



Matrices Whose Inverses are Tridiagonal, Band or Block-Tridiagonal and Their Relationship with the Covariance Matrices of a Random Markov Process

Ulan Brimkulov^a

^aKyrgyz Turkish Manas University

Abstract. The article discusses the matrices of the form A_n^1, A_n^m, A_N^m , whose inverses are: tridiagonal matrix A_n^{-1} (n - dimension of the A_n^{-m} matrix), banded matrix A_n^{-m} (m is the half-width band of the matrix) or block-tridiagonal matrix A_N^{-m} ($N = n \times m$ - full dimension of the block matrix; m - the dimension of the blocks) and their relationships with the covariance matrices of measurements with ordinary (simple) Markov Random Processes (MRP), multiconnected MRP and vector MRP, respectively. Such covariance matrices frequently occur in the problems of optimal filtering, extrapolation and interpolation of MRP and Markov Random Fields (MRF). It is shown, that the structures of the matrices A_n^1, A_n^m, A_N^m have the same form, but the matrix elements in the first case are scalar quantities; in the second case matrix elements represent a product of vectors of dimension m ; and in the third case, the off-diagonal elements are the product of matrices and vectors of dimension m . The properties of such matrices were investigated and a simple formulas of their inverses were found. Also computational efficiency in the storage and the inverse of such matrices have been considered. To illustrate the acquired results, an example on the covariance matrix inversions of two-dimensional MRP is given.

1. Introduction

Research problems of random fields and processes are faced in many applications, such as the study of spatial and temporal variability of oceanographic fields (flow velocity fields, temperature fields and sea surface height, density and salinity fields, etc.), the problems of statistical radio engineering and image fields reconstruction, and many other engineering tasks. The computing algorithms based on the least-squares method (LSM), weighted and generalized LSM (WLSM, GLSM) or Kalman-Bucy filter are usually used in estimation, filtering and interpolation of random fields which are based on the field realization measurements results. If the investigated physical fields can be approximated by simple (ordinary connected), m -connected processes, or vector Markov processes, the computing schemes utilizing tridiagonal, band and block-banded matrices are used. Therefore, in recent years, much attention is paid to the study of computational efficiency of such algorithms and the structure of the matrices included in the estimation algorithms (see e.g., [2–5, 16, 17, 24]).

2010 *Mathematics Subject Classification.* Primary 15B99; Secondary 60J28

Keywords. Best linear unbiased estimates (BLUE), simple (ordinary connected) Markov process, multiply connected (m -connected) Markov process, vector (m -dimensional) Markov process, random field filtering and parametric identification, tridiagonal matrices, banded matrices, block-tridiagonal matrices

Received: 16 July 2018; Revised: 24 April 2019; Accepted: 06 May 2019

Communicated by Fahreddin Abdullayev

Email address: ulan.brimkulov@manas.edu.kg (Ulan Brimkulov)

The computational schemes where with covariance matrices of measurements, whose inverses have diagonal structure can arise in physical random processes if they are either Markov random processes in a wide sense (that is, Gauss-Markov random processes), or when they can be approximated by Markov processes. Diagonal structures can be one of the following forms:

- (i) tridiagonal structure, in case of a simple (one-dimensional, ordinary connected) Markov process;
- (ii) band-diagonal structure for m -connected Markov process (in literature they are also referred as the Gauss-Markov processes of m -th order or m -th order Gauss-Markov process);
- (iii) block-tridiagonal structure, in the case of vector ordinary connected Markov process;
- (iv) block-band-diagonal structure, in the case of m -connected vector Markov process.

In [5], a data assimilation in large, multi-dimensional, time-dependent fields was considered. Assimilation was done by accounting the structure of measurements matrix. As a result, four efficient Kalman-Bucy filter's algorithm were built, which reduced computing costs up to 2 orders compared to known algorithms. This improvement became possible in case when the measurement errors are approximated by Markov random field (MRF). In this case, the inverse covariance matrix of measurement errors field had a band structure that allows constructing efficient algorithms. The sparse measurements are typical for tasks considered in the article (e.g., results of satellite scan). Accounting these measurements allows constructing algorithms that are more efficient compared to existing ones.

In [17] the matrices, whose inverses are banded, were considered. In this case, tridiagonal matrix represented as a Hadamard product of three matrices. This leads to very interesting result when Gauss-Markov's random process represented as the product of three independent processes: forward and backward processes with independent increments and a variance-stationary process. Here we can see the connection between matrices, entering the decomposition of the three-diagonal matrix and the processes involved in the factorization of Gauss-Markov's random process. In this sense, the positive defined symmetric matrices with banded inverses can be viewed as a representation of Gauss-Markov's random processes. The paper also considered the problem of the approximation of the covariance matrix of non-stationary general form Gaussian process. Approximation is done by covariance matrix whose inverse is a band matrix. The information loss of such approximation was estimated. This work also shows that for such matrix inversion it is necessary to know only the direct matrix elements, lying inside the band with the width L .

In [2] inversion algorithms of L-block banded matrices were obtained. Authors showed that their inverses are also L-block banded matrices. Received algorithms were applied to signal processing problems in the case of Kalman-Bucy Filtering (KBF) usage. These covariance matrices were approximated by block-band matrices. The computational complexity of the algorithm is 2 times lower compared to existing algorithms and makes the KFB algorithm feasible to solve problems of large dimension. There is a large number of papers devoted to the problems of tridiagonal, block and block-tridiagonal matrices and their inverses. As the most common and close to the subject of this article, we should note the article by G. Meurant, 1992 [19], where a detailed review and analysis of studies on the properties of inversions to symmetric tridiagonal, block-tridiagonal and banded matrices was given. In the article, author summarized many of the results in this area and showed almost complete (34 titles) bibliography of publications in this field in the period of time from 1944 to 1992. The review begins from the first publication of D. Moskovitz, 1944 [20], where analytical expressions for the inverse of tridiagonal matrices are given for 1D and 2D Poisson model problems.

In [19] there is a reference to the work of Barret, [6], which introduced the concept of "triangle property" (A matrix R has "triangle property" if $R_{ij} = \frac{(R_{ik}R_{kj})}{R_{kk}}$; a matrix with this property whose diagonal elements are nonzero elements has a tridiagonal inverse and vice versa). It should be noted that "triangle property", introduced in [6], coincides with a discrete form of the condition on the form of the covariance function $k(s, t)$ of a Markov process in the wide sense ($k(s, t) = \frac{k(s, \tau)k(\tau, t)}{k(\tau, \tau)}$, $s < \tau < t$), given in the Doob, 1953 ([12], Theorem 8.1). We focus on this work, since the results of Doob's Theorem 8.1 are supported by the results presented in the second part of this work related to the study of the relationship of the matrices whose

inverses are tridiagonal, banded or block-tridiagonal matrices with the covariance matrices of ordinary (simple) Markov processes, multiple connected Markov processes and vector Markov random processes.

Summarizing the content of these papers we can note that in all these works, matrices, the inverse of which leads to tridiagonal, banded or block-tridiagonal matrices were studied. It is shown that if the matrices are symmetric and positive definite, then they are covariance matrices of measurements of Gauss-Markov's random processes. The application of obtained results to signal processing tasks in the analysis of space-time random oceanographic fields were considered. The structure of these matrices allows obtaining efficient computational algorithms applicable to large-scale tasks. The results are coupling to specific algorithms and programs and their effectiveness tested on examples of real experiments results processing.

The author obtained results that partially go inline with the results obtained by [2–5, 16, 17, 24] in 1988-1992 independently of mentioned works. Unfortunately, these results are presented only in the form of manuscripts [8, 9] or published in the form of short abstracts of conferences in Russian (see., e.g., [10, 11]). There is only one article that was translated into English [7].

