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Singularly Perturbed Parabolic Problem with Oscillating Initial Condition

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Abstract. The aim of this paper is to construct regularized asymptotics of the solution of a singularly perturbed parabolic problem with an oscillating initial condition. The presence of a rapidly oscillating function in the initial condition has led to the appearance of a boundary layer function in the solution, which has the rapidly oscillating character of the change. In addition, it is shown that the asymptotics of the solution contains exponential, parabolic boundary layer functions and their products describing the angular boundary layers. Continuing the ideas of works [1, 3] a complete regularized asymptotics of the solution of the problem is constructed.

1. Statement of the Problem

In this paper the first boundary value problem for a parabolic equation with a small parameter on the derivatives is studied:

$$L_\varepsilon u \equiv \varepsilon\partial_t u - \varepsilon^2 a(x)\,\partial_x^2 u - b(x,t)\,u = f(x,t),$$

$$u|_{t=0} = u^0(x)\,exp\left(\frac{iS^0(x)}{\varepsilon}\right), \quad u|_{x=0} = u|_{x=1} = 0,\ u^0(0) = u^0(1) = 0, \tag{1}$$

where $(x,t) \in \Omega,\ \Omega = (0,1)\,x\,(0,T],\ \varepsilon > 0$ is small parameter, $u = u(x,t,\varepsilon)$. The problem is solved under the following conditions:

1. the given functions are sufficiently smooth,
2. $\forall x \in [0,1]$, function $a(x) > 0$, $\forall t \in [0,T]$, function $b(x,t) < 0$,
3. function $S_k(x,t)$, $k = 2,3$ are solutions of Cauchy problem:

$$i\partial_t S_2(x,t) + a(x)\left(\partial_x S_2(x,t)\right)^2 = b(x,t),\ S_2(x,t)|_{t=0} = S^0(x),$$

$$\partial_t S_3(x,t) - a(x)\left(\partial_x S_3(x,t)\right)^2 = b(x,t),\ S_3(x,t)|_{t=0} = 0,$$

where the function $S_3(x,t)$ satisfies condition $Re\,S_3(x,t) \le 0,\ \forall x,t \in \overline{\Omega}$.

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2. Regularization of the Problem

For the regularization of the problem (1) regularizing variables are introduced:

$$\tau_1 = \frac{t}{\varepsilon^2},\ \tau_k = \frac{S_k(x,t)}{\varepsilon},\ k=2,3,\ \xi_l = \frac{\varphi_l(x)}{\sqrt{\varepsilon}},\ \eta_l = \frac{\varphi_l(x)}{\sqrt{\varepsilon^3}},\ \varphi_l(x) = (-1)^{l-1}\int_{l-1}^{x}\frac{ds}{\sqrt{a(s)}},\ l=1,2 \tag{2}$$

and an extended function $\widetilde{u}(M,\varepsilon)$, $M=(x,t,\xi,\eta,\tau)$ such as:

$$\begin{gathered}\widetilde{u}(M,\varepsilon)|_{\chi=\psi(x,t,\varepsilon)} \equiv u(x,t,\varepsilon),\ \chi=(\tau,\xi,\eta),\ \tau=(\tau_1,\tau_2,\tau_3),\ \xi=(\xi_1,\xi_2),\\ \eta=(\eta_1,\eta_2),\ \psi(x,t,\varepsilon) = \left(\tfrac{t}{\varepsilon^2}, \tfrac{S_2(x,t)}{\varepsilon}, \tfrac{S_3(x,t)}{\varepsilon}, \tfrac{\varphi(x)}{\sqrt{\varepsilon}}, \tfrac{\varphi(x)}{\sqrt{\varepsilon^3}}\right).\end{gathered} \tag{3}$$

Considering (2) from (3) the following derivatives are found:

