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A Further Result on the Potential-Ramsey Number of G_1 and G_2

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Abstract. A non-increasing sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is a *graphic sequence* if it is realizable by a simple graph G on n vertices. In this case, G is referred to as a *realization* of π. Given a graph H, a graphic sequence π is *potentially H-graphic* if π has a realization containing H as a subgraph. Busch et al. (Graphs Combin., 30(2014)847–859) considered a degree sequence analogue to classical graph Ramsey number as follows: for graphs G_1 and G_2, the *potential-Ramsey number* $r_{pot}(G_1, G_2)$ is the smallest non-negative integer k such that for any k-term graphic sequence π, either π is potentially G_1-graphic or the complementary sequence $\overline{\pi} = (k-1-d_k, \ldots, k-1-d_1)$ is potentially G_2-graphic. They also gave a lower bound on $r_{pot}(G, K_{r+1})$ for a number of choices of G and determined the exact values for $r_{pot}(K_n, K_{r+1})$, $r_{pot}(C_n, K_{r+1})$ and $r_{pot}(P_n, K_{r+1})$. In this paper, we will extend the complete graph K_{r+1} to the complete split graph $S_{r,s} = K_r \vee \overline{K_s}$. Clearly, $S_{r,1} = K_{r+1}$. We first give a lower bound on $r_{pot}(G, S_{r,s})$ for a number of choices of G, and then determine the exact values for $r_{pot}(C_n, S_{r,s})$ and $r_{pot}(P_n, S_{r,s})$.

1. Introduction

Graphs in this paper are finite, undirected and simple. Terms and notation not defined here are from [1]. A non-increasing sequence $\pi = (d_1, \ldots, d_n)$ of nonnegative integers is a *graphic sequence* if it is realizable by a (simple) graph G on n vertices. In this case, G is referred to as a *realization* of π, and we write $\pi = \pi(G)$. Two well known characterizations of graphic sequences were given by Havel and Hakimi [10,9], and Erdős and Gallai [5]. Given a graph H, a graphic sequence π is *potentially H-graphic* if there exists a realization of π containing H as a subgraph. The complementary sequence of π is denoted by $\overline{\pi} = (\overline{d_1}, \ldots, \overline{d_k}) = (k-1-d_k, \ldots, k-1-d_1)$.

Degree sequence problems can be broadly classified into two types, first described as "forcible" problems and "potential" problems by A.R. Rao in [12]. In a forcible degree sequence problem, a specified graph property must exist in every realization of the degree sequence π, while in a potential degree sequence problem, the desired property must be found in at least one realization of π. Results on forcible degree sequences are often stated as traditional problems in extremal graph theory.

There are a number of degree sequence analogues to well known problems in extremal graph theory, including potentially graphic sequence analogues of the Turán problem [6,7,8], the Erdős-Sós conjecture

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[14], Hadwiger's conjecture [4,13] and the Sauer-Spencer theorem [3]. Motivated in part by this previous work, Busch et al. [2] proposed a degree sequence analogue to classical graph Ramsey number. Given two graphs G_1 and G_2 and a graphic sequence π, we write that $\pi \to (G_1, G_2)$ if either π is potentially G_1-graphic or $\overline{\pi}$ is potentially G_2-graphic. Busch et al. [2] defined the *potential-Ramsey number* of G_1 and G_2, denoted $r_{pot}(G_1, G_2)$, to be the smallest non-negative integer k such that $\pi \to (G_1, G_2)$ for any k-term graphic sequence π. Busch et al. [2] first gave a lower bound on $r_{pot}(G, K_t)$ for a number of choices of G, and then determined the exact values for $r_{pot}(K_n, K_t)$, $r_{pot}(C_n, K_t)$ and $r_{pot}(P_n, K_t)$, where K_n, C_n and P_n are the complete graph on n vertices, the cycle on n vertices and the path on n vertices, respectively. The 1-*dependence number* of a graph G, denoted $\alpha^{(1)}(G)$, is the maximum order of an induced subgraph H of G with $\Delta(H) \le 1$, where $\Delta(H)$ is the maximum degree of H.

Theorem 1.1 [2] *Let G be a graph of order n with no isolated vertices such that $\alpha^{(1)}(G) \le n-1$ and let $t \ge 2$. Then $r_{pot}(G, K_t) \ge \max\{2t+n-\alpha^{(1)}(G)-2, n+t-2\}$.*

Theorem 1.2 [2] *(1) If $n \ge t \ge 3$, then $r_{pot}(K_n, K_t) = 2n+t-4$ except when $n = t = 3$, in which case $r_{pot}(K_3, K_3) = 6$.*

(2) If $n \ge 3$ and $t \ge 2$ with $t \le \lfloor \frac{2n}{3} \rfloor$, then $r_{pot}(C_n, K_t) = n+t-2$.

(3) If $n \ge 4$ and $t \ge 3$ with $t \ge \lfloor \frac{2n}{3} \rfloor + 1$, then $r_{pot}(C_n, K_t) = 2t-2+\lceil \frac{n}{3} \rceil$.

(4) If $n \ge 6$ and $t \ge 3$, then $r_{pot}(P_n, K_t) = \begin{cases} n+t-2, & \text{if } t \le \lfloor \frac{2n}{3} \rfloor, \\ 2t-2+\lfloor \frac{n}{3} \rfloor, & \text{if } t \ge \lfloor \frac{2n}{3} \rfloor + 1. \end{cases}$

We now extend the complete graph K_{r+1} to $S_{r,s} = K_r \vee \overline{K_s}$, a complete split graph on $r+s$ vertices, where $\overline{K_s}$ is the complement of K_s and $\vee$ denotes join operation. Clearly, $S_{r,1} = K_{r+1}$. Therefore, the complete split graph $S_{r,s}$ is an extension of the complete graph K_{r+1}. In this paper, we first give a lower bound on $r_{pot}(G, S_{r,s})$ for a number of choices of G (Theorem 1.3), and then determine the exact values of $r_{pot}(C_n, S_{r,s})$ for $n \ge 3$ and $r, s \ge 1$ (Theorem 1.4–1.8) and $r_{pot}(P_n, S_{r,s})$ for $n \ge 6$ and $r, s \ge 1$ (Theorem 1.9).

