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Asymptotics of Solution to the Nonstationary Schrödinger Equation

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Abstract. The Cauchy problem with a rapidly oscillating initial condition for the homogeneous Schrödinger equation was studied in [5]. Continuing the research ideas of this work and [3], in this paper we construct the asymptotic solution to the following mixed problem for the nonstationary Schrödinger equation:

$$L_h u \equiv ih\partial_t u + h^2\partial_x^2 u - b(x,t)u = f(x,t), \qquad (x,t) \in \Omega = (0,1)\times(0,T],$$

$$u|_{t=0} = g(x),\ u|_{x=0} = u|_{x=1} = 0, \tag{1}$$

where $h > 0$ is a Planck constant, $u = u(x,t,h)$. $b(x,t), f(x,t) \in C^\infty(\bar{\Omega}), g(x) \in C^\infty[0,1]$ are given functions.

The similar problem was studied in [7, 8] when the Plank constant is absent in the first term of the equation and asymptotics of solution of any order with respect to a parameter was constructed. In this paper, we use a generalization of the method used in [7].

1. Regularization of the Problem

For regularizations of the problem (1), we will introduce the following regulating variables

$$\tau_1 = \frac{t}{h^2},\quad \tau_2 = \frac{is(x,t)}{h},\quad \xi_1 = \frac{x}{\sqrt{h}},\ \ \xi_2 = \frac{1-x}{\sqrt{h}},\ \ \eta_1 = \frac{x}{\sqrt{h^3}},\ \ \eta_2 = \frac{1-x}{\sqrt{h^3}},$$

where the existence of a smooth solution of the problem is assumed:

$$\partial_t s(x,t) - (\partial_x s(x,t))^2 - b(x,t) = 0,\ \ s(x,t)|_{t=0} = 0. \tag{2}$$

Instead of the desired function $u(x,t,h)$ we study the extended function $\tilde{u}(M,h)$, $M = (x,t,\xi,\eta,\tau), \xi = (\xi_1,\xi_2), \eta = (\eta_1,\eta_2), \tau = (\tau_1,\tau_2)$ such that its constriction by regularizing variables coincides with the desired solution:

$$\tilde{u}(M,h)|_{\chi=\psi(x,t,h)} \equiv u(x,t,h), \tag{3}$$

where $\chi = (\xi,\eta,\tau)$, $\psi(x,t,\eta) = (\frac{x}{\sqrt{h}}, \frac{1-x}{\sqrt{h}}, \frac{x}{\sqrt{h^3}}, \frac{1-x}{\sqrt{h^3}}, \frac{t}{h^2}, \frac{is(x,t)}{h})$.

Using (2), from (3) we find

$$\partial_t u \equiv (\partial_t\tilde{u} + \frac{1}{h^2}\partial_{\tau_1}\tilde{u} + \frac{i\partial_t s(x,t)}{h}\partial_{\tau_2}\tilde{u})|_{\chi=\psi(x,t,\eta)},$$

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$$\partial_x u \equiv (\partial_x \tilde{u} + \frac{1}{\sqrt{h}}\sum_{l=1}^{2}[(-1)^{l-1}[\partial_{\xi_l}\tilde{u} + \frac{1}{h}\partial_{\eta_l}\tilde{u}] + \frac{i\partial_x s}{h}\partial_{\tau_2}\tilde{u}])|_{\chi=\psi(x,t,\eta)},$$

$$\partial_x^2 u \equiv [\partial_x \tilde{u} + \frac{1}{h}\sum_{l=1}^{2}[\partial_{\xi_l}^2\tilde{u} + \frac{1}{h^2}\partial_{\eta_l}^2\tilde{u}] + \frac{1}{\sqrt{h}}L_\xi\tilde{u} + \frac{1}{\sqrt{h^3}}L_\eta\tilde{u}+$$

$$+(\frac{i\partial_x s(x,t)}{h})^2\partial_{\tau_2}^2\tilde{u} + \frac{i}{h}(2\partial_x s\partial_{x\tau_2}^2\tilde{u} + \partial_x^2 s\partial_{\tau_2}\tilde{u})]|_{\chi=\psi(x,t,\eta)}, \quad (4)$$

$$L_\xi \equiv 2\sum_{l=1}^{2}(-1)^{l-1}\partial_{x\xi_l}^2, \;\; L_\eta \equiv 2\sum_{l=1}^{2}(-1)^{l-1}\partial_{x\eta_l}^2.$$