$$\begin{gathered}\partial_t u \equiv \left(\partial_t\widetilde{u} + \tfrac{1}{\varepsilon}\partial_t S_2(x,t)\partial_{\tau_2}\widetilde{u} + \tfrac{1}{\varepsilon^2}\partial_{\tau_1}\widetilde{u} + \tfrac{1}{\varepsilon}\partial_t S_3(x,t)\partial_{\tau_3}\widetilde{u}\right)|_{\chi=\psi(x,t,\varepsilon)},\\
\partial_x u \equiv \left(\partial_x\widetilde{u} + \tfrac{\partial_x S_2}{\varepsilon}\partial_{\tau_2}\widetilde{u} + \tfrac{\partial_x S_3}{\varepsilon}\partial_{\tau_3}\widetilde{u} + \textstyle\sum_{l=1}^{2}\left[\tfrac{\varphi_l'(x)}{\sqrt{\varepsilon}}\partial_{\xi_l}\widetilde{u} + \tfrac{\varphi_l'(x)}{\sqrt{\varepsilon^3}}\partial_{\eta_l}\widetilde{u}\right]\right)|_{\chi=\psi(x,t,\varepsilon)},\\
\partial_x^2 u \equiv \left(\partial_x^2\widetilde{u} + \textstyle\sum_{k=1}^{2}\left[\left(\tfrac{\partial_x S_{k+1}}{\varepsilon}\right)^2\partial^2_{\tau_{k+1}}\widetilde{u} + \tfrac{1}{\varepsilon}\left(2\partial_x S_{k+1}\partial^2_{x\tau_{k+1}}\widetilde{u} + \partial_x^2 S_{k+1}\partial_{\tau_{k+1}}\widetilde{u}\right)\right] +\right.\\
+ \textstyle\sum_{l=1}^{2}\left(\left[\tfrac{1}{\varepsilon}\varphi_l'^2(x)\partial^2_{\xi_l}\widetilde{u} + \tfrac{1}{\varepsilon^3}\varphi_l'^2(x)\partial^2_{\eta_l}\widetilde{u}\right] +\right.\\
\left.\left. + \tfrac{1}{\sqrt{\varepsilon}}\left[2\varphi_l'(x)\partial^2_{x\xi_l}\widetilde{u} + \varphi_l''(x)\partial_{\xi_l}\widetilde{u}\right] + \tfrac{1}{\sqrt{\varepsilon^3}}\left[2\varphi_l'(x)\partial^2_{x\eta_l}\widetilde{u} + \varphi_l''(x)\partial_{\eta_l}\widetilde{u}\right]\right)\right)|_{\chi=\psi(x,t,\varepsilon)}.\end{gathered} \tag{4}$$

By virtue of (1), (3), (4) we can put the extended problem:

$$\begin{gathered}\widetilde{L_\varepsilon}\widetilde{u} \equiv \varepsilon\partial_t\widetilde{u} + \textstyle\sum_{k=1}^{2}\left[\partial_t S_{k+1}\partial_{\tau_{k+1}}\widetilde{u} + a(x)(\partial_x S_{k+1})^2\partial^2_{\tau_{k+1}}\widetilde{u}\right] - b(x,t)\widetilde{u} + \tfrac{1}{\varepsilon}\left[\partial_{\tau_1}\widetilde{u} - \Delta_\eta\widetilde{u}\right] -\\
-\varepsilon\Delta_\xi\widetilde{u} - \varepsilon\textstyle\sum_{k=1}^{2}L_{S_k}\widetilde{u} - \varepsilon\sqrt{\varepsilon}L_\xi\widetilde{u} - \sqrt{\varepsilon}L_\eta\widetilde{u} - \varepsilon^2 L_x\widetilde{u} = f(x,t),\\
\widetilde{u}|_{t=0,\tau_3=0,\tau_1=0,\tau_2=\frac{iS^0(x)}{\varepsilon}} = u^0(x)\exp\left(\tfrac{iS^0(x)}{\varepsilon}\right),\ \widetilde{u}|_{x=0,\xi_1=\eta_1=0} = \widetilde{u}|_{x=1,\xi_2=\eta_2=0} = 0,\end{gathered} \tag{5}$$

where the following notations are introduced:

$$\begin{gathered}L_{S_k} \equiv \left(2\partial_x S_{k+1}(x,t)\partial^2_{x\tau_{k+1}} + \partial_x^2 S_k(x,t)\partial_{\tau_{k+1}}\right)a(x),\\
L_\xi \equiv a(x)\textstyle\sum_{l=1}^{2}\left[2\varphi_l'(x)\partial^2_{x\xi_l} + \varphi_l''(x)\partial_{\xi_l}\right],\\
L_\eta \equiv a(x)\textstyle\sum_{l=1}^{2}\left[2\varphi_l'(x)\partial^2_{x\eta_l} + \varphi_l''(x)\partial_{\eta_l}\right],\\
L_x \equiv a(x)\partial_x^2,\ \Delta_\xi \equiv \textstyle\sum_{l=1}^{2}\partial^2_{\xi_l},\ \Delta_\eta \equiv \textstyle\sum_{l=1}^{2}\partial^2_{\eta_l}.\end{gathered}$$