Theorem 1.3 *Let G be a graph of order n with no isolated vertices such that $\alpha^{(1)}(G) \le n-1$ and let $r, s \ge 1$. Then $r_{pot}(G, S_{r,s}) \ge \max\{n+2r+s-\alpha^{(1)}(G)+\frac{-3+(-1)^{s-1}}{2}, n+r+s-\alpha(G)-1, n+r-1\}$, where $\alpha(G)$ is the independence number of G.*

Theorem 1.4 *Let $n \ge 4$, $r \ge 1$ and $s \ge 1$. If $s \le \lfloor \frac{n}{2} \rfloor$ and $r+s \le \lfloor \frac{2n}{3} \rfloor$, then $r_{pot}(C_n, S_{r,s}) = n+r-1$.*

Theorem 1.5 *Let $n \ge 4$, $r \ge 1$ and $s \ge 1$. If $s \ge \lfloor \frac{n}{2} \rfloor$ and $r \le \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$, then $r_{pot}(C_n, S_{r,s}) = \lceil \frac{n}{2} \rceil + r + s - 1$.*

Theorem 1.6 *Let $n \ge 4$, $r \ge 1$ and $s \ge 1$, where s is odd, or let $(n, r, s) = (4, 1, 4)$ or $(5, 2, 2)$ or $(4, 2, 2)$ or $(6, 3, 2)$. If $s \le \lfloor \frac{n}{2} \rfloor$ and $r+s \ge \lfloor \frac{2n}{3} \rfloor + 1$ or if $s \ge \lfloor \frac{n}{2} \rfloor$ and $r \ge \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + 1$, then $r_{pot}(C_n, S_{r,s}) = \lceil \frac{n}{3} \rceil + 2r + s - 1$.*

Theorem 1.7 *Let $n \ge 4$, $r \ge 1$ and $s \ge 2$, where s is even, and let $(n, r, s) \ne (4, 1, 4), (5, 2, 2), (4, 2, 2)$ and $(6, 3, 2)$. If $s \le \lfloor \frac{n}{2} \rfloor$ and $r+s \ge \lfloor \frac{2n}{3} \rfloor + 1$ or if $s \ge \lfloor \frac{n}{2} \rfloor$ and $r \ge \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + 1$, then $r_{pot}(C_n, S_{r,s}) = \lceil \frac{n}{3} \rceil + 2r + s - 2$.*

Theorem 1.8 *(1) $r_{pot}(C_3, S_{1,2}) = 5$, $r_{pot}(C_3, S_{1,3}) = 6$ and $r_{pot}(C_3, S_{1,s}) = s+2$ for $s \ge 4$.*

(2) If $r \ge 2$ and $s \ge 1$, where s is odd and $(r, s) \ne (2, 1)$, then $r_{pot}(C_3, S_{r,s}) = 2r+s$.

(3) If $r \ge 2$ and $s \ge 2$, where s is even and $(r, s) \ne (2, 2)$, then $r_{pot}(C_3, S_{r,s}) = 2r+s-1$.

(4) $r_{pot}(C_3, S_{2,1}) = 6$ and $r_{pot}(C_3, S_{2,2}) = 6$.

Theorem 1.9 *Let $n \ge 6$, $r \ge 1$ and $s \ge 1$.*

(1) If $s \le \lceil \frac{n}{2} \rceil - 1$ and $r+s \le \lceil \frac{2n}{3} \rceil + \frac{-1+(-1)^s}{2}$, then $r_{pot}(P_n, S_{r,s}) = n+r-1$.

(2) If $s \ge \lceil \frac{n}{2} \rceil$ and $r \le \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + \frac{-1+(-1)^s}{2}$, then $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{2} \rfloor + r + s - 1$.

(3) If $s \le \lceil \frac{n}{2} \rceil - 1$ and $r+s \ge \lceil \frac{2n}{3} \rceil + \frac{-1+(-1)^s}{2} + 1$ or if $s \ge \lceil \frac{n}{2} \rceil$ and $r \ge \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + \frac{-1+(-1)^s}{2} + 1$, then $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{3} \rfloor + 2r + s + \frac{-3+(-1)^{s-1}}{2}$.

It is easy to see that if $s = 1$, then Theorem 1.3 reduces to Theorem 1.1, Theorem 1.4 reduces to Theorem 1.2(2), Theorem 1.6 reduces to Theorem 1.2(3) and Theorem 1.9 reduces to Theorem 1.2(4).

2. Proofs of Theorem 1.3–1.9

We first prove Theorem 1.3.

Proof of Theorem 1.3. When s is odd, let $\ell = n - \alpha^{(1)}(G) - 1$ and consider $\pi = \pi(K_\ell \vee (r + \frac{s-1}{2})K_2)$, where pK_2 denotes the disjoint union of p copies of K_2. Clearly, π is unigraphic. Firstly, $\overline{\pi}$ is uniquely realized by

$(K_{2r+s-1} - (r + \frac{s-1}{2})K_2) \cup \overline{K_\ell}$ which contains no $S_{r,s}$, where $\cup$ denotes disjoint union and $K_{2r+s-1} - (r + \frac{s-1}{2})K_2$ is the graph obtained from K_{2r+s-1} by deleting $r + \frac{s-1}{2}$ independent edges. Secondly, any copy of G lying in the unique realization of π requires at least $\alpha^{(1)}(G) + 1$ vertices from the $r + \frac{s-1}{2}$ independent edges, which is impossible as any such collection of vertices would necessarily induce a subgraph of G with order at least $\alpha^{(1)}(G) + 1$ and maximum degree at most one. Hence $\pi \nrightarrow (G, S_{r,s})$. Thus $r_{pot}(G, S_{r,s}) \geq n + 2r + s - \alpha^{(1)}(G) - 1$. When s is even, let $\ell = n - \alpha^{(1)}(G) - 1$ and consider $\pi = \pi(K_\ell \vee (r + \frac{s}{2} - 1)K_2)$. Similarly, we can show that $\pi \nrightarrow (G, S_{r,s})$. Thus $r_{pot}(G, S_{r,s}) \geq n+2r+s-\alpha^{(1)}(G)-2$. Therefore, we have $r_{pot}(G, S_{r,s}) \geq n+2r+s-\alpha^{(1)}(G)+\frac{-3+(-1)^{s-1}}{2}$ for any integer $s \geq 1$.