On the basis of (1), (3), (4) for the extended function $\tilde{u}(M,h)$, we set the problem as:

$$\tilde{L}_h\tilde{u} \equiv \frac{1}{h}T_1\tilde{u} + D\tilde{u} + \sqrt{h}L_\eta\tilde{u} + hT_2\tilde{u} + h\sqrt{h}L_\xi\tilde{u} + h^2\partial_x^2\tilde{u} = f(x,t), \;\; M \in Q, \quad (5)$$

$$\tilde{u}|_{t=\tau_1=\tau_2=0} = g(x), \;\; \tilde{u}|_{x=\eta_1=\xi_1=0} = \tilde{u}|_{x=1,\eta_2=\xi_2=0} = 0,$$

where $T_1 \equiv i\partial_{\tau_1} + \sum_{l=1}^{2}\partial_{\eta_l}^2$, $T_2 \equiv i\partial_t + \sum_{l=1}^{2}\partial_{\xi_l}^2$, $D \equiv -\partial_t s\partial_{\tau_2} + (\partial_x s)^2\partial_{\tau_2}^2 + b(x,t)$. The following identity holds:

$$(\tilde{L}_h\tilde{u}(M,h))_{\chi=\psi(x,t,\eta)} \equiv u(x,t,h). \quad (6)$$

The solution of problem (5) is determined in the form of the following series

$$\tilde{u}(M,h) = \sum_{k=0}^{\infty} h^{k/2}u_k(M). \quad (7)$$

For the coefficients of this series, we obtain the following iterative problems:

$$T_1u_\nu(M) = 0, \nu = 0,1, T_1u_2(M) = f(x,t) - Du_0(M),$$

$$T_1u_k(M) = -Du_{k-2} - L_\tau u_{k-3} - T_2u_{k-4} - L_\xi u_{k-5} - \partial_x^2 u_{k-6}, \;\; k \geq 3, \quad (8)$$

$$u_0(M)|_{t=\tau_1=\tau_2=0} = g(x), u_k(M)|_{t=\tau_1=\tau_2=0} = 0, u_k|_{x=0,\xi_1=\eta_1=0} = u_k|_{x=1,\xi_2=\eta_2=0} = 0.$$

2. Solution of Iteration Problems

We introduce classes of functions in which the iterative problems are solved:

$$U_1 = \left\{u_1^1(M) : u^1 = v(x,t) + c(x,t)exp(\tau_2) + \sum_{l=1}^{2}\omega^l(x,t)erfc(\frac{\xi_l}{2\sqrt{it}})exp(\tau_2)\right\},$$

$$U_2 = \left\{u_1^2(M) : u^2 = \sum_{l=1}^{2}Y^l(N_l), \; N_l = (x,t,\tau_1,\eta), \; Y^l(N_l) \sim exp(-\frac{\eta_l^2}{4i\tau_1}), \;\; \forall\eta_l, \tau_1 \in (0,\infty)\right\}.$$

From these spaces we construct a new space:

$$U = U_1 \oplus U_2;$$

then the function $u_k(M) \in U$ has the form

$$u_k(M) = v_k(x,t) + \sum_{l=1}^{2}Y_k^l(N_l) + [c(x,t) + \sum_{l=1}^{2}\omega_k^l(x,t)erfc(\frac{\xi_l}{2\sqrt{it}})]exp(\tau_2), k \geq 0. \quad (9)$$

Theorem 2.1. *If the given functions are smooth, the problem (2) has a smooth solution and the right-hand side of the equation*

$$T_1 u_k(M) = H_k(M) \tag{10}$$

belongs to U_2, then the equation (10) is solvable in U.