It can be noted, that the results related to the tridiagonal matrices inversion and the study of Markov processes were also considered in Russian mathematical literature, namely, in literature on statistical radio engineering dedicated to various aspects of the random processes study [1, 7–12, 14, 15, 18, 21–23, 25, 26]. But there are few works in Russian that consider the use of banded and block-banded matrices in data processing algorithms. Also, the connection between Markov random processes (fields) class and covariance matrices of measurement of such fields were poorly investigated. There is a few works on the evaluation of the effectiveness of computational processing circuits of Markov fields' research results. In [22] the relationship of Markov processes and stochastic differential equations in partial derivatives were shown and numerous examples of Markov processes were given.

2. The Task of Optimum Estimating of Random Field's Mathematical Expectation

In many problems that deal with the analysis of random processes and fields based on experimental data (measurements), we are dealing with matrices, whose dimensions increase with when the number of measurements increases. That is, with increasing of the number of measurements the dimension of the matrix grows rapidly (in many cases proportional to the square of the measurements). Consider one of the most common and well-known problems of the analysis of random processes (fields) - the problem of finding the best linear unbiased estimates (BLUE) of unknown parameters of for the random field mathematical expectation model of the random field. Let a random field $Z(t)$ to be described by the model:

$$Z(t) = \eta(t) + \xi(t), t \in T, \quad (1)$$

where $\eta(t) = \eta(t, \mathbf{B}) = \mathbf{f}^T(t)\mathbf{B}$ is mathematical expectation (deterministic component of the field), described by a linear-parameterized model with the vector of known linearly-independent functions $\mathbf{f}(t) = (f_1(t), \dots, f_p(t))^T$ and the vector of unknown parameters $\mathbf{B} = (B_1, \dots, B_p)^T$; $\xi(t)$ - noise field (interference, measurement noise) with the known covariance function $k(s, t)$; T - the interval, in which the model (1) is true.

Let us assume that there was set a problem defined as follows: find the Best Linear Unbiased Estimates (BLUE) $\hat{\mathbf{B}}_n$ parameters \mathbf{B} based on discrete measurement $Z(t)$ in the points $T_n = \{t_1 < t_n < \dots < t_n | t_i \in T\}$ to find the Best Linear Unbiased Estimates (BLUE) $\hat{\mathbf{B}}_n$ parameters \mathbf{B} . The solution is well known and defined by the formula (see. e.g., [1, 23]):

$$\hat{\mathbf{B}}_n = D_n F_n K_n^{-1} Z_n \quad (2)$$

where

$$D_n = [F_n K_n^{-1} F_n^T]^{-1} \quad (3)$$

- covariance matrix of BLUE $\hat{\mathbf{B}}_n$; $F_n = [f(t_1), \dots, f(t_n)]$ - matrix of vector $\mathbf{f}(t)$ values at the measurement points T_n ; $Z_n = Z(t_i) | t_i \in T_n$ - measurement vector; K_n^{-1} - is inverse to the covariance matrix K_n for Z_n : $K_n = k(t_i, t_j) = k_{ij} (i, j = \overline{1, n}) (t_i \in T_n)$.

In spite of optimality of estimations (2), the use under large number of measurements becomes difficult or infeasible. This is due to the fact that in the expressions (2) and (3), there is the matrix K_n^{-1} , and the number of elements of matrix K_n^{-1} is increasing pro rata to the square of measurements. Thus, the process of computing and storing the matrix K_n^{-1} requires large computational costs (required that is, memory requirement is in proportional to n^2 , and the number of operations for the inversion of matrix K_n matrix inversion in is proportional to n^3 [21]).

The same matrix K_n^{-1} is used to calculate the optimal estimates for more general tasks for handling random processes based on the measurement results. For example, matrix K_n^{-1} is also a part of the formulas of filtering, interpolation and extrapolation of a random field based on the measurement results. The weighting matrix of size $(n \times n)$ is also included in the evaluation formula, when generalized least-squares method (GLSM) is used. We do not consider here the well-known results, which can be found in numerous literature, see e.g. [23], [1]. Thus, the problem of finding a class of random processes (fields), for which the covariance matrix of the measurement K_n is such, that its inverse matrix K_n^{-1} is sparse and in particular has the tridiagonal, band or block-tridiagonal structure, is an important problem. In this case, the calculation of optimal estimates, including BLUE for many problems of random processes and fields analysis is greatly simplified from a computational point of view by taking into account the structure of the covariance matrix measurement.

3. Matrices Whose Inverses are Tridiagonal, Band and Block-Tridiagonal

Below we shall consider three classes of square matrices, whose inverses are the tridiagonal, band and block-tridiagonal matrices. Note that all of these three classes of matrices have the similar structures save for the following differences

- in tridiagonal matrices, the matrix elements are formed from a scalar quantity;
- in band matrices, the elements of the matrix are the result of multiplication of matrix and vector quantities;
- in block-diagonal matrices, the elements of the matrix formed from the blocks which are the product of square matrices of smaller dimension.

3.1. A Class of Matrices Whose inverse Leads to Tridiagonal Matrices

Let matrix $A = A_n^1$ (size $n \times n$) be of the following form:

$$A_n^1 = \begin{bmatrix} a_{11} & \Lambda_{11}a_{11} & \Lambda_{12}a_{11} & \cdot & \cdot & \cdot & \Lambda_{1,n-1}a_{11} \\ \Gamma_{11}a_{11} & a_{22} & \Lambda_{22}a_{22} & \cdot & \cdot & \cdot & \Lambda_{21}a_{11} \\ \Gamma_{21}a_{11} & \Gamma_{22}a_{22} & a_{33} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1,n-1} & \Lambda_{n-1,n-1}a_{n-1,n-1} \\ \Gamma_{n-1,1}a_{11} & \Gamma_{n-1,2}a_{22} & \cdot & \cdot & \cdot & \Gamma_{n-1,n-1}a_{n-1,n-1} & a_{n,n} \end{bmatrix} \quad (4)$$

where $\Gamma_{ij} = \prod_{l=j}^i \gamma_l$ ($i \geq j$) and $\Lambda_{ij} = \prod_{l=i}^j \lambda_l$ ($j \geq i$) ($i, j = 1, 2, \dots, n - 1$), a_{ii} ($i = 1, 2, \dots, n$), γ_i, λ_i ($i = 1, 2, \dots, n - 1$) – arbitrary real numbers. Thus, the off-diagonal elements A_n^1 are determined by the expression

$$a_{ij} = \begin{cases} \Gamma_{j,i-1}a_{jj} = a_{jj} \prod_{l=j}^{i-1} \gamma_l & \text{if } i > j \\ \Lambda_{i,j-1}a_{ii} = a_{ii} \prod_{l=i}^{j-1} \lambda_l & \text{if } j > i \end{cases} \quad (i, j = \overline{1, n-1}) \quad (5)$$

Here we can formulate the following theorem.

Theorem 1. Let the matrix A_n to be of the form (4) and let $\det A_n \neq 0$, then

1. the inverse of matrix (4), A_n^{-1} , will have tridiagonal form

$$A_n^{-1} = \begin{bmatrix} \frac{\mu_1}{\alpha_1\alpha_2} & -\frac{\lambda_1}{\alpha_2} & 0 & \cdot & \cdot & \cdot & \cdot \\ -\frac{\gamma_1}{\alpha_2} & \frac{\mu_2}{\alpha_2\alpha_3} & -\frac{\lambda_2}{\alpha_3} & 0 & \cdot & \cdot & \cdot \\ 0 & -\frac{\gamma_2}{\alpha_3} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\mu_{n-1}}{\alpha_{n-1}\alpha_n} & -\frac{\lambda_{n-1}}{\alpha_n} \\ 0 & \cdot & \cdot & \cdot & 0 & -\frac{\gamma_{n-1}}{\alpha_n} & \frac{1}{\alpha_n} \end{bmatrix} \quad (6)$$

where $\alpha_i = a_{ii} - \gamma_{i-1}\lambda_{i-1}a_{i-1,i-1}$ ($i = \overline{(2, n)}$), $\alpha_1 = a_{11}$, $\mu_i = a_{i+1,i+1} - \gamma_i\lambda_{i-1}\lambda_i a_{i-1,i-1}$ ($i = \overline{(2, n-1)}$), $\mu_1 = a_{22}$.