Problem (5) is regular with respect to ε:

$$\left(\widetilde{L_\varepsilon}\widetilde{u}\right)_{\chi=\psi(x,t,\varepsilon)} \equiv L_\varepsilon u(x,t,\varepsilon),$$

so the solution of the problem (5) is searched in the form of a series:

$$\widetilde{u}(M,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^{i/2}u_i(M). \tag{6}$$

For the coefficients of this series we get the following equations:

$$T_0u_\nu \equiv \partial_{\tau_1}u_\nu - \Delta_\eta u_\nu = 0,\ \nu = 0,1,$$
$$T_0u_2 = -T_1u_0 + f(x,t),$$
$$T_0u_3 = -T_1u_1 + L_\eta u_0,$$
$$T_0u_4 = -T_1u_2 + L_\eta u_1 - T_3u_0 - \sum_{k=1}^{2} L_{S_k}u_0 + L_\xi u_0,$$
$$T_0u_i = -T_1u_{i-2} + L_\eta u_{i-3} - T_3u_{i-4} - \sum_{k=1}^{2} L_{S_k}u_{i-4} + L_\xi u_{i-4} + L_x u_{i-6},$$
$$u_0(M)|_{t=0} = u^0(x)\,exp\left(\frac{iS^0(x)}{\varepsilon}\right),\quad u_i|_{t=0} = u_i|_{x=0} = u_i|_{x=1} = 0, \tag{7}$$

where $T_1 \equiv \sum_{k=1}^{2}\left[\partial_t S_{k+1}\partial_{\tau_{k+1}} - a(x)(\partial_x S_{k+1})^2\partial^2_{\tau_{k+1}}\right] + b(x,t),\ T_3 \equiv \partial_t - \Delta_\xi.$

3. Solution of Iteration Problems

The iterative problems are solved (5) in the class of functions $U = U_1 \otimes U_2$:

$$U_1 = \left\{u_k(N) : u_k(N) = \sum_{l=1}^{2} Y_k^l(N_l),\left|Y_k^l(N_l)\right| < c\exp\left(-\frac{\eta_l^2}{8\tau_1}\right)\right\},$$
$$U_2 = \left\{u_k(M) : u_k(M) = v_k(x,t) + c_k^1(x,t)\exp(i\tau_2) + c_k^2(x,t)\exp(\tau_3) + \right.$$
$$\left. + \sum_{l=1}^{2}\left[p^{2,l}(x,t)\,erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\exp(i\tau_2) + p^{3,l}(x,t)\,erfc\left(\frac{\xi_l}{2\sqrt{t}}\right)\exp(\tau_3)\right]\right\},$$
$$v_k(x,t),\ c^l(x,t),\ d^l(x,t),\ p^l(x,t),\ q^l(x,t) \in C^\infty\left(\overline{\Omega}\right),\ erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp^{-s^2}ds.$$

Theorem 3.1. *Let be* $\left|H^l(N)\right| < c\exp\left(-\frac{\eta_l^2}{4\tau_1}\right)$*, then the problem* $\partial_\eta Y_k^l(N) = \partial^2_{\eta_l}Y_k^l(N) + H^l(N),\ Y_k^l|_{\tau_1=0} = 0,\ Y_k^l|_{\eta_l=0} = d_k^l(x,t)$ *has a solution satisfying* $\left|Y_k^l(N)\right| < c\exp\left(-\frac{\eta_l^2}{4\tau_1}\right)$.

The proof of this theorem was carried out in [[4], Theorem 2].
Satisfying the function $u_i(M) \in U$ and considering that the $t = \tau_3 = 0$ function $erfc(\infty) = 0$ the value of the function $p_i^{k,l}(x,t)|_{t=0}$ is chosen arbitrarily and we get:

$$c_0^1(x,t)|_{t=0} = u^0(x),\ c_i^2(x,t)|_{t=0} = -v(x,0),\ p_i^{k,l}(x,t)|_{t=0} = \widetilde{p}_i^{k,l}(x),\ k = 2,3,\ l = 1,2,$$
$$Y_i^l(N_l)|_{\eta_l=0} = d_i^l(x,t),\ d_i^l(x,t)|_{x=l-1} = -v_i(l-1,t),\ p^{2,l}(x,t)|_{x=l-1} = -c^1(l-1,t),\ p^{3,l}(x,t)|_{x=l-1} = -c^2(l-1,t),$$
$$Y_i^l(N_l)|_{t=\tau_1=0} = 0. \tag{8}$$

$\widetilde{p}^{k,l}(x)$ -are arbitrary functions.

