



## About Uniqueness of Solutions of Fredholm Linear Integral Equations of the First Kind in the Axis

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**Abstract.** In this work, we apply the method of integral transformation to prove uniqueness theorems for the new class of Fredholm linear integral equations of the first kind in the axis.

### 1. Introduction

Various problems of the theory of integral equations of the first kind were studied in [1-14]. But fundamental results for Fredholm integral equations of the first kind were obtained in [11, 12], where regularizing operators in the sense of M.M. Lavrent'ev were constructed. Results on non-classical Volterra integral equations of the first kind can be found in [1]. In [2, 6], problems of regularization, uniqueness and existence of solutions for Volterra integral and operator equations of the first kind are studied. In [13], for linear Volterra integral equations of the first and the third kind with smooth kernel, the existence of multiparameter family of solutions was proved. In [4, 5], on the basis of the theory of Volterra integral equations of the first kind, various inverse problems were studied. In [8], uniqueness theorems were proved and regularizing operators in the sense of Lavrent'ev were constructed for systems of linear Fredholm integral equations of the third kind. In [10], problems of uniqueness and stability of solutions for linear integral equations of the first kind with two independent variables were investigated. In [3, 9], based on a new approach, the existence and uniqueness of solutions of Fredholm integral equations and the system linear Fredholm integral equations of the third kinds were studied.

In the present paper, on the basis of the method of integral transformation, uniqueness theorems for the new class of linear Fredholm integral equations of the first kind in the axis are proved.

### 2. The Linear Fredholm Integral Equations of the First Kind

Consider the linear Fredholm integral equations of the first kind

$$\int_{-\infty}^{\infty} K(t,s)u(s)ds = f(t), t \in R = (-\infty, +\infty), \quad (1)$$

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where the  $u(t)$  is the desired function on  $R$ , the given function  $f(t)$  is the continuous on  $R$ ,

$$K(t, s) = \begin{cases} A(t, s), & -\infty < s \leq t < \infty, \\ B(t, s), & -\infty < t \leq s < \infty, \end{cases} \tag{2}$$

the given functions  $A(t, s)$  and  $B(s, t)$  are continuous on the set  $G = \{(t, s) : -\infty < s \leq t < \infty\}$ .

Let  $C(R)$  denote the space of all functions continuous on  $R$ . Here  $C(G)$  denote the space of all functions continuous on  $G$ .

We introduce the notation

$$H(t, s) = A(t, s) + B(s, t), (t, s) \in G. \tag{3}$$

Assume that the following conditions are satisfied:

(i)  $H(t, s), H'_t(t, s), H'_s(t, s), H''_{ts}(t, s) \in C(G), \alpha(t) = \lim_{s \rightarrow -\infty} H(t, s), t \in R,$   
 $\alpha(t) \in C(R), \lim_{t \rightarrow \infty} \alpha(t) = a_0 \in R, a_0 \geq 0, \alpha'(t) = \lim_{s \rightarrow -\infty} H'_t(t, s) \leq 0$   
 for all  $t \in R, H''_{ts}(t, s) \leq 0$  for all  $(t, s) \in G, \alpha'(t) \in L_1(R),$   
 $\beta(s) = \lim_{t \rightarrow \infty} H'_s(t, s) \geq 0$  for all  $s \in R, \beta(s) \in C(R) \cap L_1(R);$

(ii)  $\sup_{(t,s) \in G} |H(t, s)| \leq L < \infty, \sup_{t \in R} \int_{-\infty}^t |H'_s(t, s)| ds \leq M < \infty, H''_{ts}(t, s) \in L_1(G),$   
 $\sup_{t \in [s, \infty)} |H'_s(t, s)| \leq \gamma(s) \in L_1(R);$

(iii) At least one of the following three conditions holds:

- 1)  $\alpha'(t) < 0$  for almost all  $t \in R;$
- 2)  $\beta(s) > 0$  for almost all  $s \in R;$
- 3)  $H''_{ts}(t, s) < 0$  for almost all  $(t, s) \in G.$

**Theorem 2.1.** *Let conditions (i), (ii) and (iii) be satisfied. Then the solution of the integral equation (1) is unique in the space  $L_1(R)$ .*

*Proof.* Let  $u(t) \in L_1(R)$  be a solution of the integral equation (1). By virtue of (2), we can write the integral equation (1) in the form

$$\int_{-\infty}^t A(t, s)u(s)ds + \int_t^{\infty} B(t, s)u(s)ds = f(t), t \in R. \tag{4}$$