In order to show that $r_{pot}(G, S_{r,s}) \geq n+r+s-\alpha(G)-1$, we let $\ell = n-\alpha(G)-1$ and consider $\pi = \pi(K_\ell \vee \overline{K_{r+s-1}})$, which is unigraphic. Firstly, $\overline{\pi}$ is uniquely realized by $K_{r+s-1} \cup \overline{K_\ell}$ which contains no $S_{r,s}$. Secondly, any copy of G lying in the unique realization of π requires at least $\alpha(G) + 1$ vertices from the $\overline{K_{r+s-1}}$, which is impossible as any such collection of vertices would necessarily induce a subgraph of G with order at least $\alpha(G) + 1$ and maximum degree zero. Hence $\pi \nrightarrow (G, S_{r,s})$. Thus $r_{pot}(G, S_{r,s}) \geq n + r + s - \alpha(G) - 1$.

We now consider $\pi = \pi(K_{n-1} \cup \overline{K_{r-1}})$, which is unigraphic. Clearly, $\pi \nrightarrow (G, S_{r,s})$. Thus, $r_{pot}(G, S_{r,s}) \geq n + r - 1$. □

In order to prove Theorem 1.4–1.9, we need some useful lemmas as follows. For a subgraph H of graph G and a vertex v in G, $N_H(v)$ denotes those neighbors of v lying in H and we let $d_H(v) = |N_H(v)|$. Moreover, for $S \subseteq V(G)$, we denote $N_H(S) = \cup_{v \in S} N_H(v)$.

Lemma 2.1 [11] *Let $n \geq 3$ and $\pi = (d_1, \dots, d_n)$ be a graphic sequence with $d_3 \geq 2$. Then π is potentially C_3-graphic if and only if $\pi \neq (2^4), (2^5)$, where the symbol x^y in a sequence stands for y consecutive terms x.*

Lemma 2.2 [2] *Let $n \geq 4$, $r, s \geq 1$, $k = \max\{\lceil \frac{n}{3} \rceil + 2r + s + \frac{-3+(-1)^{s-1}}{2}, \lceil \frac{n}{2} \rceil + r + s - 1, n + r - 1\}$ and $\pi = (d_1, \dots, d_k)$ be a graphic sequence. Suppose that π has a realization G containing a cycle $C = v_0 v_1 \cdots v_{m-1}$ with $m \geq n$, and amongst all such realizations let m be minimum. If $m > n$, then (1) C is induced; (2) $d_G(x) = 0$ for each $x \in V(G) \setminus V(C)$.*

Lemma 2.3 *Let $n \geq 4$, $r, s \geq 1$, $k = \max\{\lceil \frac{n}{3} \rceil + 2r + s + \frac{-3+(-1)^{s-1}}{2}, \lceil \frac{n}{2} \rceil + r + s - 1, n + r - 1\}$ and $\pi = (d_1, \dots, d_k)$ be a graphic sequence. Let G be a realization of π containing a longest cycle $C = v_1 v_2 \cdots v_m$ with $m \leq n - 1$ and suppose that G has the maximum circumference amongst all realizations of π. Denote $H = G \setminus V(C)$. Then*

(1) [2] H is acyclic.

(2) [2] If $\Delta(H) \geq 2$, then the unique non-trivial component of H is a star H_1. Moreover, if $x \in V(H)$ is the center of H_1, then $d_H(x) = \Delta(H)$, m is even and x is adjacent to either all odd index vertices or all even index vertices of C.

(3) If $\Delta(H) = 1$, then $N_C(u) = N_C(u')$ for any two distinct vertices $u, u' \in V(H)$ with $d_H(u) = d_H(u') = 1$.

(4) [2] If $\Delta(H) = 1$, denote $R = N_C(u)$ and $R^+ = \{v_{i+1} | v_i \in R\}$, where $u \in V(H)$ with $d_H(u) = 1$, then $v_{i\pm1}, v_{i\pm2} \notin R$ for any $v_i \in R$, R^+ is an independent set of G, and $xy \notin E(G)$ for any $x \in R^+$ and $y \in V(H)$ with $d_H(y) = 0$.

(5) If $\Delta(H) = 1$, then $|N_C(x) \setminus R| \leq 1$ for each $x \in V(H)$ with $d_H(x) = 0$.

(6) If $\Delta(H) = 1$, $R \neq \emptyset$ and $r + s \leq |V(H)| \leq 2r + s - 1$, then $\pi \rightarrow (C_n, S_{r,s})$ or $2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| \leq 2r - 2$.

(7) If $\Delta(H) \leq 1$ and H contains p isolated vertices with $p \geq r$, then $\pi \rightarrow (C_n, S_{r,s})$.

Proof. (3) Let $xx' \in E(H)$. For $v_i \in V(C)$, if $v_i x \in E(G)$ and $v_i x' \notin E(G)$, then exchange the edges xx' and $v_i v_{i+1}$ for the nonedges $v_{i+1} x$ and $v_i x'$, we obtain a realization of π containing a cycle $v_1 \cdots v_i x v_{i+1} \cdots v_m v_1$ of length $m + 1$, a contradiction. Hence, if $v_i x \in E(G)$, then $v_i x' \in E(G)$. This implies that $N_C(x) \subseteq N_C(x')$. Similarly, we have $N_C(x') \subseteq N_C(x)$. Thus $N_C(x) = N_C(x')$. For $yy' \in E(H)$ with $yy' \neq xx'$, if $v_i x \in E(G)$ and $v_i y \notin E(G)$, then exchange the edges $xx', yy', v_i v_{i+1}$ for the nonedges $v_{i+1} x, x'y', v_i y$, we obtain a realization of π containing a cycle $v_1 \cdots v_i x v_{i+1} \cdots v_m v_1$ of length $m + 1$, a contradiction. Hence, if $v_i x \in E(G)$, then $v_i y \in E(G)$. This implies that $N_C(x) \subseteq N_C(y)$. Similarly, we have $N_C(y) \subseteq N_C(x)$. Thus $N_C(x) = N_C(y)$. Therefore, $N_C(u) = N_C(u')$ for any two distinct vertices $u, u' \in V(H)$ with $d_H(u) = d_H(u') = 1$.