Theorem 3.2. *Suppose that conditions 1)-3) are satisfied and* $h(M) \in U_2$ *then the problem:*

$$T_0u_k = h(M),\ u_k(M)|_{t=0} = u_k^0(x)\exp\left(\frac{iS^0(x)}{\varepsilon}\right),\ u_k|_{x=l-1,\xi_l=0,\eta_l=0} = 0,\ l = 1,2, \tag{9}$$

under additional conditions:

a) $L_\xi u_k = 0,\ L_\eta u_k = 0,$
b) $T_1u_{k-1} - T_3u_{k-4} - \sum_{l=1}^{2} L_{s_l}u_{k-4} + L_x u_{k-6} \in U_1$ *has a unique solution in* U.

Proof. The free member of iterative problems has the form (7). We substitute the function $u_i(M) \in U$ in the free term:

$$h(M) = T_1 u_{k-2} + L_\eta u_{k-3} - T_3 u_{k-4} - \sum_{k=1}^{2} L_{s_k} u_{k-4} + L_\xi u_{k-5} + L_x u_{k-6} =$$
$$\partial_t v_{k-4}(x,t) + \partial_t c^1_{k-4} \exp(i\tau_2) + \partial_t c^2_{k-4} \exp(\tau_3) +$$
$$+ \sum_{l=1}^{2} \left[\partial_t Y^l_{k-4} + \partial_t p^{2,l}_{k-4}(x,t)\, erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \exp(i\tau_2) + \partial_t p^{3,l}_{k-4}(x,t)\, erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \exp(\tau_3) \right] +$$
$$+ b(x,t) \Big[v_{k-2}(x,t) + c^1_{k-2}(x,t) \exp(i\tau_2) + c^2_{k-2}(x,t) \exp(\tau_3) +$$
$$+ \sum_{l=1}^{2} \left(Y^l_{k-2} + p^{2,l}_{k-2}(x,t)\, erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \exp(i\tau_2) + p^{3,l}_{k-4}(x,t)\, erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \exp(\tau_3) \right) \Big] +$$
$$+ \sum_{l=1}^{2} \left[i\partial_t S_2(x,t) + a(x) (\partial_x S_2)^2 \right] p^{2,l}_{k-2}(x,t)\, erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \exp(i\tau_2) +$$
$$+ \left[\partial_t S_3(x,t) + a(x) (\partial_x S_3)^2 \right] \exp(\tau_3)\, p^{3,l}_{k-2}(x,t)\, erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) +$$
$$+ \left[i\partial_t S_2(x,t) + a(x) (\partial_x S_2)^2 \right] c^1_{k-2}(x,t) \exp(i\tau_2) +$$
$$+ \left[\partial_t S_3(x,t) - a(x) (\partial_x S_3)^2 \right] \exp(\tau_3)\, c^2_{k-2}(x,t) + L_\eta u_{k-3} - \sum_{k=1}^{2} L_{s_k} u_{k-4} + L_\xi u_{k-5} + a(x) L_x u_{k-6}.$$

From here ensuring the existence of a solution of equation:

$$T_0 u_k \equiv \sum_{l-1}^{2} T_0 Y^l_k(N_l) = h(M) \qquad (A).$$

We suppose that:

$$b(x,t)\, v_{k-2}(x,t) = -\partial_t v_{k-4}(x,t), \ \partial_t Y^l_{k-4} = L_x Y^l_{k-6},$$
$$\begin{cases} \partial_t c^1_{k-4}(x,t) = \left[2i\partial_x S_2(x,t)\, \partial_x c^1_{k-4} + i\partial_x^2 S_2(x,t)\, c^1_{k-4}(x,t) \right] a(x) + a(x)\, \partial_x^2 c^1_{k-6}(x,t), \\ \partial_t c^2_{k-4}(x,t) = \left[2\partial_x S_3(x,t)\, \partial_x c^2_{k-4} + \partial_x^2 S_3(x,t)\, c^2_{k-4}(x,t) \right] + a(x)\, \partial_x^2 c^1_{k-6}(x,t), \end{cases} \tag{10}$$

$$\begin{cases} \partial_t p^{2,l}_{k-4} = ia(x) \left[2\partial_x S_2(x,t)\, \partial_x p^{2,l}_{k-4} + \partial_x^2 S_2(x,t)\, p^{2,l}_{k-4}(x,t) \right] + a(x)\, \partial_x^2 p^{2,l}_{k-6}(x,t), \\ \partial_t p^{3,l}_{k-4} = a(x) \left[2\partial_x S_3(x,t)\, \partial_x p^{3,l}_{k-4} + \partial_x^2 S_3(x,t)\, p^{3,l}_{k-4}(x,t) \right] + a(x)\, \partial_x^2 p^{3,l}_{k-6}(x,t), \end{cases} \tag{11}$$