Multiplying both sides of the equation (4) by  $u(t)$  and integrating over the domain  $R$ , we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^t A(t, s)u(s)u(t)dsdt + \int_{-\infty}^{\infty} \int_s^{\infty} B(s, t)u(t)u(s)dt ds = \int_{-\infty}^{\infty} f(t)u(t)dt. \tag{5}$$

Applying Dirichlet's formulas to (5) and taking into account notation (3), we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^t H(t, s)u(s)u(t)dsdt = \int_{-\infty}^{\infty} f(t)u(t)dt. \tag{6}$$

We shall introduce the notation

$$z(t, s) = \int_s^t u(\tau)d\tau, (t, s) \in G. \tag{7}$$

Then from (7), we have

$$d_s z(t, s) = -u(s)ds, z(t, s)u(t)dt = \frac{1}{2}d_t(z^2(t, s)). \tag{8}$$

Let us transform the integral on the left hand of the identity (6). Taking into account (7), (8) and integrating by parts, we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^t H(t, s)u(s)u(t)dsdt = \int_{-\infty}^{\infty} \alpha(t)z(t, -\infty)u(t)dt + \int_{-\infty}^{\infty} \int_{-\infty}^t H'_s(t, s)z(t, s)u(t)dsdt.$$

Hence, applying Dirichlet’s formula, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^t H(t, s)u(s)u(t)dsdt = \frac{1}{2} \int_{-\infty}^{\infty} \alpha(t)d_t(z^2(t, -\infty)) + \frac{1}{2} \int_{-\infty}^{\infty} \left[ \int_s^{\infty} H'_s(t, s)d_t(z^2(t, s)) \right] ds = \frac{1}{2}a_0z^2(+\infty, -\infty) - \frac{1}{2} \int_{-\infty}^{\infty} \alpha'(t)z^2(t, -\infty)dt + \frac{1}{2} \int_{-\infty}^{\infty} \beta(s)z^2(\infty, s)ds - \frac{1}{2} \int_{-\infty}^{\infty} \int_s^{\infty} H''_{ts}(t, s)z^2(t, s)dt ds. \tag{9}$$

Taking into account (6) and applying Dirichlet’s formula from (9) we obtain

$$\frac{1}{2}a_0z^2(\infty, -\infty) - \frac{1}{2} \int_{-\infty}^{\infty} \alpha'(t)z^2(t, -\infty)dt + \frac{1}{2} \int_{-\infty}^{\infty} \beta(s)z^2(\infty, s)ds - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^t H''_{ts}(t, s)z^2(t, s)dsdt = \int_{-\infty}^{\infty} f(t)u(t)dt. \tag{10}$$

Suppose that  $f(t) = 0$  for  $t \in R$ . Then, taking into account conditions (i), (ii) and (iii), we see that (10) implies  $\int_{-\infty}^t u(\tau)d\tau = 0, t \in R$  or  $\int_s^{\infty} u(\tau)d\tau = 0, s \in R$  or

$$\int_s^t u(\tau)d\tau = 0, (t, s) \in G.$$

Therefore,  $u(t) = 0$  for all  $t \in R$ . □

**Remark 2.2.** If  $B(s, t) = 0$  for all  $(t, s) \in G$ , then the integral equation (1) is a Volterra linear integral equation of the first kind. In this case the assertion of the theorem is true for  $H(t, s) = A(t, s), \forall (t, s) \in G$ .

**Remark 2.3.** If  $A(t, s) = 0$  for all  $(t, s) \in G$ , then the integral equation (1) is a Volterra linear integral equation of the first kind. In this case the assertion of the theorem is true for  $H(t, s) = B(s, t), \forall (t, s) \in G$ .