(5) Assume $v_j, v_k \in N_C(x) \setminus R$ with $k \geq j + 1$ for $x \in V(H)$ with $d_H(x) = 0$. Let $x_1 y_1 \in E(H)$, if $k - j = 1$, then π contains a cycle with length $m + 1$, a contradiction. If $k - j \geq 2$, then exchange the edges $x_1 y_1, v_k x, v_j v_{j+1}$ for the nonedges $v_k y_1, v_j x_1, v_{j+1} x$, we obtain a realization of π which contains a cycle $v_1 \cdots v_j x v_{j+1} \cdots v_1$ of length $m + 1$, a contradiction.

(6) Note that $|E(C \setminus R)| \geq m - 2|R|$. If $m - 2|R| \geq \ell - \lfloor \frac{s}{2} \rfloor$, then we can use $\ell - \lfloor \frac{s}{2} \rfloor$ edges in C to breakout the $\ell - \lfloor \frac{s}{2} \rfloor$ edges in H and create a realization of π in which there are at least r isolated vertices in H, implying that $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $m - 2|R| \leq \ell - \lfloor \frac{s}{2} \rfloor - 1$, we can use the $m - 2|R|$ edges to breakout the $m - 2|R|$ edges in H

and obtain a realization of π in which there are $2(m-2|R|)+p+(\ell-(m-2|R|)-\lceil \frac{s}{2} \rceil) = \ell-\lceil \frac{s}{2} \rceil+(m-2|R|)+p$ isolated vertices in H. If $\ell-\lceil \frac{s}{2} \rceil+(m-2|R|)+p \geq r$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $\ell-\lceil \frac{s}{2} \rceil+(m-2|R|)+p \leq r-1$. On the other hand, by Lemma 2.3(4), then R^+ along with the p isolates in H and $\ell-\lceil \frac{s}{2} \rceil$ vertices from $\ell-\lceil \frac{s}{2} \rceil$ edges in H forms an independent set in G. If $\ell-\lceil \frac{s}{2} \rceil+|R^+|+p = \ell-\lceil \frac{s}{2} \rceil+|R|+p \geq r$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $\ell-\lceil \frac{s}{2} \rceil+|R|+p \leq r-1$, then $(\ell-\lceil \frac{s}{2} \rceil+(m-2|R|)+p)+(\ell-\lceil \frac{s}{2} \rceil+|R|+p) \leq 2r-2$, i.e., $2\ell-2\lceil \frac{s}{2} \rceil+2p+m-|R| \leq 2r-2$.

(7) Clearly, $|V(H)| = |G|-|V(C)| \geq (n+r-1)-(n-1) = r$. Let S be the set of r isolated vertices in H. If $|N_C(S)| \leq \lceil \frac{n}{2} \rceil - 1$, then $|G|-|N_C(S)\cup S| \geq (\lceil \frac{n}{2} \rceil+r+s-1)-(\lceil \frac{n}{2} \rceil+r-1) = s$, implying that $\overline{G}$ contains $S_{r,s}$, i.e., $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $|N_C(S)| \leq \lceil \frac{n}{3} \rceil + r + \frac{-3+(-1)^{s-1}}{2}$, then $|G|-|N_C(S)\cup S| \geq (\lceil \frac{n}{3} \rceil+2r+s+\frac{-3+(-1)^{s-1}}{2})-(\lceil \frac{n}{3} \rceil+2r+\frac{-3+(-1)^{s-1}}{2}) \geq s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $|N_C(S)| \geq \lceil \frac{n}{2} \rceil$ and $|N_C(S)| \geq r+\lceil \frac{n}{3} \rceil+\frac{-3+(-1)^{s-1}}{2}+1$. By $\lceil \frac{n}{2} \rceil = \lfloor \frac{n-1}{2} \rfloor + 1 \geq \lfloor \frac{m}{2} \rfloor + 1$ and the maximum of m, there are two consecutive vertices (say v_1, v_2) on C and $x, x' \in S$ $(x \neq x')$ so that $v_1x, v_2x' \in E(G)$, and hence $r \geq 2$. By $r+\lceil \frac{n}{3} \rceil+\frac{-3+(-1)^{s-1}}{2}+1 \geq r+1$, there are $y \in S$ and $v, v' \in V(C)$ $(v \neq v')$ so that $vy, v'y \in E(G)$. Assume $N_C(x) = \{v_1\}$ and $N_C(x') = \{v_2\}$. Then $y \neq x, x'$. If $N_C(y) \cap \{v_1, v_2\} = \emptyset$, then exchange the edges $v_1x, v_2x', vy, v'y$ for the nonedges $v_1y, v_2y, vx, v'x'$, we obtain a realization of π containing a cycle $v_1yv_2\cdots v_mv_1$ of length $m+1$, a contradiction. If $N_C(y) \cap \{v_1, v_2\} \neq \emptyset$, without loss of generality, we let $v = v_1$, then exchange the edges $v_2x', v'y$ for the nonedges $v_2y, v'x'$, we obtain a realization of π containing a cycle $v_1yv_2\cdots v_mv_1$ of length $m+1$, a contradiction. Hence $|N_C(x)| \geq 2$ or $|N_C(x')| \geq 2$. For $v \in V(C)\setminus\{v_1\}$, if $vx \in E(G)$ and $vx' \notin E(G)$, then exchange the edges vx, v_2x' for the nonedges v_2x, vx', we obtain a realization of π containing a cycle $v_1xv_2\cdots v_mv_1$ of length $m+1$, a contradiction. Similarly, we have that for $v \in V(C)\setminus\{v_2\}$, if $vx' \in E(G)$, then $vx \in E(G)$. So, we conclude that $N_C(x)\setminus\{v_1\} = N_C(x')\setminus\{v_2\}$.