2. The determinant of any corner of the sub-matrix A_i^1 , including the determinant of the complete matrix A_n^1 , can be calculated by the expression

$$\det A_i^1 = \prod_{l=1}^i a_l, (i = \overline{(1, n)}).$$

The proof of the theorem uses the method of induction and recursive procedure of matrix inversion by method of step-by-step bordering (the proof if proof is provided in Appendix).

Note 1. The matrix (4) can also be written as

$$A_n^1 = \begin{bmatrix} \alpha_{11} & \lambda_1\alpha_{11} & \lambda_2\alpha_{12} & \cdot & \cdot & \cdot & \lambda_{n-1}a_{1,n-1} \\ \gamma_1a_{11} & a_{22} & \lambda_2a_{22} & \cdot & \cdot & \cdot & \lambda_{n-1}a_{2,n-1} \\ \gamma_2\alpha_{21} & \gamma_2\alpha_{22} & \alpha_{33} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \alpha_{n-1,n-1} & \lambda_{n-1}\alpha_{n-1,n-1} & \cdot \\ \gamma_{n-1}a_{n-1,1} & \gamma_{n-1,2}a_{n-1,2} & \cdot & \cdot & \gamma_{n-1}a_{n-1,n-1} & \alpha_{nn} & \cdot \end{bmatrix} \quad (7)$$

where $a_{ii}(i = \overline{(1, n)})$, $\gamma_i, \lambda_i(i = 1, 2, \dots, n-1)$ - arbitrary real numbers, $a_{ij}(i = \overline{(1, n-1)}|i \neq j)$ are as determined in (5). Expressing $a_{31}(a_{13})$ through $a_{21}(a_{12})$; $a_{41}(a_{14})$ through $a_{31}(a_{13})$, etc., the matrix (7) can be written in the form (4).

Note 2. The Matrix (4) is completely determined by $3n - 2$ elements, which are included into the three central diagonals A_n^1 . Other elements of A_n^1 cancel each other while inversion.

Note 3. The Matrix (4) (or (7)) is just one form of the square matrices representation, whose inverse are tridiagonal matrices. Many papers devoted to matrices research questions, whose inverses are tridiagonal matrices (see, for instance, [15] and other). Representation of matrix A_n^1 in the form (4) (or (7)) is convenient because the results are easily generalized to the matrices whose inverses are banded or block-tridiagonal matrices.

3.2. The Class of Matrices Whose inverse Leads to Banded Matrices with the Half-Band's Width m

The results of Theorem 1 can be generalized to the band matrices with $1 \leq m \leq n-1$, where n - dimension of the reversible matrix; m - the half-width band of band inverse matrix.

To formulate the main theorem of this section, a number of new notations is required. For ease of perception, we divide the square matrix A_i (i - order of matrix) into the sub-matrix as follows:

Note 6. The matrix A_n^m can be written in a form similar to (7) with the replacement of the off-diagonal scalar elements a_{ij} ($i, j = \overline{(1, n-1)} | i \neq j$) and γ_i, λ_i ($i = \overline{(1, n-1)}$) by vectors $\mathbf{a}_{[i],j}^T$ ($i = \overline{(1, n-1)}$; $j = \overline{(i-1, n-1)}$), $\mathbf{a}_{i,[j]}^T$ ($i = \overline{(j-1, n-1)}$; $j = \overline{(n-1)}$) and $\boldsymbol{\gamma}_i^T, \boldsymbol{\lambda}_i$ ($i = \overline{(1, n-1)}$), respectively. These vectors have a length i for $i \leq m$ and length m for $m < i \leq n-1$.

Taking into account the Note 6, the proof of Theorem 2 fully repeats the proof of Theorem 1 with the replacement of scalar values γ_i, λ_i ($i = \overline{(1, n-1)}$) and a_{ij}, a_{ji} ($i, j = \overline{(1, n-1)} | i \neq j$) with vectors $\boldsymbol{\gamma}_i^T, \boldsymbol{\lambda}_i$ ($i = \overline{(1, n-1)}$) and $\mathbf{a}_{[i],j}, \mathbf{a}_{i,[j]}^T$ ($i, j = \overline{(1, n-1)} | i \neq j$).

Note 7. From formulas (12) - (15) it is clear that for the inversion of matrix of the form A_n^m it is enough to know its elements, lying inside the band with a width of $2m + 1$. Other elements of A_n^m cancel each other out during inversion. In other words, the matrix A_n^m is completely determined by its elements lying inside the band width $2m + 1$.

Note 8. The matrix A_n^m depends on $w^* = (2m + 1)n - m(m + 1)$ of arbitrary selected values $\{a_{ii}, \overline{(i = 1, n)}\}, \{\boldsymbol{\gamma}_i^*, \boldsymbol{\lambda}_i^*, i = \overline{(1, w)}\}$, or, in other words, has w^* independent elements. Other elements of A_n^m are directly connected with them.

If $m = n - 1$, the number of independent values which depend on the elements of the matrix A_n^m becomes equal to n^2 , we come to a matrix of general form $A_n^m = A_n^{n-1} = A_n$. In this case, the inverse matrix is completely filled, i.e. it has n^2 nonzero elements.

If $m = 1$, then the value $w^* = 3n - 2$. For $A_n^m = A_n^1$ its inverse matrix $A_n^{-m} = A_n^{-1}$ will be tridiagonal. For $m = 0$, matrix $A_n^m = A_n^0$ and inverse of this matrix, $(A_n^0)^{-1}$, will be of a diagonal form.

Note 9. Matrices of the form A_n^m can be stored in memory in a compact form. It is sufficient to introduce the vectors $\mathbf{a}_{ii} = \{a_{ii}, (i = \overline{1, n})\}, \boldsymbol{\gamma}_w^*$ and $\boldsymbol{\lambda}_w^*$ in memory, i.e. matrices $\mathfrak{J}_{n-1,m}, \mathfrak{K}_{m,n-1}$, which require $w^* \leq n^2$ memory cells. With the help of vectors $\mathbf{a}_{ii}, \boldsymbol{\gamma}_w^*$ and $\boldsymbol{\lambda}_w^*$, every element of the matrix A_n^m can easily be calculated, if necessary. Thus, if $m \ll n$, the gain in the amount of required memory can be reduced considerably.

3.3. The Class of Matrices Whose inverse Leads to a Block Tridiagonal Matrix

Let the matrix A_N of size $(N \times N)$ have a form: $A_N = A_N^m = [A_{ij}]_{i,j=1}^n$, where A_{ij} - square sub-matrices (blocks) of size $(m \times m)$; ($N = n \times m$).

