x, \sigma t) d_\alpha t d_\beta x$$

$$= \int_0^\infty e^{-p\frac{x^\beta}{\beta}} \left(\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} u(\gamma x, \sigma t) d_\alpha t \right) d_\beta x$$

$$= \frac{1}{\sigma^\alpha} \int_0^\infty e^{-p\frac{x^\beta}{\beta}} \left(\int_0^\infty e^{-s\frac{\chi^\alpha}{\sigma^\alpha \alpha}} u(\gamma x, \chi) d_\alpha \chi \right) d_\beta x$$

$$= \frac{1}{\sigma^\alpha} \int_0^\infty e^{-p\frac{x^\beta}{\beta}} U\left(\gamma x, \frac{s}{\sigma^\alpha} \right) d_\beta x$$

$$= \frac{1}{\sigma^\alpha \gamma^\beta} \int_0^\infty e^{-p\frac{\tau^\beta}{\gamma^\beta \beta}} U\left(\tau, \frac{s}{\sigma^\alpha} \right) d_\beta \tau$$

$$= \frac{1}{\gamma^\beta \sigma^\alpha} U\left(\frac{p}{\gamma^\beta}, \frac{s}{\sigma^\alpha} \right).$$

iv. The order of differentiation and integration can be changed, with respect to convergence properties of the improper integral involved. So we can differentiate with respect to p, s under the integral sign. Hence,

$$\frac{\partial^{m+n} U(p,s)}{\partial p^m \partial s^n} = \int_0^\infty \frac{\partial^m}{\partial p^m} e^{-p\frac{x^\beta}{\beta}} \left[\int_0^\infty \frac{\partial^n}{\partial s^n} \left(e^{-s\frac{t^\alpha}{\alpha}} u(x,t) \right) d_\alpha t \right] d_\beta x.$$

If differentiation is iterated with respect to p and s, led to the following equation

$$\frac{\partial^{m+n} U(p,s)}{\partial p^m \partial s^n} = (-1)^{m+n} \mathscr{L}_t^\alpha \mathscr{L}_x^\beta \left[\frac{x^{m\beta}}{\beta^m} \frac{t^{n\alpha}}{\alpha^n} u(x,t) \right].$$

□

Lemma 2.5. ([15]) *β-th and α-th order CDLT of conformable fractional partial derivatives can be expressed*

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [_x D_\beta u(x,t)] = pU(p,s) - U(0,s), \tag{6}$$

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [_t D_\alpha u(x,t)] = sU(p,s) - U(p,0). \tag{7}$$

Now considering the mixed fractional partial derivatives with CDLT of

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [_x D_{\beta t} D_\alpha (u(x,t))] = psU(p,s) - pU(p,0) - sU(0,s) + U(0,0). \tag{8}$$

Proof. The proof can be obtained with the aid of the definition of conformable fractional integral and the Theorem 2.2 in [10]. □

Theorem 2.6. ([15]) *Let $u(x,t) \in C^l(\mathbb{R}^+ \times \mathbb{R}^+)$ with $l = \max(m,n)$, where $0 < \alpha, \beta \leq 1$ and $m, n \in \mathbb{N}$. Also regard the CLT of $u(x,t)$, $_x D_\beta^{(i)} u(x,t)$ and $_t D_\alpha^{(j)} u(x,t)$ $i = 1, ..., m$, $j = 1, ..., n$ can be obtained. Then*

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [_x D_\beta^{(m)} u(x,t)] = p^m U(p,s) - p^{m-1} U(0,s) - \sum_{i=1}^{m-1} p^{m-1-i} \mathscr{L}_t^\alpha [_x D_\beta^{(i)} U(0,t)], \tag{9}$$

$$\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [_t D_\alpha^{(n)} u(x,t)] = s^n U(p,s) - s^{n-1} U(p,0) - \sum_{j=1}^{n-1} s^{n-1-j} \mathscr{L}_x^\beta [_t D_\alpha^{(j)} U(x,0)]. \tag{10}$$

By the same procedure, CDLT applied version of the conformable mixed partial derivatives can be obtained as

$$\begin{aligned}
\mathscr{L}_t^\alpha \mathscr{L}_x^\beta [_x D^{(m)\beta}{}_t D^{(n)\alpha} (u(x,t))] = p^m s^n \Big(& U(p,s) - s^{-1} U(p,0) \\
& - p^{-1} U(0,s) - \sum_{j=1}^{n-1} s^{-j-1} \mathscr{L}_x^\beta [_t D^{(j)\alpha} U(x,0)] \\
& - \sum_{i=1}^{m-1} p^{-i-1} \mathscr{L}_t^\alpha [_x D^{(i)\beta} U(0,t)] \\
& + \sum_{j=1}^{n-1} s^{-j-1} p^{-1} {}_t D^{(j)\alpha} U(0,0) \\
& + \sum_{i=1}^{m-1} s^{-1} p^{-i-1} {}_x D^{(i)\beta} U(0,0) \\
& + \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} s^{-j-1} p^{-i-1} {}_t D^{(j)\alpha} {}_x D^{(i)\beta} U(0,0) \\
& + p^{-1} s^{-1} U(0,0) \Big),
\end{aligned} \tag{11}$$

where $_x D_\beta^{(m)} u(x,t)$, $_t D_\alpha^{(n)} u(x,t)$ indicates m, n times successive CFD of function $u(x,t)$ in r β-th and α-th order respectively.