$$L_\xi u_{k-5} = a(x) \sum_{l=1}^{2} \left[2\varphi_l'(x)\, \partial_x p^{j,l}_{k-5}(x,t) + \varphi_l''(x)\, p^{j,l}_{k-5}(x,t) \right] \left(erfc\left(\frac{\xi_l}{2\sqrt{t}}\right) \right)_{\xi_l} \exp(\tau_3) = 0,$$
$$L_\eta u_{k-4} = \sum_{l=1}^{2} L_\eta Y^l_{k-3}(N_l) = 0, \tag{12}$$

then with condition a) from Theorem 3.2 we obtain the free term $h(M) = \sum_{l=1}^{2} b(x,t)\, Y^l_{k-2} \in U_1$. By Theorem 3.1 the equation (A) with a free term $h(M) \in U_1$ is solvable and its solution is representable in the form of:

$$Y^l_k(N_l) = d^l_k(x,t)\, erfc\left(\frac{\eta_l}{2\sqrt{\tau_1}}\right) + I^l_k(N_l), \tag{13}$$

where $I^l_k(N_l) = p^l_k I(\eta_l, \tau_1)$ and the estimate $\left| Y^l_k(N) \right| < c \exp\left(-\frac{\eta_l^2}{4\tau_1}\right)$ is fair. We substitute (13) into (10) and with noticing $\left| I(\eta_l,\ \tau_1) \right| \leq c \exp\left(-\frac{\eta_l^2}{8\tau_1}\right), \left| erfc\left(\frac{\xi_l}{2\sqrt{\tau_1}}\right) \right| < c \exp\left(-\frac{\eta_l^2}{8\tau_1}\right)$ with respect to $d^l_{k-4}(x,t)$ we obtain the problem $\partial_t d^l_k(x,t) = -\partial_t p^l_k(x,t) + L_x q^l_k(x,t),\ d^l_k(x,t)|_{t=0} = \widetilde{d^l_k}(x)$, where $\widetilde{d^l_k}(x)$ is arbitrary function. This choice is dictated by the fact that the function $erfc\left(\frac{\eta_l}{2\sqrt{\tau_1}}\right)$ at $t = \tau_1 = 0$ vanishes. So we choose the value of the multiplier $d^l_{k-4}(x,t)|_{t=0}$ arbitrarily. Arbitrary functions $\widetilde{d^l_k}(x)$ allow to vanish expression $L_\eta Y_{k-4}(N_l)$.

Solutions of equations with respect to $p_i^{k,l}(x,t)$ from (11) under the initial conditions from (8) will contain arbitrary functions $\widetilde{p}_i^{k,l}(x,t)$. This arbitrary functions $\widetilde{p}_i^{k,l}(x,t)$ which included in the function $p_i^{k,l}(x,t)$ allow to vanish expression $L_\xi u_i$. In this case, with respect to $\widetilde{p}_i^{k,l}$ we obtain the differential equations which are solved under the initial condition from (8). The equation (8) is solved under the initial condition from (8). In this way $u(M) \in U$ is uniquely determined. The theorem is proved. □

By using (3.1) sequentially are defined $u_i(M)$, $i = 0, 1, ..., 2n$, functions, i.e partial sum of the $u_{\varepsilon_n}(M)$ series (6).
Taking into account (7), we substitute a partial sum into problem (5) and make a narrowing by regularizing functions, then with respect to the remainder term:

$$R_{\varepsilon n}(x,t,\varepsilon) = u(x,t,\varepsilon) - u_{\varepsilon n}(x,t,\psi(x,t,\varepsilon),\varepsilon),$$

the problem is obtained:

$$L_\varepsilon R_{\varepsilon n} = \varepsilon^{n+\frac{1}{2}} g_n(x,t,\varepsilon),\ R_{\varepsilon n}|_{t=0} = R_{\varepsilon n}|_{x=0} = R_{\varepsilon n}|_{x=1} = 0.$$

Analogously to [2], it can be shown that:

$$\begin{aligned} &\|R_{\varepsilon n}(x,t,\varepsilon)\| < c\varepsilon^{n+\frac{1}{2}}, \\ &\forall x,t \in \Omega, n = 0,1,2,..., \end{aligned} \tag{14}$$

for sufficiently small $\varepsilon > 0$.

Theorem 3.3. *Suppose that the conditions 1)-3) are satisfied. Then constructed partial sum (6) is an asymptotic solution of problem (1), i.e the estimate (14) is fair .*

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