Let non-diagonal sub-matrices A_{ij}^m to be determined by following expressions (compare with (5)):

$$A_{ij} = \begin{cases} \Gamma_{j,i-1} A_{jj} = [\prod_{l=j}^{i-1} \Gamma_l] A_{jj}, & \text{if } i \geq j, \\ A_{jj} \Lambda_{i,j-1} = A_{ii} \prod_{l=i}^{j-1} \Lambda_l, & \text{if } j \geq i, \end{cases} \quad (i, j = \overline{1, n-1}) \tag{16}$$

where Λ_i ($i = \overline{1, n-1}$) and Γ_i ($i = \overline{1, n-1}$) - matrices of real elements of size $(m \times m)$; square (sub)matrices Λ_{ij} and Γ_{ij} ($i, j = \overline{1, n-1}$) of order n can also be written as follows:

$$\begin{cases} \Gamma_{i,j} = \Gamma_j \Gamma_{j+1} \dots \Gamma_{i-1} \Gamma_i, & \text{if } i \geq j, \\ \Lambda_{i,j} = \Lambda_i \Lambda_{i+1} \dots \Lambda_{j-1} \Lambda_j, & \text{if } j \geq i, \end{cases} \quad (i, j = \overline{1, n-1}) \tag{17}$$

Let the matrix $A = A_N^m$ has a form:

$$A_N^m = \begin{bmatrix} A_{11} & A_{11}\boldsymbol{\Lambda}_{11} & A_{11}\boldsymbol{\Lambda}_{12} & \dots & \dots & A_{11}\boldsymbol{\Lambda}_{1,n-1} \\ \boldsymbol{\Gamma}_{11}A_{11} & A_{22} & A_{22}\boldsymbol{\Lambda}_{22} & \dots & \dots & A_{22}\boldsymbol{\Lambda}_{2,n-1} \\ \boldsymbol{\Gamma}_{21}A_{11} & \boldsymbol{\Gamma}_{22}A_{22} & A_{33} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{\Gamma}_{n-1,1}A_{1,1} & \boldsymbol{\Gamma}_{n-1,2}A_{2,2} & \dots & \dots & A_{n-1,n-1} & A_{n-1,n-1}\boldsymbol{\Lambda}_{n-1,n-1} \\ \dots & \dots & \dots & \dots & \boldsymbol{\Gamma}_{n-1,n-1}A_{n-1,n-1} & A_{nn} \end{bmatrix} \tag{18}$$

Elements (blocks) (18) depend only on $(3n - 2)m^2$ scalar quantities. These quantities are sub-matrices A_{ii} ($i = \overline{(1, n)}$), Λ_{ij} and Γ_{ij} ($i, j = \overline{(1, n-1)}$).

Taking into account given notations we can formulate the following theorem.

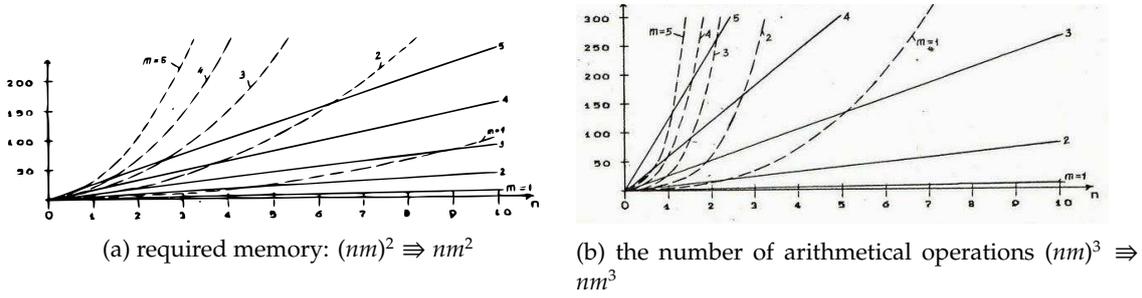


Figure 1: The required amount of memory (a) and the number of arithmetic operations (b), which is necessary for inversion of the matrix of the form A_N^m (continuous lines) and general matrices (dashed lines).

3. $C_{ii} (i = \overline{1, n}) \Rightarrow (n - 1)[m^2(m + 1)];$

4. $C_{i,i+1} (i = \overline{1, n - 1}) \Rightarrow (n - 1)m^2.$

Thus, the calculation of the matrix A_N^{-m} requires only $nm^2(4m + 3) - m^2(11m + 7)/2$ multiplications. In addition, there are $\approx (nm^2)$ operations on $\Omega_i (i = \overline{1, n})$ submatrices inversion. The last expression shows that the number of multiplications is proportional to n and m^3 .

By calculating the exact number of additions and subtractions required for inversion of A_N^m , one obtains an expression $(5n - 6, 5)m^3 - (2n - 2, 5)m^2 + (n - 1)m$, which is also proportional to n and m^3 .

It is known [21], that the number of arithmetic operations required for general matrix inversion of the size $(n \times m) \times (n \times m)$ is proportional to $(n \times m)^3$. Thus, for large values of the ratio n/m (that usually takes place in the tasks, considered in this study), accounting the structure of the covariance matrix of the observed Markov process gives an opportunity to simplify the calculation of the required estimates.

The ratio of the number of non-zero elements of the symmetric matrix A_N^m to the number of the elements of the filled symmetric matrix with different values of n and m is shown in Table 1. From the Table 1 it can be seen that the gain in the required memory amount practically independent of m and proportional to n .

Fig. 1 shows memory saving graphs and the number of arithmetic operations for inversion of matrix A_N^m with respect to n and m .

The ratio of the number of elements of the filled matrix $(N \times N)$ to the number of non-zero elements of the matrix A_N^m with an allowance of their symmetry in given in Table 1:

n/m	5	10	50	100	500	1000
1	1.67	2.89	12.82	25.38	125.38	250.38
2	1.53	2.76	12.75	25.25	125.25	250.25
3	1.48	2.72	12.71	25.21	125.21	250.21
4	1.46	2.70	12.69	25.19	125.19	250.19
5	1.44	2.68	12.68	25.18	125.18	250.18

Table 1: The ratio of the number of elements of the filled matrix $(N \times N)$ to the number of non-zero elements of the matrix A_N^m with an allowance of their symmetry

5. The Relationships of Matrices A_n^1, A_n^m, A_N^m with the Covariance Matrices of Measurements of Ordinary, m -Connected and Vector Markov Processes

5.1. The Covariance Matrix of Markov Process Measurements in a Wide-Sense

Let the observed process $Z(t)$ be a Markov process in the wide-sense (hereinafter referred to as the "Markov process"). This means that the covariance function $k(s, t)$ of the process $Z(t)$ satisfies the condition

(8).

$$k(s, t) = \frac{k(s, \tau)k(\tau, t)}{k(\tau, \tau)} (s < \tau < t) \tag{23}$$

From the Doob Theorem [12] it can be assumed that the condition (23) is not only necessary but also sufficient, i.e. positive definite function $k(s, t)$ is the covariance function of a Markov process only in case when it satisfies the condition (23).

Let the values of γ_i ($i = 1, 2, \dots$) to be defined as follows:

$$\gamma_i = \frac{k_{i,i+1}}{k_{ii}} \tag{24}$$

where $k_{ij} = k(t_i, t_j)$ – values of the covariance function of process $Z(t)$ at the points t_i and $t_j, t_j > t_i$. Thus, γ_i are the coefficients of the covariance of neighboring points reduced to a dispersion quantity in the points with a lower coordinate value. For stationary random processes $\gamma_i = \rho_{i,i+1}$, i.e. γ_i - correlation coefficients between adjacent measurement process.

Taking into account (23) and (24) we can formulate the following theorem.

Theorem 4. 1. Covariance matrix K_n of measurements of a Markov process $Z_M(t)$ at points $T_n = \{t_1 < t_2 < \dots < t_n \mid t_i \in T\}$ is a special case of the matrix (4) for $\Lambda_{ji} = \Gamma_{ij} = \prod_{l=j}^i \gamma_l$ ($i \geq j$), ($i, j = \overline{1, n-1}$), where γ_i is as previously defined in (24).