Proof. The proof follows from Lemma 2.1. □

Functions $f(x,t)$	Conformable Double Laplace Transform $f(p,s)$
ab	$\frac{ab}{ps}$
xt	$\beta^{\frac{1}{\beta}}\alpha^{\frac{1}{\alpha}}\frac{\Gamma(1+\frac{1}{\beta})\Gamma(1+\frac{1}{\alpha})}{p^{\frac{\beta+1}{\beta}}s^{\frac{\alpha+1}{\alpha}}}$
$\frac{x^{\beta}}{\beta}\frac{t^{\alpha}}{\alpha}$	$\frac{1}{p^2s^2}$
$\frac{x^{m\beta}}{\beta}\frac{t^{n\alpha}}{\alpha}$, m,n are natural numbers	$\frac{m!n!}{p^{m+1}s^{n+1}}$
$e^{\frac{x^{\beta}}{\beta}+\frac{t^{\alpha}}{\alpha}}$	$\frac{1}{(s-1)(p-1)}$
$e^{\frac{x^{\beta}}{\beta}+\frac{t^{\alpha}}{\alpha}}\frac{x^{m\beta}}{\beta}\frac{t^{n\alpha}}{\alpha}$, m,n are natural numbers	$\frac{m!n!}{(p-1)^{m+1}(s-1)^{n+1}}$
$\cos\left(\omega\frac{x^{\beta}}{\beta}\right)\cos\left(\omega\frac{t^{\alpha}}{\alpha}\right)$	$\frac{ps}{(w^2+s^2)(w^2+p^2)}$
$\sin\left(\omega\frac{x^{\beta}}{\beta}\right)\sin\left(\omega\frac{t^{\alpha}}{\alpha}\right)$	$\frac{w^2}{(w^2+s^2)(w^2+p^2)}$
$e^{\frac{x^{\beta}}{\beta}+\frac{t^{\alpha}}{\alpha}}\sinh\left(\frac{x^{\beta}}{\beta}\right)\sinh\left(\frac{t^{\alpha}}{\alpha}\right)$	$\frac{1}{(p-2)p(s-2)s}$
$e^{\frac{x^{\alpha}}{\alpha}+\frac{t^{\beta}}{\beta}}\cosh\left(\frac{x^{\alpha}}{\alpha}\right)\cosh\left(\frac{t^{\beta}}{\beta}\right)$	$\frac{(p-1)(s-1)}{(p-2)p(s-2)s}$

Table 1: CDLT of some basic functions.

3. Illustrative Examples

Example 3.1. Regard the following time-space conformable FPDE system

$$\begin{cases} D_t^{\alpha}u(x,t) - D_x^{\beta}v(x,t) + u(x,t) + v(x,t) &= 0, \\ D_t^{\alpha}v(x,t) - D_x^{\beta}u(x,t) + u(x,t) + v(x,t) &= 0 \end{cases} \tag{12}$$

with the conditions

$$\begin{aligned} u(0,t) &= \sinh\left(-\frac{t^{\alpha}}{\alpha}\right), \\ u(x,0) &= \sinh\left(\frac{x^{\beta}}{\beta}\right), \\ v(0,t) &= \cosh\left(-\frac{t^{\alpha}}{\alpha}\right), \\ v(x,0) &= \cosh\left(\frac{x^{\beta}}{\beta}\right), \end{aligned} \tag{13}$$

where $0 < \beta \leq 1$, $0 < \alpha \leq 1$, $x > 0$, $t > 0$, D_t^{α}, D_x^{β} denotes α-th and β-th order CFD of function $u(x,t)$ and $v(x,t)$. Firstly employing the the CDLT to Eq. (12)

$$\begin{aligned} sU(p,s) - U(p,0) - (pV(p,s) - V(0,s)) + V(p,s) + U(p,s) &= 0 \\ sV(p,s) - V(p,0) - (pU(p,s) - U(0,s)) + V(p,s) + U(p,s) &= 0, \end{aligned} \tag{14}$$

where $U(p, s)$, $V(p, s)$ show the double conformable Laplace transformed version of the functions $u(x, t)$ and $v(x, t)$ respectively. When we implement the CLT to the given initial conditions given in (13) arises

$$\mathscr{L}_t^\alpha[u(0, t)] = U(0, s) = -\frac{1}{s^2 - 1},$$

$$\mathscr{L}_x^\beta[u(x, 0)] = U(p, 0) = \frac{1}{p^2 - 1},$$

$$\mathscr{L}_t^\alpha[v(0, t)] = V(0, s) = \frac{s}{s^2 - 1},$$

$$\mathscr{L}_x^\beta[v(x, 0)] = V(p, 0) = \frac{p}{p^2 - 1}.$$

(15)