Proof. We substitute the function $u_k(M) \in U$ from (9) into (10); then, with respect to $Y_k^l(N_l)$, we obtain the equation

$$T_{1l} Y_k^l(N_l) = H_k(M), \;\; T_{1l} \equiv i\partial_{\tau_1} - \partial^2_{\eta_l}.$$

Since the right-hand side of $H_k(M) \in U_2$, this equation, with the appropriate boundary conditions, has a solution of the form

$$Y_k^l(N_l) = d_k^l(x,t) erfc(\frac{\eta_l}{2\sqrt{i\tau_1}}) + \frac{2}{\sqrt{i\pi}} \int_0^t \int_0^\infty \frac{H_k^l(\cdot)}{\sqrt{\tau_1 - \tau}} [exp(-\frac{(\eta_l - s)^2}{4i(\tau_1 - \tau)}) - exp(-\frac{(\eta_l + s)^2}{4i(\tau_1 - \tau)})] ds d\tau$$

The theorem is proved. □

Theorem 2.2. *Let the conditions of Theorem 2.1 hold. Then equation (10) under additional conditions*

$$1) \;\; u_k(M)|_{t=\tau_1=\tau_2=0} = g(x), \;\; u_k(M)|_{x=l-1,\xi_l=\eta_l=0} = 0, \;\; l = 1, 2,$$

$$2) \;\; H(M) \equiv -Du_{k-2} - L_\eta u_{k-3} - T_2 u_{k-4} - L_\xi u_{k-5} - \partial_x^2 u_{k-6} \in U_2,$$

$$3) \;\; L_\eta u_k = 0, \;\; L_\xi u_k = 0$$

has a unique solution.

Proof. By Theorem 2.1, equation (10) has solutions $u_k(M) \in U$. Since the function $u_k(M)$ satisfies conditions 1), we obtain

$$Y_k^l(N_l)|_{t=\tau_1=0} = 0, \;\; Y_k^l(N_l)|_{\eta_l=0} = d_k^l(x,t),$$

$$d_k^l(x,t)|_{x=l-1} = -v_k(l-1,t), \;\; d_k^l(x,t)|_{t=0} = d_k^{l,0}(x)$$

$$\omega_k^l(x,t)|_{t=0} = \omega_k^{l,0}(x), \;\; \omega_k^l(x,t)|_{x=l-1,\xi_l=0} = -c_k(l-1,t), \;\; l = 1, 2. \tag{11}$$

There $d_k^{l,0}(x)$, $\omega_k^{l,0}(x)$ are arbitrary functions.

We calculate the actions of the operators $D, L_\eta, T_2, L_\xi, \partial_x^2$ on the function $u_k(M) \in U$ with allowance for (2), and we obtain

$$Du_{k-2}(M) = b(x,t) Y_{k-2}^l + b(x,t) v_{k-2}(x,t),$$

$$L_\eta u_{k-3}(M) = 2 \sum_{l=1}^{2} (-1)^{l-1} \partial^2_{x\eta_l} Y_{k-3}^l,$$

$$T_2 u_{k-4}(M) = i\partial_t v_{-4} + i\partial_t Y_{k-4}^l + [\sum_{l=1}^{2} i\partial_t \omega_{k-4}^l erfc(\frac{\xi_l}{2\sqrt{it}}) + i\partial_t c_{k-4}(x,t)] exp(\tau_2), \tag{12}$$

$$L_\xi u_{k-5}(M) = 2 \sum_{l=1}^{2} (-1)^{l-1} \partial_x \omega_{k-5}^l(x,t) \partial_{\xi_l}(erfc(\frac{\xi_l}{2\sqrt{it}})),$$

$$L_x u_{k-6}(M) = \partial_x^2 v_{k-6}(x,t) + \sum_{l=1}^{2} \partial_x Y_{k-6}^l + [\partial_x^2 c_k + \sum_{l=1}^{2} \partial_x^2 \omega_{k-6}^l(x,t) erfc(\frac{\xi_l}{2\sqrt{it}})] exp(\tau_2).$$

Using these relations and ensuring condition 2), we set

$$L_\xi u_{k-5}(M)=0,\;\; L_\eta u_{k-3}(M)=0,$$

$$b(x,t)v_{k-2}(x,t)+i\partial_t v_{k-4}(x,t)+\partial_x^2 v_{k-6}=0,\;\; \partial_x\omega_{k-5}^l(x,t)=0,$$

$$i\partial_t c_{k-4}(x,t)+\partial_x^2 c_{k-6}(x,t)=0,\;\; i\partial_t\omega_{k-4}^l(x,t)+\partial_x^2\omega_{k-6}^l(x,t)=0,\;\; i\partial_t Y_{k-4}^l+\partial_x^2 Y_{k-6}^l=0.$$