2. Elements of K_n^{-1} are defined by expression (6), taking into account that $\lambda_i = \gamma_i$ ($i = \overline{1, n-1}$) and $a_{ii} = k_{ii}$ ($i = \overline{1, n}$). At the same time

$$\alpha_i = k_{ii} - \gamma_{i-1}^2 k_{i-1,i-1}, \quad (i = \overline{1, n}), \alpha_1 = k_{11}, \tag{25}$$

$$\mu_i = k_{i+1,i+1} - \gamma_{i-1}^2 \gamma_i^2 k_{i-1,i-1}, \quad (i = \overline{2, n-1}), \mu_1 = k_{22}. \tag{26}$$

Theorem 5. (Inverse) Any symmetric positive definite matrix of the form A_n^1 is the covariance matrix of the Markov process measurements.

The proofs of Theorems 4 and 5 are given in [7] (see in [7] Theorem 2 and Note 3).

Note 10. As a consequence of the note 2, the matrix K_n^1 is completely determined by the elements of its two diagonals (i.e., elements at the main diagonal and at positions parallel to the main diagonal, above or below it). In other words, the matrix K_n^1 depends only on the dispersion (variance) values in the measuring n points and the $(n - 1)$ coefficients of the covariance between adjacent measurement points. The covariance matrix of the measurements of the Markov process is completely determined by small number $(2n - 1)$ of its elements. Therefore, for effective solution of the Markov random processes of statistics problems it is sufficient a priori knowledge of the mentioned elements of the covariance matrix.

5.2. The Covariance Function and the Covariance Matrix of Measurement of m -Connected Markov Process

Let $Z(t)$ be an m -connected Markov process. This means that the covariances between discrete measurements of the process $Z(t)$ satisfy the condition

$$k(t_i, t_j) = k_{ij} = \mathbf{k}_{i,[j-1]}^T K_m^{-1} [j - 1] \mathbf{k}_{[j-1],j}, \tag{27}$$

where $t_i < \dots < t_{j-m} < t_{j-m+1} < \dots < t_{j-1} < t_j$. The condition (2) can be obtained from (23) by converting it to the matrix-vector notation .

In (27) the following notations are used:

- $\mathbf{k}_{i,[j-1]}^T = (k_{i,j-m}, k_{i,[j-m+1]}, \dots, k_{i,j-1})$ - m - dimensional row-vector values $k(s, t)$ at the points $T_m[j - 1] = t_{j-m}, t_{j-m+1}, \dots, t_{j-1}$ and at the point t_i ;

- $\mathbf{k}_{[j-1],j} = (k_{j-m,j}, k_{j-m+1,j}, \dots, k_{j-1,j})$ - m - dimensional column-vector values $k(s, t)$ at the points $T_m[j-1]$ and in the point t_i (in other words $k_{[i],l}$ is m -dimensional vector of covariance measurements at points $T_m[i]$ with measurement at point $t_l \in T_n$);
- $K_m[j-1]$ - ($m \times m$) covariance matrix of vector values of $Z(t)$ at the points $T_m[j-1]$, i.e. $K_m[j-1] = [k(t_s, t_l)]$ ($s, l = \overline{j-m, j-1}$).

A graphical illustration of notation to the formula (27) is shown on Fig. 2.

Suppose that the m - connected Markov process $Z_M(t)$ was measured at the points T_n . Let K_n be the covariance matrix of these measurements.

For the matrix K_n , the vector $k_{[i],j}$ ($i \geq j$) in (27) can be interpreted as a set of elements of j - th column of the $i - m + 1$ to i for $i \geq m$ or from 1 to i under $i < m$. Thus, the dimension of the vector $k_{[i],j}$ will be equal to m for $i \geq m$ and will be equal i for $i < m$. Thus, matrix $K_m[i]$ can be interpreted as a diagonal sub-matrix K_n , located at the intersection of rows and columns of the same indices from $i - m + 1$ to i for $i \geq m$ or from 1 to i for $i < m$. Thus, the size of sub-matrix $K_m[i]$ will be equal $m \times m$ for $i \geq m$ and $i \times i$ for $i < m$.

Let the vectors Γ_i ($i = \overline{1, n-1}$) be defined as follows:

$$\Gamma_i = K_m^{-1}[i] \mathbf{k}_{[i],i+1}, \quad (i = \overline{1, n-1}). \tag{28}$$

Obviously, the dimension of the vector Γ_i will be equal m for $i \geq m$ and i for $i < m$. Taking into account the given notation we can formulate the following theorem.

Theorem 6. 1) The covariance matrix K_n for measurements at points T_n of m -connected Markov process is a special case A_n^m when $a_{ii} = k_{ii}$ ($i = \overline{1, n}$), $\gamma_{im}^T = \lambda_{mi} = \gamma_i$ ($i = \overline{1, n-1}$), $A_{im} = A_{mi}^T = K_{im}$ where K_{im} - sub-matrices of the matrices K_i ($i = \overline{1, n}$), represent their right m - column of a size $(i \times m)$ for $i \geq m$ and the size $(i \times i)$ for $1 < i < m$.

2) The matrix K_n^m , inverse to K_n^m , is a band with a half-width band equal to m , whose elements are defined by expressions

$$\alpha_i = k_{ii} - \mathbf{k}_{i,[i-1]}^T K_m^{-1}[i-1] \mathbf{k}_{[i-1],i} = k_{ii} - \mathbf{k}_{i,[i-1]}^T \Gamma_{i-1}, \quad (i = \overline{1, n}), \tag{29}$$

$$c_{ii} = \frac{1}{\alpha_i} + \sum_{k=0}^w \frac{\gamma_{m-k,i+k}^2}{\alpha_{i+k+1}}, \quad (i = \overline{1, n}), \tag{30}$$

$$c_{i,i+k} = -\frac{\gamma_{i+k-1,m-k+1}}{\alpha_{i+k}} + \sum_{j=k}^w \frac{\gamma_{i+j,m-j} \gamma_{i+j,m+k-j}}{\alpha_{i+j+1}}, \quad c_{i+k,i} = c_{i,i+k}, \quad (i = \overline{1, n-1}); \quad k = \overline{1, m}), \tag{31}$$

where $w = m - 1$ for $i \leq n - m$ and $w = n - i - 1$ for $i > n - m$.

Note 11. If in (30) and (31), the calculated upper limit becomes smaller than the lower one, then the summarizing should not be executed, i.e. in $j, k > w$ and $j, k > m$, the second term on the right side of mentioned formula is set to 0.

5.3. The Covariance Function and Covariance Matrix of the Measurement of m -Dimensional (Vector) Markov Process

The matrix of covariance function $K(s, t)$ of the vector of Markov process $Z_M(t)$ satisfies the conditions

$$K(s, t) = K(s, \tau) K^{-1}(\tau, \tau) K(\tau, t), \tag{32}$$

where $s < \tau < t$ or $s > \tau > t$. (Condition (32) is a necessary and sufficient condition to determine a Markov process in a wide-sense.)