Using the equalities given in (15) and after making some algebraic calculations in Eqn. (14) we acquire

$$U(p, s) = \frac{-p + s}{(p^2 - 1)(s^2 - 1)},$$

$$V(p, s) = \frac{-1 + ps}{(p^2 - 1)(s^2 - 1)}.$$

Hence the unknown functions $u(x, t)$ and $v(x, t)$ can be obtained as

$$u(x, t) = \sinh\left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha}\right),$$

$$v(x, t) = \cosh\left(\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha}\right).$$

Example 3.2. Let observe the time-space fractional non-homogenous partial differential equation system given as

$$\begin{cases} D_t^\alpha u(x, t) - D_x^\beta v(x, t) - u(x, t) + v(x, t) &= -2, \\ D_t^\alpha v(x, t) + D_x^\beta u(x, t) - u(x, t) + v(x, t) &= -2 \end{cases}$$

(16)

via the conditions

$$u(0, t) = e^t + 1, v(0, t) = e^{-t} - 1,$$
$$u(x, 0) = e^x + 1, v(x, 0) = e^x - 1$$

(17)

with $0 < \beta \le 1$, $0 < \alpha \le 1$, $x > 0$, $t > 0$, D_t^α, D_x^β symbolizes the α and β order CFD of functions $u(x, t)$ and $v(x, t)$. Operating the CDLT for Eq. (16) produces

$$sU(p, s) - U(p, 0) - (pV(p, s) - V(0, s)) - U(p, s) + V(p, s) = \frac{-2}{ps},$$

$$sV(p, s) - V(p, 0) - (pU(p, s) - U(0, s)) - U(p, s) + V(p, s) = \frac{-2}{ps}.$$

(18)

Thereafter using CLT for the conditions (17)

$$\mathscr{L}_t^\alpha[u(0, t)] = U(0, s) = \frac{1}{s - 1} + \frac{1}{s}, \mathscr{L}_t^\alpha[V(0, t)] = V(0, s) = \frac{1}{s + 1} - \frac{1}{s} +,$$

$$\mathscr{L}_x^\beta[u(x, 0)] = U(p, 0) = \frac{1}{p - 1} + \frac{1}{p}, \mathscr{L}_t^\alpha[V(x, 0)] = V(p, 0) = \frac{1}{p - 1} - \frac{1}{p}.$$

(19)

Associating all the obtained results (18),(19) and making some algebraic regulations yield

$$U(p,s) = \frac{-1 + p + s - 2ps}{ps(p-1)(s-1)},$$

$$V(p,s) = \frac{-1 + p - s}{ps(p-1)(s+1)}.$$

Thus we can get the functions $u(x,t)$ and $v(x,t)$ as

$$u(x,t) = e^{\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}} + 1,$$

$$v(x,t) = e^{\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha}} - 1.$$

Example 3.3. Now let us to discuss the following FPDE including higher order and mixed order partial derivatives

$$\begin{cases} D_x^{2\beta} u(x,t) & = D_t^\alpha D_x^\beta u(x,t) + D_t^{2\alpha} u(x,t) w(x,t), \\ D_t^{2\alpha} v(x,t) & = D_t^\alpha D_x^\beta v(x,t) + D_x^{2\beta} v(x,t) u(x,t) \end{cases} \tag{20}$$

with the initial conditions

$$u(x,0) = e^{-\frac{x^\beta}{\beta}} + e^{\frac{x^\beta}{\beta}} + \cos\left(\frac{x^\beta}{\beta}\right) + \sin\left(\frac{x^\beta}{\beta}\right),$$

$$u(0,t) = e^{-\frac{t^\alpha}{\alpha}} + e^{\frac{t^\alpha}{\alpha}} + \cos\left(\frac{t^\alpha}{\alpha}\right) + \sin\left(\frac{t^\alpha}{\alpha}\right),$$

$$D_t^\alpha u(x,0) = -e^{-\frac{x^\beta}{\beta}} + e^{\frac{x^\beta}{\beta}} + \cos\left(\frac{x^\beta}{\beta}\right) - \sin\left(\frac{x^\beta}{\beta}\right),$$

$$D_x^\beta u(0,t) = -e^{-\frac{t^\alpha}{\alpha}} + e^{\frac{t^\alpha}{\alpha}} + \cos\left(\frac{t^\alpha}{\alpha}\right) - \sin\left(\frac{t^\alpha}{\alpha}\right),$$