With such a choice of the functions entering into the function $u_k(M)$, equation (10) takes the form

$$T_{1l}Y_k^l(N_l)=b(x,t)Y_{k-2}^l(N_l),$$

of the solution, which, under the boundary conditions from (11), can be written in the form

$$Y_k^l(N_l)=d_k^l(x,t)erfc(\frac{\eta_l}{2\sqrt{\tau_1 i}})+\frac{1}{2\sqrt{i}}\int_0^{\tau_1}\int_0^\infty\frac{b(x,t)Y_{k-2}^l(\cdot)}{\sqrt{\tau_1-\tau}}[exp(-\frac{(\eta_l-s)^2}{4i(\tau_1-\tau)})-exp(-\frac{(\eta_l+s)^2}{4(\tau_1-\tau)i})]d\tau ds. \quad (13)$$

The function $d_k^l(x,t)$ stands with the factor of the function $erfc(\frac{\eta_l}{2\sqrt{i\tau_1}})$. Since $erfc(\frac{\eta_l}{2\sqrt{i\tau_1}})|_{\tau_1=0}=0$ is the value of the function $d_k^l(x,t)$ for $t=0$ arbitrarily chosen and this arbitrary function ensures the condition $L_\eta Y_{k-3}^l(N_l)=0$. The initial condition for this equation is determined from the relation

$$Y_{k-3}^l(N_l)|_{x=l-1,\eta_l=0}=d_k^l(x,t)|_{x=l-1}=-v_{k-3}(l-1,t).$$

Thus the function $Y_k^l(N_l)$ is uniquely defined. Solving equations (12) with the corresponding initial conditions from (11). The function $\omega_k^l(x,t)$ is expressed in terms of an arbitrary function $\omega_k^{l,0}(x)$, which ensures the condition $L_\xi u_k(M)=0$. This uniquely determines all functions occurring in $u_k(M)$ from (9). The theorem is proved. □

We solve the iterative problems (8) in the class of functions U. By Theorem 2.1, problem (8) for $k=0,1$ has a solution of the form (9) if the function $Y_k^l(N_l)$ is a solution of equation

$$i\partial_{\tau_1}Y_\nu^l=\partial_{\eta_l}^2Y_\nu^l,\;\; \nu=0,1 \quad (14)$$

for initial and boundary conditions in (8):

$$Y_\nu^l(N_l)|_{\tau_1=0}=Y_\nu^l(N_l)|_{\eta_l=0}=d_\nu^l(x,t)=-v_\nu(l-1,t),\;\; d_\nu^l(x,t)|_{t=0}=d_\nu^{l,0}(x),$$

$$c_0(x,0)=g(x)-v_0(x,0),\;\; \omega_\nu^l(x,t)|_{t=0}=\tilde{\omega}_\nu^{l,0}(x),\;\; c_1(x,0)=v_1(x,0),$$

$$\omega_\nu^l(x,t)|_{x=l-1,\xi_l=0}=-c_\nu(l-1,t). \quad (15)$$

The solution of equation (13) with boundary conditions (14) has the form

$$Y_\nu^l(N_l)=d_\nu^l(x,t)erfc(\frac{\eta_l}{2\sqrt{i\tau_1}}). \quad (16)$$

For $\tau_1=0$, we have $erfc(\frac{\eta_l}{2\sqrt{i\tau_1}})=0$; therefore, by its factor we chose an arbitrary function $d_k^l(x,t)$ and the function $d_\nu^{l,0}(x)$ is taken as the value for $t=0$. Following Theorem 2.2, this function will be used to make zero $L_\eta u_k(M)=0$. We substitute (14) into the equation for $Y_k^l(N_l)$ from (12); then, with respect to $d_\nu^l(x,t)$, we obtain equation

$$\partial_t d_{k-4}^l(x,t)+\partial_x^2 d_{k-6}^l(x,t)=0.$$

Solving it under the initial condition $d_{k-4}^l(x,t)|_{t=0}=d_{k-4}^{l,0}(x)$, we define

$$d_{k-4}^l(x,t)=d_{k-4}^{l,0}(x)+P_{k-6}^l(x,t). \quad (17)$$

Now substitute in $L_\eta u_k(M)$, then taking into account (17) with respect to $d^{l,0}_{k-4}(x)$, we obtain a differential equation. The initial condition for it is determined from the relation with respect to $Y^l_\nu(N_l)$ is the one entering into (14)

$$d^l_{k-4}(x,t)|_{x=l-1} = (d^{l,0}_{k-4}(x) + P^l_{k-6}(x,t))|_{x=l-1} = -v_{k-4}(l-1,t). \tag{18}$$