Let us now try to find a general view of the covariance matrix of measurement at the points T_n of the vector Markov process. Let K_N be a block covariance matrix measurement at points T_n of m -dimensional

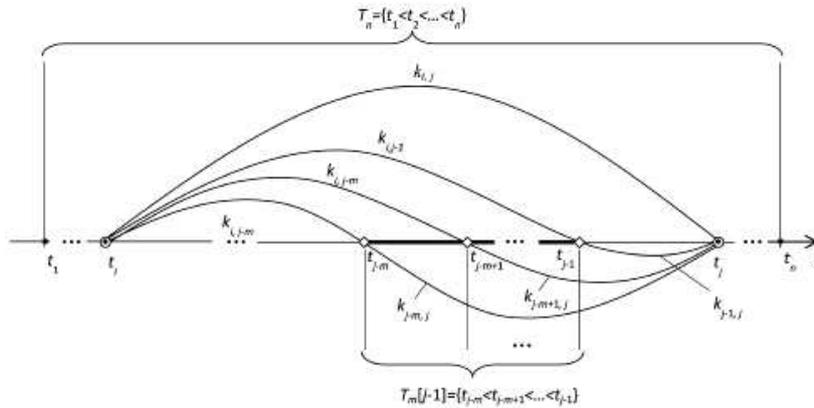


Figure 2: Graphical illustration of the formula (27)

vector process, consisting of n^2 blocks K_{ij} ($i, j = \overline{1, n}$). Blocks, in their own turn, represent the covariance matrix measurement of size $m \times m$ of components of the vector $Z_M(t)$. Let us define the square (sub) matrices Γ_i ($i = \overline{1, n-1}$) and Γ_{ij} ($i, j = \overline{1, n-1}$) of order m as follows:

$$\Gamma_i = K_{ii}^{-1} K_{i,i+1}, \tag{33}$$

$$\Gamma_{ij} = \begin{cases} \Gamma_i \Gamma_{i+1} \dots \Gamma_{j-1} \Gamma_j, & \text{for } i \geq j, \\ \Gamma_j^T \Gamma_{j+1}^T \dots \Gamma_{i+1}^T \Gamma_i^T, & \text{for } j \leq i, \end{cases} \quad (i, j = \overline{1, n-1}). \tag{34}$$

Taking into account the given notations, it is possible to formulate the following theorem.

Theorem 7. 1. Covariance matrix $K_N = K_N^m$ of measurements of m -dimensional Markov process $Z_M(t)$ at points T_n is positive defined and is a special case of (18) for $A_{ii} = K_{ii}$, ($i = \overline{1, n}$) $\Lambda_{ij} = \Gamma_{ji}$ ($i, j = \overline{1, n-1}$). Elements K_N^m depend (taking into account its symmetry) only on $(2n-1)m^2$ scalar quantities, which are the elements of sub-matrix K_{ii} ($i = \overline{1, n}$) and Γ_i ($i = \overline{1, n-1}$).

2. The inverse matrix $C_N^m = (K_N^m)^{-1} = K_N^{-m}$, divided into $(m \times m)$ blocks C_{ij}^m , has a block-tridiagonal form with non-zero elements:

$$C_{ii}^m = A_{i+1}^{-1} M_i A_i^{-1}, \quad (i = \overline{1, n-1}), \quad C_{nn}^m = A_n^{-1}, \tag{35}$$

$$C_{i,i+1}^m = -\Gamma_i A_{i+1}^{-1}, \quad C_{i+1,i}^m = -A_{i+1}^{-1} \Gamma_i^T = C_{i,i+1}^m, \tag{36}$$

where

$$A_i = K_{ii} - \Gamma_{i-1}^T K_{i-1,i-1} \Gamma_{i-1}, \quad (i = \overline{2, n}), \quad A_1 = K_{11}, \tag{37}$$

$$M_i = K_{i+1,i+1} - \Gamma_i^T \Gamma_{i-1}^T K_{i-1,i-1} \Gamma_{i-1} \Gamma_i, \quad (i = \overline{2, n-1}), \quad M_1 = K_{22} \tag{38}$$

The proof of Theorem 7 is similar to that of Theorem 4.

6. An Example of Covariance Matrix Inversion for Vector Markov Process

Despite of their external inconvenience, the formula retrieved above is easy to use in practical calculations. We will show it by the example of the covariance matrix inversion of the 2D Markov process, when the covariance function of the analyzed process is given.

Below, some examples of getting of general expressions for submatrices K_{ii} ($i = \overline{1, n}$), $K_{i,i+1}$ and Γ_i ($i = \overline{1, n - 1}$) of covariance matrix K_N^m of measurements at points T_n for the 2D vector process are given. Also, general expression for the sub-matrices A_i ($i = \overline{1, n}$) and M_i ($i = \overline{1, n - 1}$) was found, with the help of which we can easily calculate the non-zero sub-matrices of inverse matrix K_N^{-m} , without resorting to the standard procedures of matrix inversion (formulas (33)-(38)).

Note that in the general case, the separate components of the vector Markov process may be a Markov and non-Markov. Relationships between the components can also be Markov, non-Markov and semi-Markov, i.e. when mutual covariance function $k_{ij}(s, t)$ component Z_i and Z_j ($i, j = \overline{1, m}$) m - dimensional vector process satisfy (23) for $s < \tau < t$ and not satisfy for $s > \tau > t$.

Example 8. Let us consider a 2D non-stationary Markov process $Z(t) = (Z_1(t), Z_2(t))^T$, defined by the covariance matrix function

$$K(s, t) = [k_{ij}(s, t)](i, j = \overline{1, 2})$$

with elements

$$K_{11}(s, t) = \sigma_1^2 \min(s, t),$$

$$K_{22}(s, t) = \frac{\sigma_2^2}{2\alpha} [\exp(-\alpha|s - t|) - \exp(-\alpha(s + t))]$$

$$K_{12} = \frac{\sigma_1\sigma_1}{\alpha} \begin{cases} \exp(-\alpha(t - s)) - \exp(-\alpha t), & \text{for } s < t; \\ 1 - \exp(-\alpha t), & \text{for } s > t. \end{cases}$$

$$K_{12} = \frac{\sigma_1\sigma_1}{\alpha} \begin{cases} 1 - \exp(-\alpha s), & \text{for } s > t; \\ \exp(-\alpha(s - t)) - \exp(-\alpha s), & \text{for } s < t. \end{cases}$$

It is possible to verify that the matrix function $K(s, t)$ satisfy (32), and its elements satisfy (23), both at $s < \tau < t$, and $s > \tau > t$. Thus, $Z(t)$ and its components are Markov and Markov related processes in a wide-sense.

Note 12 ([22]). Under a normal distribution, the centered component $Z(t)$ coincides with the two-dimensional Markov process representing a solution for $t_0 = 0$ and $Z_1(0) = Z_2(0) = 0$ of system of stochastic differential equations (SDE) $\frac{dZ_1(t)}{dt} = \sigma_1 N(t)$, $\frac{dZ_2(t)}{dt} = -\alpha Z_1(t) + \sigma_2 N(t)$, excitation by normal white noise $N(t)$ with unit variance.

Let us write the expression for the submatrices K_{ii} and $K_{i,i+1}$ ($t_{i+1} > t_i$) (see.(33)) of the covariance matrix K_N , which affects the formation of non-zero submatrices of inverse matrix K_N^{-m} :

$$K_{ii} = \frac{1}{\alpha} \begin{bmatrix} \alpha\sigma_1^2 t_i & \sigma_1\sigma_2(1 - \exp(-\alpha t_i)) \\ \sigma_1\sigma_2(1 - \exp(-\alpha t_i)) & \sigma_2^2(1 - \exp(-2\alpha t_i))/2 \end{bmatrix}$$

$$K_{i,i+1} = \frac{1}{\alpha} \begin{bmatrix} \alpha\sigma_1^2 t_i & \sigma_1\sigma_2(\exp(-\alpha(t_{i+1} - t_i)) - \exp(-\alpha t_{i+1})) \\ \sigma_1\sigma_2(1 - \exp(-\alpha t_i)) & \sigma_2^2(\exp(-\alpha(t_{i+1} - t_i)) - (\exp(-\alpha(t_i - t_{i+1})))) \end{bmatrix}; (i = \overline{1, n - 1}).$$