$$u(0,0) = 3,$$

$$v(x,0) = -e^{-\frac{x^\beta}{\beta}} - e^{\frac{x^\beta}{\beta}} + \cos\left(\frac{x^\beta}{\beta}\right) + \sin\left(\frac{x^\beta}{\beta}\right), \tag{21}$$

$$v(0,t) = -e^{-\frac{t^\alpha}{\alpha}} - e^{\frac{t^\alpha}{\alpha}} + \cos\left(\frac{t^\alpha}{\alpha}\right) + \sin\left(\frac{t^\alpha}{\alpha}\right),$$

$$D_t^\alpha v(x,0) = e^{-\frac{x^\beta}{\beta}} - e^{\frac{x^\beta}{\beta}} + \cos\left(\frac{x^\beta}{\beta}\right) - \sin\left(\frac{x^\beta}{\beta}\right),$$

$$D_x^\beta v(0,t) = e^{-\frac{t^\alpha}{\alpha}} - e^{\frac{t^\alpha}{\alpha}} + \cos\left(\frac{t^\alpha}{\alpha}\right) - \sin\left(\frac{t^\alpha}{\alpha}\right),$$

$$v(0,0) = -1,$$

where all the derivatives are in conformable sense and $D_x^{2\beta}, D_t^{2\alpha}$ means two times conformable derivative of functions $u(x,t)$ and $v(x,t)$. Utilizing the CDLT definition and Lemma 2.1 and Theorem 2.3 for Eqns. (20) produces folowing equalities

$$p^2 U(p,s) - pU(0,s) - D_x^\beta U(0,s) - (psU(p,s) - pU(p,0) - sU(0,s) + U(0,0))$$

$$- (s^2 U(p,s) - sU(p,0) - D_t^\alpha U(p,0)) - V(p,s) = 0,$$

$$s^2 V(p,s) - sV(p,0) - D_t^\alpha V(p,0) - (psV(p,s) - pV(p,0) - sV(0,s) + V(0,0)) \tag{22}$$

$$- (p^2 V(p,s) - pV(0,s) - D_x^\beta V(0,s)) - U(p,s) = 0,$$

where $U(p,s), V(p,s)$ are the transformed versions of functions $u(x,t), v(x,t)$ respectively. Again performing CLT for the initial conditions that given in Eqns. (21), we have

$$
\begin{aligned}
U(p,0) &= \frac{-1 + p + p^2 + 3p^3}{p^4 - 1}, \\
U(0,s) &= \frac{-1 + s + s^2 + 3s^3}{s^4 - 1}, \\
D_t^\alpha U(p,0) &= \frac{3 - p + p^2 + p^3}{p^4 - 1}, \\
D_x^\beta U(0,s) &= \frac{3 - s + s^2 + s^3}{s^4 - 1}, \\
U(0,0) &= 3, \\
V(p,0) &= \frac{1 - 3p + p^2 - p^3}{p^4 - 1}, \\
V(0,s) &= \frac{-1 - 3s + s^2 - s^3}{s^4 - 1}, \\
D_t^\alpha V(p,0) &= \frac{-1 - p - 3p^2 + p^3}{p^4 - 1}, \\
D_x^\beta V(0,s) &= \frac{-1 - s - 3s^2 + s^3}{s^4 - 1}, \\
V(0,0) &= -1.
\end{aligned}
\tag{23}
$$

Adding up all the obtained data such as Eqns. (23) and (22) results as

$$
\begin{aligned}
U(p,s) &= \frac{1}{(p-1)(s-1)} + \frac{1}{ps + p + s + 1} + \frac{ps + p + s - 1}{(p^2 + 1)(s^2 + 1)}, \\
V(p,s) &= -\frac{1}{(p-1)(s-1)} - \frac{1}{ps + p + s + 1} + \frac{ps + p + s - 1}{(p^2 + 1)(s^2 + 1)}.
\end{aligned}
\tag{24}
$$

Thus unknown functions can be obtained as

$$
\begin{aligned}
u(x,t) &= e^{\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}} + e^{-\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha}} + \sin\left(\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}\right) + \cos\left(\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}\right), \\
v(x,t) &= -e^{\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}} - e^{-\frac{x^\beta}{\beta} - \frac{t^\alpha}{\alpha}} + \sin\left(\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}\right) + \cos\left(\frac{x^\beta}{\beta} + \frac{t^\alpha}{\alpha}\right).
\end{aligned}
\tag{25}
$$

4. Conclusion

The CDLT provides a powerful method for analyzing fractional partial differential equation systems. It can be easily seen that the theorems that described here can be further generated for other type of functions and relations. These relations can be used to calculate new conformable Laplace transform pairs in fractional calculus.

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