Thus the function $Y^l_\nu(N_l)$ is uniquely defined. Consider equation (8) for $k = 2$. Assuring solvability in U, according to Theorem 2.1, we require condition

$$F_2(M) = f(x,t) - Du_0 \in U_2; \tag{19}$$

then equation (8), $k = 2$ is solvable if $Y^l_2(N_l)$ and is a solution of the equation

$$i\partial_{\tau_1} Y^l_2 = \partial^2_{\eta_l} Y^l_2 + F_2(N_l).$$

Providing condition (19), following Theorem 2.2, we obtain

$$b(x,t)v_0(x,t) = -f(x,t); \tag{20}$$

the right-hand side is rewritten as

$$F_2(N_l) = -b(x,t)Y^l_0(N_l).$$

Equation (20) has the solution of the form (13) under the appropriate conditions from (14). In the next step, the right-hand side of equation (8), with $k = 3$, has the form

$$F_3(M) = -Du_1 - L_\eta u_0.$$

According to Theorems 2.1 and 2.2, we get

$$L_\eta u_0 = 2\sum_{l=1}^{2}(-1)^{l-1}\partial_x d^{l,0}_0(x)\partial_{\eta_l}(erfc(\frac{\eta_l}{2\sqrt{i\tau_1}})) = 0,\ or\ (d^{l,0}_0(x))' = 0$$

$$v_1(x,t) = 0.$$

Whence we determine

$$d^{l,0}_0(x) = -v_0(l-1,t),$$

the value of d is determined in the next step from the problem

$$\partial_t d^l_0(x,t) = 0,\ \ d^l_0(x,t)|_{t=0} = d^{l,0}_0(x).$$

Notice that the function $u_k(M)$ with odd indices vanishes. Indeed, the free term of the next iteration equation for $k = 4$ has the form

$$F_4(M) = -Du_2 - L_\eta u_1 - T_1 u_0.$$

By Theorems 2.1 and 2.2, this equation has a solution in U if

$$-b(x,t)v_2(x,t) = \partial_t v_0(x,t),$$

$$\partial_t d^l_0(x,t) = 0,\ d^l_0(x,t)|_{t=0} = d^{l,0}_0(x),$$

$$(d^l_1(x,t))'_x = 0,\ \ d^l_1(x,t)|_{x=l-1} = -v_1(l-1,t),\ \ d^l_1(x,t)|_{t=0} = d^{l,0}_1(x).$$

$$\partial_t \omega^l_0(x,t) = 0,\ \ \omega^l_0(x,t)|_{t=0} = \omega^{l,0}_0(x),\ \ \partial_t c_0(x,t) = 0,\ \ c_0(x,t)|_{t=0} = g(x) - v_0(x,0),$$

$$\omega^l_0(x,t)|_{x=l-1} = -c_0(l-1,t).$$

Taking into account that $v_1(x,t) = 0$, we find $d^l_1(x,t) = 0$, and from the remaining problems we define $v_2(x,t), \omega^l_0(x,t), c_0(x,t)$. Further, repeating this process, we successively determine all the coefficients of the partial sum.

Lemma 2.3. *For the function*

$$erfc(\frac{\xi}{2\sqrt{it}}) = \frac{2}{\sqrt{\pi}}\int_{\frac{\xi}{2\sqrt{it}}}^{\infty} exp(-s^2)ds$$

it holds

$$erfc(\frac{\xi}{2\sqrt{it}}) < cexp(-\frac{\xi^2}{4it}).$$

Proof. We make the change of variables $s = y + \frac{\xi}{2\sqrt{it}}$, $dy = ds$, and considering that $\frac{1}{\sqrt{i}} = \frac{2}{\sqrt{2}}(1-i)$ we get

$$erfc(\frac{\xi}{2\sqrt{it}}) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} exp(-y^2 - \frac{\xi}{\sqrt{it}}y - \frac{\xi^2}{4it})dy =$$