In this sub-matrix K_{ii}^{-1} will have the form:

$$K_{ii}^{-1} = \frac{1}{\alpha \det K_{ii}} \begin{bmatrix} \sigma_{22}(1 - \exp(-2\alpha t_i))/2 & -\sigma_1\sigma_2(1 - \exp(-\alpha t_i)) \\ -\sigma_1\sigma_2(1 - \exp(-\alpha t_i)) & \alpha\sigma_1^2 t_i \end{bmatrix}; (i = \overline{1, n})$$

where

$$\det K_{ii} = \frac{\sigma_1^2\sigma_2^2}{\alpha} \left[\frac{t_i}{2}(1 - \exp(-2\alpha t_i)) - \frac{1}{\alpha}(1 - \exp(-\alpha t_i))^2 \right]$$

To present the matrix K_N^m in the form of K_N^m it is necessary to calculate sub-matrix Γ_i (34):

$$\Gamma_i = K_{ii}^{-1}K_{i,i+1} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(-\alpha(t_{i+1} - t_i)) \end{bmatrix} = \begin{bmatrix} \gamma_{1i} & 0 \\ 0 & \gamma_{2i} \end{bmatrix}, (i = \overline{1, n-1})$$

where $\gamma_{1i} = 1, \gamma_{2i} = \exp(-\alpha(t_{i+1} - t_i))$.

Prior to calculation of non-zero submatrices K_N^{-m} it is necessary calculate A_i and $A_i^{-1} (i = \overline{1, n})$ (see formula (37)):

$$A_i = \frac{1}{a} \begin{bmatrix} \alpha\sigma_1^2(t_i - t_{i1}) & \sigma_1\sigma_2(1 - \gamma_{2,i-1}) \\ \sigma_1\sigma_2(1 - \gamma_{2,i-1}) & \sigma_2^2(1 - \sigma_{2,i-1}^2)/2 \end{bmatrix}; (i = \overline{2, n})$$

$$A_1 = K_{11} = \frac{1}{a} \begin{bmatrix} \alpha\sigma_1^2 t_1 & \sigma_1\sigma_2(1 - \exp(-\alpha t_1)) \\ \sigma_1\sigma_2(1 - \exp(-\alpha t_1)) & \sigma_{22}(1 - \exp(-2\alpha t_1))/2 \end{bmatrix}; (t_0 = 0)$$

It is easy to calculate

$$A_i^{-1} = \frac{1}{\alpha \det A_i} \begin{bmatrix} \sigma_2^2(1 - \sigma_{2,i-1}^2)/2 & -\sigma_1\sigma_2(1 - \sigma_{2,i-1}) \\ -\sigma_1\sigma_2(1 - \sigma_{2,i-1}) & \alpha\sigma_1^2(t_i - t_{i-1}) \end{bmatrix} (i = \overline{2, n})$$

where

$$\det A_{ii} = \frac{\sigma_1^2\sigma_2^2}{\alpha} \left[\frac{1}{2}(t_i - t_{i-1})(1 - \sigma_{2,i-1}^2) - \frac{1}{\alpha}(1 - \sigma_{2,i-1})^2 \right];$$

$$A_1^{-1} = \frac{1}{\alpha \det A_1} \begin{bmatrix} \sigma_2^2(1 - \exp(-2\alpha t_1))/2 & -\sigma_1\sigma_2(1 - \exp(-\alpha t_1)) \\ -\sigma_1\sigma_2(1 - \exp(-\alpha t_1)) & \alpha\sigma_1^2 t_1 \end{bmatrix}; (i = \overline{1, n})$$

and

$$\det A_1 = \frac{\sigma_1^2\sigma_2^2}{a} \left[\frac{t_1}{2}(1 - \exp(-2\alpha t_1)) - \frac{1}{a}(1 - \exp(-\alpha t_1))^2 \right]$$

$$M_i = \frac{1}{\alpha} \begin{bmatrix} \alpha\sigma_1^2(t_{i+1} - t_{i-1}) & \sigma_1\sigma_2(1 - \gamma_{2,i-1}\gamma_{2,i}) \\ \sigma_1\sigma_2(1 - \gamma_{2,i-1}\gamma_{2,i}) & \gamma_{2,i-1}^2(1 - \gamma_{2,i-1}^2\gamma_{2,i}^2)/2 \end{bmatrix}; (i = \overline{2, n}).$$

$$M_1 = \frac{1}{\alpha} \begin{bmatrix} \alpha\sigma_1^2 t_2 & \sigma_1\sigma_2(1 - \exp(-\alpha t_2)) \\ \sigma_1\sigma_2(1 - \exp(-\alpha t_2)) & \sigma_2^2(1 - \exp(-2\alpha t_2)) \end{bmatrix}; (t_0 = 0)$$

Simplified expressions are obtained for a uniform measurements T_n^τ . Let $t_0 = 0, t_1 = \tau, t_2 = 2\tau$ and etc. Let $\sigma_1\sigma_2 = \sigma$. Then $\Gamma_1 = \Gamma_2 = \dots = \Gamma(n-1) = \Gamma; M_1 = M_2 = \dots = M_{n-1} = M$;

$$A = \frac{\sigma^2}{\alpha} \begin{bmatrix} \alpha\tau & (1 - \gamma) \\ (1 - \gamma) & (1 - \gamma^2)/2 \end{bmatrix}; A^{-1} = \frac{\sigma^2}{\alpha \det A} \begin{bmatrix} (1 - \gamma^2)/2 & -(1 - \gamma) \\ -(1 - \gamma) & \alpha\tau \end{bmatrix};$$

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}; M = \frac{\sigma^2}{\alpha} \begin{bmatrix} 2\alpha\tau & (1 - \gamma^2) \\ (1 - \gamma^2) & (1 - \gamma^4)/2 \end{bmatrix};$$

where $\gamma = \exp(-\alpha\tau)$

$$\det A = (\sigma^4/\alpha^2)(1 - \sigma)[\alpha\tau(1 + \gamma)/2 - (1 - \gamma)]$$

Then K_N^{-m} can be represented as a block-tridiagonal matrix consisting of n^2 blocks of size $(m \times m)$:

$$K_N^{-m} = A^{-1} = \left| \begin{array}{ccccccc} M & A\Gamma & & & & & \\ -\Gamma^T A & M & -A\Gamma & & & & 0 \\ & -\Gamma^T A & M & -A\Gamma & & & \\ & & -\Gamma^T A & \ddots & \ddots & & \\ & & & \ddots & \ddots & -A\Gamma & \\ & 0 & & & -\Gamma^T A & M & -A\Gamma \\ & & & & & -\Gamma^T A & A \end{array} \right| A^{-1}$$