We claim that $|N_C(z)\setminus(N_C(x)\cup\{v_2\})| \leq 1$ for $z \in V(S)\setminus\{x, x'\}$. To the contrary, let $v, v' \in N_C(z)\setminus(N_C(x)\cup\{v_2\})$ with $v \neq v'$. If $N_C(z)\cap\{v_1, v_2\} = \emptyset$, then exchange the edges $vz, v'z, v_1x, v_2x'$ with the nonedges $v_1z, v_2z, vx, v'x'$, we obtain a realization of π which contains a cycle $v_1zv_2\cdots v_mv_1$ of length $m+1$, a contradiction. If $N_C(z)\cap\{v_1, v_2\} \neq \emptyset$, without loss of generality, we let $v_1 \in N_C(z)$, then exchange the edges $v'z, v_2x'$ with the nonedges $v_2z, v'x'$, we obtain a realization of π which contains a cycle $v_1zv_2\cdots v_mv_1$ of length $m+1$, a contradiction.

Since $|N_C(S)| \geq r+\lceil \frac{n}{3} \rceil+\frac{-3+(-1)^{s-1}}{2}+1$ and $|V(S)\setminus\{x, x'\}| = r-2$, $|N_C(x)| = |N_C(x')| \geq |N_C(S)|-(r-2)-1 \geq \lceil \frac{n}{3} \rceil+\frac{-3+(-1)^{s-1}}{2}+2$. If $v_3 \in N_C(x)$ or $v_m \in N_C(x)$, then G clearly contains a cycle of length $m+1$, a contradiction. Hence $v_3, v_m \notin N_C(x)$. Let $v_p, v_{p+q} \in N_C(x)\setminus\{v_1\}$ so that q is the minimum. Then $4 \leq p \leq p+q \leq m-1$. If $q = 1$, then G clearly contains a cycle of length $m+1$, a contradiction. If $q = 2$, by $N_C(x)\setminus\{v_1\} = N_C(x')\setminus\{v_2\}$, then $v_pxv_1v_m\cdots v_{p+2}x'v_2v_3\cdots v_{p-1}v_p$ is a cycle of length $m+1$, a contradiction. Hence $q \geq 3$. If $n \not\equiv 0 \pmod 3$, then $\lceil \frac{n}{3} \rceil \leq |N_C(x)| \leq \lceil \frac{m-4}{3} \rceil + 1 \leq \lceil \frac{n-2}{3} \rceil$ (by $v_2 \notin N_C(x)$ and $m \leq n-1$), a contradiction. If $n \equiv 0 \pmod 3$ and $m \leq n-2$, then $\lceil \frac{n}{3} \rceil \leq |N_C(x)| \leq \lceil \frac{m-4}{3} \rceil+1 \leq \lceil \frac{n-3}{3} \rceil$, a contradiction. Assume $n \equiv 0 \pmod 3$ and $m = n-1$. If s is odd, then $\lceil \frac{n}{3} \rceil+1 \leq |N_C(x)| \leq \lceil \frac{m-4}{3} \rceil+1 \leq \lceil \frac{n-2}{3} \rceil$, a contradiction. Assume that s is even. If $r \geq 3$, we take $z \in S\setminus\{x, x'\}$. If $|N_C(z)\setminus(N_C(x)\cup\{v_2\})| = 0$, then $|N_C(x)| = |N_C(x')| \geq |N_C(S)|-(r-3)-1 \geq \lceil \frac{n}{3} \rceil+1$ and $\lceil \frac{n}{3} \rceil+1 \leq |N_C(x)| \leq \lceil \frac{m-4}{3} \rceil+1 \leq \lceil \frac{n-2}{3} \rceil$, a contradiction. If $|N_C(z)\setminus(N_C(x)\cup\{v_2\})| = 1$, let $N_C(z)\setminus(N_C(x)\cup\{v_2\}) = \{v_j\}$, where $3 \leq j \leq m$, then $N_C(z)\setminus\{v_j\} = N_C(x)\setminus\{v_1\} = N_C(x')\setminus\{v_2\}$. To the contrary, let $v' \in N_C(x)\setminus\{v_1\}$ and $v' \notin N_C(z)$, exchange the edges $v'x, v_2x', v_jz$ with the nonedges $v'z, v_2x, v_jx'$, we obtain a realization of π containing a cycle $v_1xv_2\cdots v_mv_1$ of length $m+1$, a contradiction. Thus $\lceil \frac{n}{3} \rceil \leq |N_C(x)| \leq \lceil \frac{m-5}{3} \rceil+1 = \lceil \frac{n-3}{3} \rceil$ (by $v_2, v_j \notin N_C(x)$ and $m = n-1$), a contradiction. Assume $r = 2$. If $k \geq \lceil \frac{n}{3} \rceil+s+3$ and $|N_C(S)| \leq \lceil \frac{n}{3} \rceil+1$, then $|G|-|N_C(S)\cup S| \geq (\lceil \frac{n}{3} \rceil+s+3)-(\lceil \frac{n}{3} \rceil+3) = s$, and $\overline{\pi}$ is potentially $S_{2,s}$-graphic. If $k \geq \lceil \frac{n}{3} \rceil+s+3$ and $|N_C(S)| \geq \lceil \frac{n}{3} \rceil+2$, then $\lceil \frac{n}{3} \rceil+1 \leq |N_C(S)|-1 = |N_C(x)| \leq \lceil \frac{m-4}{3} \rceil+1 \leq \lceil \frac{n-2}{3} \rceil$, a contradiction. If $k = \lceil \frac{n}{3} \rceil+s+2$, then $n = 6, m = 5, s \geq 4$. Since $|G| = k = s+4$ is even, there is $z \in V(H)\setminus\{x, x'\}$ with $d_H(z) = 0$. Note that $N_C(x) = \{v_1, v_4\}$ and $N_C(x') = \{v_2, v_4\}$ (by $|N_C(x)| = |N_C(x')| \geq \lceil \frac{6}{3} \rceil = 2$). If $v_3 \in N_C(z)$, then $v_2 \notin N_C(z)$, exchange the edges v_2x', v_3z with the nonedges v_2z, v_3x', we obtain a realization of π which contains a cycle $v_1v_2v_3x'v_4v_5v_1$ of length 6, a contradiction. Hence $v_3 \notin N_C(z)$. Similarly, $v_5 \notin N_C(z)$. Thus $N_C(z) \subseteq \{v_1, v_4\}$ or $N_C(z) \subseteq \{v_2, v_4\}$. Without loss of generality, we let $N_C(z) \subseteq \{v_1, v_4\}$, then there are $(s+4)-4 = s$ vertices in G which are not adjacent to x and z, implying that $\overline{\pi}$ is potentially $S_{2,s}$-graphic. □