$$= \frac{2}{\sqrt{\pi}}exp(-\frac{\xi^2}{4it})\int_0^{\infty} exp(-y^2 - \frac{\xi}{\sqrt{t}}\frac{2}{\sqrt{2}}(1-i)y)dy =$$

$$= \frac{2}{\sqrt{\pi}}exp(-\frac{\xi^2}{4it})\int_0^{\infty} exp(-y^2 - \sqrt{\frac{2}{t}}\xi y + \sqrt{\frac{2}{t}}i\xi y)dy =$$

$$= \frac{2}{\sqrt{\pi}}exp(-\frac{\xi^2}{4it})\int_0^{\infty} exp(-y^2 - \sqrt{\frac{2}{t}}\xi y)\left[cos(\sqrt{\frac{2}{t}}\xi y) + isin(\sqrt{\frac{2}{t}}\xi y)\right]dy.$$

Using Hölder's inequality we have

$$erfc(\frac{\xi}{2\sqrt{it}}) \leq \frac{2}{\sqrt{\pi}}exp(-\frac{\xi^2}{4it})\left(\int_0^{\infty}\left|exp(-y^2 - \sqrt{\frac{2}{t}}\xi y)\right|dy\right)^{\frac{1}{2}} \times$$

$$\times\left(\int_0^{\infty}\left|exp(-y^2 - \sqrt{\frac{2}{t}}\xi y)\right|\left|cos(\sqrt{\frac{2}{t}}\xi y) + isin(\sqrt{\frac{2}{t}}\xi y)\right|^2 dy\right)^{\frac{1}{2}} =$$

$$= \frac{2}{\sqrt{\pi}}exp(-\frac{\xi^2}{4it})\int_0^{\infty} exp(-y^2 - \sqrt{\frac{2}{t}}\xi y)dy.$$

Replacing the integral by the formula 7.4.2 of [1], we find

$$erfc(\frac{\xi}{2\sqrt{it}}) \leq \frac{2}{\sqrt{\pi}}exp(-\frac{\xi^2}{4it})\frac{\sqrt{\pi}}{2}exp(\frac{\xi^2}{2t})\int_{\xi/\sqrt{2t}}^{\infty} e^{-s^2}ds.$$

Using inequality 4 from §4.8.5 in [4], we obtain

$$erfc(\frac{\xi}{2\sqrt{it}}) \leq exp(-\frac{\xi^2}{4it})\frac{1}{\sqrt{(\pi-2)^2\frac{\xi^2}{2t} + \pi} + 2\sqrt{\frac{\xi}{\sqrt{2t}}}} = cexp(-\frac{\xi^2}{4it}).$$

□

Lemma 2.4. *Let*

$$F(\xi,t) \leq cexp(-\frac{\xi^2}{4it}). \qquad (L-1)$$

Then for the integral

$$I(\xi,t)=\frac{2}{\sqrt{i\pi}}\int_0^t\int_0^\infty\frac{F(s,\tau)}{\sqrt{t-\tau}}[exp(-\frac{(\xi-s)^2}{4i(t-\tau)})-exp(-\frac{(\xi+s)^2}{4i(t-\tau)})]dsd\tau \qquad \text{(L-2)}$$

we have

$$I(\xi,t)\le cexp(-\frac{\xi^2}{4it}). \qquad (L-3)$$

Proof. Consider

$$I(\xi,t)=\frac{2}{\sqrt{i\pi}}\int_0^t\int_0^\infty\frac{F(s,\tau)}{\sqrt{t-\tau}}\left[exp(-\frac{(\xi-s)^2}{4i(t-\tau)})-exp(-\frac{(\xi+s)^2}{4i(t-\tau)})\right]dsd\tau=$$

$$=\left|\frac{\xi\pm s}{2\sqrt{i(t-\tau)}}=z\ \ dz=\pm\frac{ds}{2\sqrt{i(t-\tau)}},\quad \pm s=-\xi+2\sqrt{i(t-\tau)}z\right|=$$

$$=\frac{4}{\sqrt{\pi}}\int_0^t\left[\int_{\frac{\xi}{2\sqrt{i(t-\tau)}}}^{-\infty}F(\xi-2\sqrt{i(t-\tau)}z,\tau)e^{-z^2}dz-\int_{\frac{\xi}{2\sqrt{i(t-\tau)}}}^{\infty}F(-\xi+2\sqrt{i(t-\tau)}z,\tau)e^{-z^2}dz\right]d\tau.$$