In contrast to the scalar case, for vector processes there are various possible ways of formation of the covariance matrix of the measurement (CMM). The results, obtained above, relate to the case when CMM formed of n^2 submatrices of the size $(m \times m)$, representing measurements m component of vector process in a given point. But it is possible to form a CMM so that it will consist of m^2 blocks of the size $(n \times n)$, representing measurement of one component of the field at the points T_n^r . The general form of the matrix K_N^{-m} for this case have a form:

$$K_N^{-m} = A^{-1} = \left| \begin{array}{cccc|cccc} A_1 & A_2 & & & \Gamma_1 & \Gamma_2 & & \\ A_2 & A_1 & A_2 & & \Gamma_3 & \Gamma_1 & \Gamma_2 & 0 \\ & A_2 & A_1 & \dots & & \Gamma_3 & \Gamma_1 & \dots \\ & & \dots & \ddots & & \dots & \ddots & \dots \\ & 0 & \dots & \dots & 0 & \dots & \dots & \Gamma_1 & \Gamma_2 \\ & & & & & & & \Gamma_1 & 1 \\ \hline \Gamma_1 & \Gamma_3 & & & \Gamma_4 & \Gamma_5 & & \\ \Gamma_2 & \Gamma_1 & \Gamma_3 & & \Gamma_5 & \Gamma_4 & \Gamma_5 & 0 \\ & \Gamma_2 & \Gamma_1 & \dots & & \Gamma_5 & \Gamma_4 & \dots \\ & & \dots & \ddots & & \dots & \ddots & \dots \\ & 0 & \dots & \dots & 0 & \dots & \dots & \Gamma_4 & \Gamma_5 \\ & & & & & & & \Gamma_5 & \frac{\Gamma_1}{2} \end{array} \right|$$

where $A_1 = 2\alpha\tau$; $A_2 = -\alpha\tau$; $\Gamma_1 = 1 - \gamma^2$; $\Gamma_2 = -\gamma(1 - \gamma)$; $\Gamma_3 = -(1 - \gamma)$; $\Gamma_4 = (1 - \gamma^4)/2$; $\Gamma_5 = -\gamma(1 - \gamma^2)/2$ i.e. matrix K_N^{-m} consists of m^2 tridiagonal blocks of the size $(n \times n)$.

7. Conclusion

1. In this paper, forms of square matrix, whose inverses are tridiagonal, band or block-tridiagonal matrices, convenient for usage, have been represented. In the work they are designated as matrices: A_n^1 , where n - dimension of a matrix, 1 – half-width of the band; where m - half-width of the band; and A_N^m , where N - the dimension of the matrix, m - dimension of the blocks ($N = n \times m$).
2. Although there are many works devoted to the study of such matrices, no common approach and a general (unique) matrix structure was proposed. In our work it was shown that for all three classes of matrices, a common approach and a common (unique) matrix structure can be applied. Moreover, in the first case, the matrix elements A_n^1 are formed of the scalar quantities; in the second case, A_n^m - of vectors of dimensions m ; A_n^m where m - half-width of the band and in the third case, A_N^m of square blocks (sub-matrices) of dimension m .
3. For matrices of a given structure, a simple inversion formulas were found. It was shown that the elements of inverse matrices depend only on:

- $3n - 2$ elements included in 3 central diagonals for the matrix A_n^1 ;
 - $(2m + 1)n - m(m + 1)$ elements lying inside the band of the width $2m + 1$ (band's half-width equals m) for the matrix A_n^m ;
 - $(3n - 2)(m \times m)$ elements for the matrix $A_N^m (N = n \times m)$.
4. It is shown that if the matrices A_n^1 , A_n^m and $A_N^m (N = n \times m)$ are symmetric and positive definite, they are covariance matrices of measurements of simply (ordinary connected), multiple of the connectivity m and m - dimensional vector Markov processes in a wide-sense, respectively.
 5. It is shown that the covariance matrix of the measurement (CMM) of ordinary connected Markov process in a wide-sense depends only on the variance value at the measuring points and the coefficients of the covariance between adjacent measurement points. Accordingly, for multiply connected Markov process CMM depends on the variance and coefficients of covariance between points standing from each other by an amount equal to or less than the connectivity of process m .
 6. The obtained results allow simplifying the solution of many problems of random processes statistics. In particular, it dramatically simplifies the computational complexity of the estimating tasks, filtering and interpolation of random processes and fields using BLUE and GLSE, which have been using inversion of covariance matrix of measurements as a weighting matrix.

There are other related problems such as approximation of an arbitrary random process by Markov m -connected process; case sparse covariance matrices with number of non-zero elements being greater than $3n - 2$ elements, arranged in arbitrary positions; constructing recurrent algorithms. As a future study, a number of other interesting problems of processing the results of measurements of random processes are planned to be considered.

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APPENDIX: PROOF OF THEOREM 1

The validity of Theorem 1 is shown by induction, using a recursive procedure by means of a serial (step-by-step, successive) matrix bordering (Faddeev-Faddeeva bordering , see. [13]).

Firstly, let us consider the i -th step of the recurrent procedure of matrix inversion. According to [13], if

$$A_{i+1} = \begin{bmatrix} A_i & \alpha_{i+1} \\ \alpha_{i+1}^T & \alpha_{i+1,i+1} \end{bmatrix}; \quad i = 1, 2, 3, \dots$$

where $\alpha_{i+1}^T = (a_{i+1,1}, \dots, a_{i+1,i})$; $\alpha_{i+1} = (a_{1,i+1}, \dots, a_{i,i+1})^T$ - bordering a row vector and a column vector respectively of i -elements length,

$$A_{i+1} = \begin{bmatrix} A_i^{-1} + \frac{\mu_{i+1}v_{i+1}}{\alpha_{i+1}} & \frac{(-\mu_{i+1})}{\alpha_{i+1}} \\ \frac{(-v_{i+1})}{\alpha_{i+1}} & \frac{1}{\alpha_{i+1}} \end{bmatrix} \quad i = 1, 2, 3, \dots \tag{39}$$

where $\alpha_{i+1} = \alpha_{i+1,i+1} - \alpha_{i+1}^T \mu_{i+1} = \alpha_{i+1,i+1} - v_{i+1} \alpha_{i+1}$; $\mu_{i+1} = A_i^{-1} \alpha_{i+1}$; $v_{i+1} = \alpha_{i+1}^T A_i^{-1}$.

Suppose that the matrix A_i^{-1} of size $i \times i$ found in the previous $(i-1)$ -step recurrent procedure ($1 \leq i \leq n-1$) has tridiagonal form corresponding to (6), and the i -dimensional vectors α_{i+1} and α_{i+1}^T of the forms:

$$\alpha_{i+1}^T = (\alpha_{i+1}^*)^T = \left[\prod_{l=1}^i \gamma_l \alpha_{11}, \prod_{l=2}^i \gamma_l \alpha_{22}, \dots, \gamma_i \alpha_{ii} \right] \tag{40}$$

$$\alpha_{i+1} = \alpha_{i+1}^* = \left[\prod_{l=1}^i \lambda_l \alpha_{11}, \prod_{l=2}^i \lambda_l \alpha_{22}, \dots, \lambda_i \alpha_{ii} \right]$$

Carrying out the necessary calculations, we obtain the following formula:

$$\begin{aligned} \mu_{i+1} &= A_i^{-1} \alpha_{i+1} = A_i^{-1} \alpha_{i+1}^* = [0, 0, \dots, 0, \lambda_i] = u_{i+1}^* \\ v_{i+1} &= \alpha_{i+1}^T A_i^{-1} = \alpha_{i+1}^{*T} A_i^{-1} = [0, 0, \dots, 0, \gamma_i]^T = v_{i+1}^* \end{aligned} \tag{41}$$

Substituting the values u_{i+1} and v_{i+1} from (41) to (39) and performing all necessary operations, we see that the matrix A_{i+1}^{-1} will also be tridiagonal, and the elements A_{i+1}^{-1} correspond to (6). Thus, if the initial matrix, which begins the process of recurrent inverse is tridiagonal and bordering vectors $\alpha_l, \alpha_l^T, l = \overline{i+1, n}$ are selected from the corresponding rows and columns of (4), all subsequent $\{A_l^{-1}, l = \overline{i+1, n}\}$ matrices, including the latest, are tridiagonal.

Successive (step-by-step) calculation of $A_1^{-1}, A_2^{-1}, A_3^{-1}$ for the matrix $A_i^1 (i = \overline{1, 3})$ shows that the matrix A_3^{-1} is tridiagonal, i.e. the initial part of the procedure (A1) for the matrices (4) also leads to a tridiagonal matrix. This completes the proof of Theorem 1.