### 3. Examples

**Example 3.1.** Consider the integral equation

$$\int_{-\infty}^t A(t, s)u(s)ds + \int_t^{\infty} B(t, s)u(s)ds = f(t), t \in R, \tag{11}$$

where

$$A(t, s) = \begin{cases} -\frac{c}{a(b-a)} \left[ e^{bt}e^{-a(t-s)} - \frac{2b}{a+b}e^{as} \right], -\infty < s \leq t \leq 0, \\ \frac{c}{a(a+b)} e^{-a(t-s)}e^{-bt}, -\infty < s \leq t < \infty, t \geq 0, \end{cases} \tag{12}$$

$$B(t, s) = \begin{cases} \frac{cd}{ab}(e^{bs} - 2), -\infty < t \leq s \leq 0, \\ -\frac{cd}{ab}e^{-bs}, -\infty < t \leq s < \infty, s \geq 0, \end{cases} \tag{13}$$

$a, b, c$  and  $d$  are real parameters,  $a > 0, b > 0, c > 0, d < 0, a \neq b$ . Then taking into account (12) and (13) from (3) we have

$$H(t, s) = \begin{cases} -\frac{c}{a(b-a)} \left[ e^{bt} e^{-a(t-s)} - \frac{2b}{a+b} e^{as} \right] + \frac{cd}{ab} (e^{bt} - 2), & -\infty < s \leq t \leq 0, \\ \frac{c}{a(a+b)} e^{-a(t-s)} e^{-bt} - \frac{cd}{ab} e^{-bt}, & -\infty < s \leq t < \infty, t \geq 0, \end{cases} \tag{14}$$

$$H'_t(t, s) = \begin{cases} -\frac{c}{a} e^{bt} e^{-a(t-s)} + \frac{cd}{a} e^{bt}, & -\infty < s \leq t \leq 0, \\ -\frac{c}{a} e^{-a(t-s)} e^{-bt} + \frac{cd}{a} e^{-bt}, & -\infty < s \leq t < \infty, t \geq 0, \end{cases} \tag{15}$$

$$H'_s(t, s) = \begin{cases} -\frac{c}{b-a} \left[ e^{bt} e^{-a(t-s)} - \frac{2b}{a+b} e^{as} \right], & -\infty < s \leq t \leq 0, \\ \frac{c}{a+b} e^{-a(t-s)} e^{-bt}, & -\infty < s \leq t < \infty, t \geq 0, \end{cases} \tag{16}$$

$$H''_{ts}(t, s) = -c e^{-b|t|} e^{-a(t-s)}, (t, s) \in G \tag{17}$$

$$\alpha(t) = \lim_{s \rightarrow -\infty} H(t, s) = \begin{cases} \frac{cd}{ab} (e^{bt} - 2), & t \leq 0, \\ -\frac{cd}{ab} e^{-bt}, & t \geq 0, \end{cases} \tag{18}$$

$$\alpha'(t) = \lim_{s \rightarrow -\infty} H'_t(t, s) = \begin{cases} \frac{cd}{a} e^{bt}, & t \leq 0, \\ \frac{cd}{a} e^{-bt}, & t \geq 0, \end{cases} \tag{19}$$

$$\beta(s) = \lim_{t \rightarrow \infty} H'_s(t, s) = 0, s \in R, \tag{20}$$

$$a_0 = \lim_{t \rightarrow \infty} \alpha(t) = 0. \tag{21}$$

From (14) and (16), we have

$$L = \sup \left\{ \frac{c}{a|b-a|} \left( 1 + \frac{2b}{a+b} \right) + \frac{2c|d|}{ab}, \frac{c}{a(a+b)} + \frac{c|d|}{ab} \right\}, \tag{22}$$

$$M = \sup \left\{ \frac{c}{a|b-a|} \left( 1 + \frac{2b}{a+b} \right), \frac{c}{a(a+b)} \right\}, \tag{23}$$

$$\gamma(s) = \begin{cases} \frac{c}{|b-a|} (e^{bs} + \frac{2b}{a+b} e^{as}), & s \leq 0, \\ \frac{c}{a+b} e^{-bs}, & s \geq 0, \end{cases} \tag{24}$$

Then taking into account (14)-(24), we can verify that conditions (i), (ii) and (iii) are satisfied for system (11). Therefore the solution of the integral equation (11) is unique in the space  $L_1(R)$ .