With regard to (L-3) we rewrite this as

$$I(\xi,t)\le\frac{4c}{\sqrt{\pi}}\int_0^t[-\int_{-\infty}^{\frac{\xi}{2\sqrt{i(t-\tau)}}}exp(-z^2-\frac{(\xi-2\sqrt{i(t-\tau)}z)^2}{4it})dz-$$

$$-\int_{\frac{\xi}{2\sqrt{i(t-\tau)}}}^{\infty}exp(-z^2-\frac{(-\xi+2\sqrt{i(t-\tau)}z)^2}{4it})dz]d\tau\le$$

$$\le\frac{4c}{\sqrt{\pi}}\int_0^t[\int_{-\infty}^{\frac{\xi}{2\sqrt{i(t-\tau)}}}exp(-z^2-\frac{\xi^2-4\sqrt{i(t-\tau)}z\xi+4i(t-\tau)z^2}{4i\tau})dz+$$

$$+\int_{\frac{\xi}{2\sqrt{i(t-\tau)}}}^{\infty}exp(-z^2-\frac{\xi^2-4\sqrt{i(t-\tau)}z\xi+4i(t-\tau)z^2}{4i\tau})dz]=$$

$$=\frac{4c}{\sqrt{\pi}}\int_0^t\int_{-\infty}^{\infty}exp(-\frac{4z^2i\tau+\xi^2-4\sqrt{i(t-\tau)}z\xi+4itz^2-4i\tau z^2}{4i\tau})dzd\tau=$$

$$=\frac{4c}{\sqrt{\pi}}\int_0^t exp(-\frac{\xi^2}{4i\tau})\int_{-\infty}^{\infty}exp(-\frac{t}{\tau}z^2+\frac{\sqrt{t-\tau}}{\sqrt{i}\tau}\xi z)dzd\tau.$$

Using the formula 3.323.3 from [2] we obtain

$$I(\xi,t)\sim\frac{4c}{\sqrt{\pi}}\int_0^t exp(-\frac{\xi^2}{4i\tau})exp(\frac{\frac{t-\tau}{i\tau^2}\xi^2}{4\frac{t}{\tau}})\frac{\sqrt{\pi}}{\sqrt{\frac{t}{\tau}}}d\tau=$$

$$=\frac{4c}{\sqrt{\pi}}\int_0^t exp(-\frac{\xi^2}{4i\tau})exp(\frac{(t-\tau)\tau\xi^2}{4i\tau^2t})\sqrt{\frac{\tau}{t}}d\tau=$$

$$=4c\int_0^t exp(-\frac{\xi^2}{4i\tau}+\frac{(t-\tau)\xi^2}{4i\tau t})\sqrt{\frac{\tau}{t}}d\tau=$$

$$=4c\int_0^t\sqrt{\frac{\tau}{t}}exp(-\frac{\xi^2}{4i\tau}+\frac{\xi^2}{4i\tau}-\frac{\xi^2}{4it})d\tau=$$

$$= 4c\frac{1}{\sqrt{t}}exp(-\frac{\xi^2}{4it})\int_0^t \sqrt{\tau}d\tau = cexp(-\frac{\xi^2}{4it}).$$

□

$$u_{n,h}(M) = \sum_{k=0}^{n} h^k u_{2k}(M) \qquad (L-4)$$

Producing a restriction by means of the regularizing functions, on the basis of (6), for the remainder term

$$R_n(x,t,h) = u(x,t,\epsilon) - u_n(M)|_{\chi=\psi(x,t,h)}$$

we obtain the problem

$$L_\epsilon R_n = h^{n+1}g_{2n}(x,t,h), \;\; R_n(x,t,h)|_{t=0} = R_n(x,t,h)|_{x=0} = R_n(x,t,h)|_{x=1} = 0,$$

where $\left|g_{2n}(x,t,h)\right| < c$. Using the maximum principle and following [6]. We get the estimate

$$|R_n(x,t,h)| < ch^{n+1}.$$

Theorem 2.5. *Let the given functions be sufficiently smooth. Then the problem (1) has an asymptotic solution that is representable in the form (L-4) for* $\chi = \psi(x,t,\eta)$ *and for all* $n = 0,1,2,...,$ $0 < h < h_0$ *holds.*

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