Lemma 2.4 [15] *Let $n \geq r+1$ and $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_r \geq r+s-1$ and $d_{r+s} \geq r$. If $d_i \geq 2r + (s-1) - i$ for $i = 1, \ldots, r+s-1$, then π is potentially $S_{r,s}$-graphic.*

Proof of Theorem 1.4. By $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha^{(1)}(C_n) = \lfloor \frac{2n}{3} \rfloor$ (see [2]), it is easy to get from Theorem 1.3 that $r_{pot}(C_n, S_{r,s}) \geq n + r - 1$.

Let $\pi = (d_1, \ldots, d_k)$ be a graphic sequence with $k = n + r - 1$. We now prove that $\pi \to (C_n, S_{r,s})$. If no realization of π contains a cycle, by Lemma 2.1, then $d_3 \leq 1$. Let G be a realization of π. Then $|G| = n+r-1 \geq (\lfloor \frac{2n}{3} \rfloor + \lceil \frac{n}{3} \rceil) + r - 1 \geq \lceil \frac{n}{3} \rceil + 2r + s - 1 \geq 2r + 2$. Let $v_1, v_2 \in V(G)$ so that $d_G(v_1) = d_1$ and $d_G(v_2) = d_2$. Then in G, each vertex of $V(G) \setminus \{v_1, v_2\}$ has degree at most one. By $|V(G) \setminus \{v_1, v_2\}| \geq 2r$, we can choose an independent set $S \subseteq V(G) \setminus \{v_1, v_2\}$ of G with $|S| = r$. Then $|N_G(S)| \leq r$. Since $|G| - |S \cup N_G(S)| \geq n - r - 1 = \lfloor \frac{2n}{3} \rfloor + \lceil \frac{n}{3} \rceil - r - 1 \geq s$, it is easy to see that $\overline{G}$ contains $S_{r,s}$ as a subgraph. In other words, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Suppose that there is a realization G of π containing a cycle $C = v_0 v_1 \cdots v_{m-1}$ with $m \geq n$, and amongst all such realizations let m be minimum. If $m = n$ then we are done, so further assume that $m \geq n+1$. Then C is induced by Lemma 2.2(1).

Assume first that $m = n + 1$. Then $r \geq 2$. By Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{r-2}}$, where $C = C_{n+1}$. Then $\overline{K_{r-2}} \cup \{v_1, v_3\}$ is an independent set of size r in G. By $n \geq 4$, there are $m - 5 = n - 4 = \lceil \frac{n}{3} \rceil + \lfloor \frac{2n}{3} \rfloor - 4 \geq \lceil \frac{n}{3} \rceil + r + s - 4 \geq s$ vertices which are not adjacent to each vertex of $\overline{K_{r-2}} \cup \{v_1, v_3\}$ in G, implying that $\overline{G}$ contains $S_{r,s}$, i.e., $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Suppose that $m = n + 2$. Then $r \geq 3$ and $n \geq 6$. By Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{r-3}}$, where $C = C_{n+2}$. Then $\overline{K_{r-3}} \cup \{v_1, v_3, v_5\}$ is an independent set of size r in G. By $r \geq 3$ and $m \geq 8$, there are $m - 7 = n - 5 = \lceil \frac{n}{3} \rceil + \lfloor \frac{2n}{3} \rfloor - 5 \geq \lceil \frac{n}{3} \rceil + r + s - 5 \geq s$ vertices which are not adjacent to each vertex of $\overline{K_{r-3}} \cup \{v_1, v_3, v_5\}$ in G, implying that $\overline{G}$ contains $S_{r,s}$, i.e., $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

If $m \geq n + 3$, then replace the induced C_m in G with a copy of $C_{m-3} \cup C_3$, contradicting the choice of m. Hence, we assume that every realization of π has circumference at most $n - 1$. Let G be a realization of π containing a longest cycle $C = v_1 v_2 \cdots v_m$ with $m \leq n-1$ and suppose that G has the maximum circumference amongst all realizations of π. Let $H = G \setminus V(C)$. Then $|V(H)| = |G| - |V(C)| \geq (n + r - 1) - (n - 1) = r$.

Claim 1 $\Delta(H) \leq 1$.

Proof of Claim 1. To the contrary, we assume $\Delta(H) \geq 2$. By Lemma 2.3 (1) and (2), the unique non-trivial component of H is a star H_1. Moreover, if $x \in V(H)$ is the center of H_1, then $d_H(x) = \Delta(H)$, m is even and x is adjacent to either all odd index vertices or all even index vertices of C. Without loss of generality, $v_i x \in E(G)$ if and only if i is even. Let x' be an neighbor of x in H. If x' is adjacent to v_2, then $v_1 v_2 x' x v_4 \cdots v_m v_1$ is a cycle of length $m + 1$ in G, a contradiction. Hence x' is not adjacent to v_2. We now exchange the edges xx' and $v_1 v_2$ with the nondeges $v_1 x$ and $v_2 x'$, and obtain a realization of π containing a cycle $v_1 x v_2 v_3 \cdots v_m v_1$ of length $m + 1$, a contradiction. □

Claim 2 If $\Delta(H) = 0$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 2. Clearly, $V(H)$ is an independent set of G. By Lemma 2.3(7), $\pi \to (C_n, S_{r,s})$. □

Claim 3 If $\Delta(H) = 1$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 3. Let H contain $2\ell \geq 2$ vertices with degree one and p isolated vertices. If $p \geq r$, by Lemma 2.3(7), then $\pi \to (C_n, S_{r,s})$. Assume $p \leq r - 1$. By Lemma 2.3(3), $N_C(u) = N_C(u')$ for any two distinct vertices $u, u' \in V(H)$ with $d_H(u) = d_H(u') = 1$. Denote $R = N_C(u)$, where $u \in V(H)$ with $d_H(u) = 1$, and let $x_i y_i$, $1 \leq i \leq \ell$ be the (disjoint) edges in H.