**Example 3.2.** Consider the integral equation

$$\int_{-\infty}^t A(t, s)u(s)ds = f(t), t \in R, \tag{25}$$

where

$$A(t, s) = \begin{cases} -\frac{c}{a(b-a)} \left[ e^{bt} e^{-a(t-s)} - \frac{2b}{a+b} e^{as} \right] + \frac{cd}{ab} (e^{bt} - 2), & -\infty < s \leq t \leq 0, \\ \frac{c}{a(a+b)} e^{-a(t-s)} e^{-bt} - \frac{cd}{ab} e^{-bt}, & -\infty < s \leq t < \infty, t \geq 0, \end{cases} \tag{26}$$

$a, b, c$  and  $d$  are real parameters,  $a > 0$ ,  $b > 0$ ,  $c > 0$ ,  $d < 0$ ,  $a \neq b$ . Then taking into account (25), from (3) we have

$$H(t, s) = A(t, s), (t, s) \in G. \quad (27)$$

Then taking into account (26), (27) and (14)-(24), we can verify that conditions (i), (ii) and (iii) are satisfied for the integral equation (25). Therefore the solution of the integral equation (25) is unique in the space  $L_1(R)$ .

**Example 3.3.** Consider the integral equation

$$\int_t^\infty B(t, s)u(s)ds = f(t), t \in R, \quad (28)$$

where

$$B(s, t) = \begin{cases} -\frac{c}{a(b-a)} \left[ e^{bt} e^{-a(t-s)} - \frac{2b}{a+b} e^{as} \right] + \frac{cd}{ab} (e^{bt} - 2), & -\infty < s \leq t \leq 0, \\ \frac{c}{a(a+b)} e^{-a(t-s)} e^{-bt} - \frac{cd}{ab} e^{-bt}, & -\infty < s \leq t < \infty, t \geq 0, \end{cases} \quad (29)$$

$a, b, c$  and  $d$  are real parameters,  $a > 0$ ,  $b > 0$ ,  $c > 0$ ,  $d < 0$ ,  $a \neq b$ . Then taking into account (28), from (3) we have

$$H(t, s) = B(s, t), (t, s) \in G. \quad (30)$$

Then taking into account (29), (30) and (14)-(24), we can verify that conditions (i), (ii) and (iii) are satisfied for the integral equation (28). Therefore the solution of the integral equation (28) is unique in the space  $L_1(R)$ .

## References

- [1] A.S. Apartsyn, Nonclassical Linear Volterra Equations of the First Kind, VSP, Utrecht, The Netherlands, 2003.
- [2] A. Asanov, Regularization, Uniqueness and Existence of Solutions of Volterra Equations of the First Kind, VSP, Utrecht, The Netherlands, 1998.
- [3] A. Asanov, K.B. Matanova, R.A. Asanov, A class of linear and nonlinear Fredholm integral equations of the third kind, Kuwait J. Sci. 44 (2017) 17–28.
- [4] A.L. Bukhgeim, Volterra Equations and Inverse Problems, VSP, Utrecht, The Netherlands, 1999.
- [5] A.M. Denisov, Elements of the Theory of Inverse Problems, VSP, Utrecht, The Netherlands, 1999.
- [6] M.I. Imanaliev, A. Asanov, Solutions of system of nonlinear Volterra integral equations of the first kind, Soviet Math. Dokl. 309 (1989) 1052–1055.
- [7] M.I. Imanaliev, A. Asanov, Solutions of systems of Volterra nonlinear integral equations of the third kind, Doklady Math. 76 (2007) 490–493.
- [8] M.I. Imanaliev, A. Asanov, Solutions of system of Fredholm linear integral equations of the third kind, Doklady Math. 81 (2010) 115–118.
- [9] M.I. Imanaliev, A. Asanov, R.A. Asanov, Solutions to systems of linear Fredholm integral equations of the third kind with multipoint singularities, Doklady Math. 95 (2017) 235–239.
- [10] M.I. Imanaliev, A. Asanov, Z.A. Kadenova, A class of linear integral equations of the first kind with two independent variables, Doklady Math. 89 (2014) 98–102.
- [11] M.M. Lavrent'ev, The integral equations of the first kind, Doklady Akad. Nauk SSSR 127 (1959) 31–33.
- [12] M.M. Lavrent'ev, V.G. Romanov, S.P. Shishatskii, Ill-Posed Problems of Mathematical Physics and Analysis, Providence R.I., American Mathematical Society, 1986.
- [13] N.A. Magnitskii, Volterra linear integral equations of the first and third kinds, Zh. Vychisl. Mat. Matem. Fiziki 19 (1979) 970–989.
- [14] S.P. Shishatskii, A. Asanov, E.R. Atamanov, Uniqueness Problems for Degenerating Equations and Nonclassical Problems, VSP, Utrecht, The Netherlands, 2001.