Firstly, suppose that $R = \emptyset$. If $|V(H)| \geq 2r + s$, then we can choose an independent set S of H with $|S| = r$. Moreover, by $|V(H)| - |S \cup N_H(S)| \geq 2r + s - 2r = s$, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r + s \leq |V(H)| \leq 2r + s - 1$, then $m \geq (n + r - 1) - (2r + s - 1) \geq \lceil \frac{n}{3} \rceil$. For each $i = 1, \ldots, \min\{\ell, m\}$, we exchange the edges $x_i y_i, v_i v_{i+1}$ for the nonedges $v_i x_i, v_{i+1} y_i$ to obtain at least r isolated vertices in H, implying that $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $|V(H)| \leq r + s - 1$, then $m \geq (n + r - 1) - (r + s - 1) \geq \lceil \frac{n}{3} \rceil + r$. By Lemma 2.3(5), $d_C(x) = |N_C(x)| \leq 1$ for each $x \in V(H)$ with $d_H(x) = 0$. Thus by $\frac{|V(H)|}{2} \leq \frac{\lfloor \frac{2n}{3} \rfloor - 1}{2} \leq \lceil \frac{n}{3} \rceil + r \leq m$, for each $i = 1, \ldots \ell$, we can exchange the edges $x_i y_i, v_i v_{i+1}$ for the nonedges $v_i x_i, v_{i+1} y_i$. Finally, we obtain a realization of π so that $V(H)$ is an independent set and $d_C(x) \leq 1$ for each $x \in V(H)$. By $|V(C)| \leq n - 1$ and $|V(H)| \geq r$, we take $S \subseteq V(H)$ with $|S| = r$. Clearly, there are at least $|G| - 2r = (n + r - 1) - 2r = n - r - 1 \geq \lceil \frac{n}{3} \rceil + s - 1 \geq s$ vertices which are not adjacent to each vertex in S. This implies that $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Now assume that $R \neq \emptyset$. If $|V(H)| \geq 2r+s$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $r+s \leq |V(H)| \leq 2r+s-1$. By Lemma 2.3(6), we may assume $2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| \leq 2r-2$. By $2\ell + p + m = n+r-1$ and $r+s \leq \lfloor \frac{2n}{3} \rfloor$, we have $0 \geq 2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| - (2r-2) \geq (n+r-1) - (2r-2) - (s+1) + p - |R| \geq \lceil \frac{n}{3} \rceil + p - |R|$. By Lemma 2.3(4), $|R| \leq \lfloor \frac{m}{3} \rfloor \leq \lfloor \frac{n-1}{3} \rfloor$. This implies that $\lceil \frac{n}{3} \rceil + p \leq \lfloor \frac{n-1}{3} \rfloor$, a contradiction.

If $|V(H)| \leq r+s-1$, then $m \geq \lceil \frac{n}{3} \rceil + r$. By Lemma 2.3(5), $|N_C(x) \setminus R| \leq 1$ for each $x \in V(H)$ with $d_H(x) = 0$. Since $p \leq r-1$, we have $\ell \geq \lceil \frac{r-p}{2} \rceil$. By $m \geq \lceil \frac{n}{3} \rceil + r$, $|R| \leq \lfloor \frac{m}{3} \rfloor$ and $\lceil \frac{n}{3} \rceil \geq \frac{\lfloor \frac{2n}{3} \rfloor}{2} \geq \frac{r}{2}$, then $m - 2|R| \geq m - 2\lfloor \frac{m}{3} \rfloor \geq \lceil \frac{m}{3} \rceil \geq \lceil \frac{\lceil \frac{n}{3} \rceil + r}{3} \rceil \geq \lceil \frac{\frac{r}{2}+r}{3} \rceil = \lceil \frac{r}{2} \rceil$, and hence we can use $\lceil \frac{r-p}{2} \rceil$ edges of C to breakout $\lceil \frac{r-p}{2} \rceil$ edges of H and obtain a realization of π in which H contains at least $p + 2\lceil \frac{r-p}{2} \rceil \geq r$ isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \leq |R| + r \leq \lfloor \frac{m}{3} \rfloor + r \leq \lceil \frac{n}{3} \rceil + r - 1$. Then $|G| - |S \cup N_C(S)| \geq (n+r-1) - (\lceil \frac{n}{3} \rceil + 2r - 1) \geq s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. □

Proof of Theorem 1.5. By Theorem 1.3, $r_{pot}(C_n, S_{r,s}) \geq \lceil \frac{n}{2} \rceil + r + s - 1$. Let $\pi = (d_1, \ldots, d_k)$ be a graphic sequence with $k = \lceil \frac{n}{2} \rceil + r + s - 1$. We now prove that $\pi \to (C_n, S_{r,s})$. If no realization of π contains a cycle, by Lemma 2.1, then $d_3 \leq 1$. Let G be a realization of π. By $\lceil \frac{n}{2} \rceil - r \geq \lceil \frac{n}{2} \rceil - (\lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor) = \lceil \frac{n}{3} \rceil$, i.e., $\lceil \frac{n}{2} \rceil \geq \lceil \frac{n}{3} \rceil + r$, we have $|G| = \lceil \frac{n}{2} \rceil + r + s - 1 \geq \lceil \frac{n}{3} \rceil + 2r + s - 1 \geq 2r + 2$. Let $v_1, v_2 \in V(G)$ so that $d_G(v_1) = d_1$ and $d_G(v_2) = d_2$. Then in G, each vertex of $V(G) \setminus \{v_1, v_2\}$ has degree at most one. By $|V(G) \setminus \{v_1, v_2\}| \geq 2r$, we can choose an independent set $S \subseteq V(G) \setminus \{v_1, v_2\}$ of G with $|S| = r$. Then $|N_G(S)| \leq r$. Since $|G| - |S \cup N_G(S)| \geq \lceil \frac{n}{2} \rceil + s - r - 1 \geq (\lceil \frac{n}{3} \rceil + r) + s - r - 1 \geq s$, it is easy to see that $\overline{G}$ contains $S_{r,s}$ as a subgraph. In other words, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Suppose that there is a realization G of π containing a cycle $C = v_0 v_1 \cdots v_{m-1}$ with $m \geq n$, and amongst all such realizations let m be minimum. If $m = n$ then we are done, so further assume that $m \geq n+1$. Then C is induced by Lemma 2.2(1).

Firstly, assume $m = n+1$. By Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{r+s-\lfloor \frac{n}{2} \rfloor - 2}}$, where $C = C_{n+1}$. If $r = 1$, then $n \geq 5$. Since there are $|G| - 3 = \lceil \frac{n}{2} \rceil + s - 3 \geq s$ vertices which are not adjacent to v_1 in G, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r \geq 2$, then $n \geq 9$. Since $r + s - \lfloor \frac{n}{2} \rfloor - 2 \geq r - 2$, we have that $\{v_1, v_3\}$ in C along with $r-2$ vertices in $\overline{K_{r+s-\lfloor \frac{n}{2} \rfloor - 2}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r+3) = \lceil \frac{n}{2} \rceil + s - 4 \geq s$ vertices which are not adjacent to each vertex in S, implying that $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

If $m = n+2$, by Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{r+s-\lfloor \frac{n}{2} \rfloor - 3}}$, where $C = C_{n+2}$. If $r = 1$, then $n \geq 5$. Since there are $|G| - 3 = \lceil \frac{n}{2} \rceil + s - 3 \geq s$ vertices which are not adjacent to v_1 in G, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r = 2$, then $n \geq 9$. Since there are $|G| - 5 = \lceil \frac{n}{2} \rceil + s - 4 \geq s$ vertices which are not adjacent to v_1 and v_3 in G, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r \geq 3$, then $n \geq 15$, and hence $\{v_1, v_3, v_5\}$ along with $r-3$ vertices in $\overline{K_{r+s-\lfloor \frac{n}{2} \rfloor - 3}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r+4) = \lceil \frac{n}{2} \rceil + r + s - 1 - (r+4) = \lceil \frac{n}{2} \rceil + s - 5 \geq s$ vertices which are not adjacent to each vertex in S, implying that $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

If $m \geq n+3$, then replace the induced C_m in G with a copy of $C_{m-3} \cup C_3$, contradicting the choice of m. Hence, we assume that every realization of π has circumference at most $n-1$. Let G be a realization of π containing a longest cycle $C = v_1 v_2 \cdots v_m$ with $m \leq n-1$ and suppose that G has the maximum circumference amongst all realizations of π. Let $H = G \setminus V(C)$. Then $|V(H)| = |G| - |V(C)| \geq (\lceil \frac{n}{2} \rceil + r + s - 1) - (n-1) = r + s - \lfloor \frac{n}{2} \rfloor \geq r$.

Claim 1 $\Delta(H) \leq 1$.

Proof of Claim 1. The proof is similar to that of Claim 1 of Theorem 1.4. □

Claim 2 If $\Delta(H) = 0$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 2. Clearly, $V(H)$ is an independent set of G. By Lemma 2.3(7), $\pi \to (C_n, S_{r,s})$. □

Claim 3 If $\Delta(H) = 1$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 3. Let H contain $2\ell \geq 2$ vertices with degree one and p isolated vertices. If $p \geq r$, by Lemma 2.3(7), then $\pi \to (C_n, S_{r,s})$. Assume $p \leq r-1$. By Lemma 2.3(3), $N_C(u) = N_C(u')$ for any two distinct vertices $u, u' \in V(H)$ with $d_H(u) = d_H(u') = 1$. Denote $R = N_C(u)$, where $u \in V(H)$ with $d_H(u) = 1$, and let $x_i y_i$, $1 \leq i \leq \ell$ be the (disjoint) edges in H.

Firstly, suppose that $R = \emptyset$. If $|V(H)| \geq 2r+s$, then we can choose an independent set S of H with $|S| = r$. Moreover, by $|V(H)| - |S \cup N_H(S)| \geq 2r + s - 2r = s$, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r+s \leq |V(H)| \leq 2r+s-1$, then $m \ge (\lceil \frac{n}{2} \rceil + r + s - 1) - (2r + s - 1) \ge \lceil \frac{n}{3} \rceil$. For each $i = 1, \ldots, \min\{\ell, m\}$, we exchange the edges x_iy_i, v_iv_{i+1} for the nonedges $v_ix_i, v_{i+1}y_i$ to obtain at least r isolated vertices in H, implying that $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $|V(H)| \le r + s - 1$, then $m \ge (\lceil \frac{n}{2} \rceil + r + s - 1) - (r + s - 1) \ge \lceil \frac{n}{3} \rceil + r \ge \lceil \frac{r}{2} \rceil$. By Lemma 2.3(5), $d_C(x) = |N_C(x)| \le 1$ for each $x \in V(H)$ with $d_H(x) = 0$. For each $i = 1, \ldots, \min\{\ell, \lceil \frac{r}{2} \rceil\}$, we can exchange the edges x_iy_i, v_iv_{i+1} for the nonedges $v_ix_i, v_{i+1}y_i$, and obtain a realization of π in which H contains at least r isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \le r$. Then $|G| - |S \cup N_C(S)| \ge (\lceil \frac{n}{2} \rceil + r + s - 1) - 2r \ge \lceil \frac{n}{3} \rceil + s - 1 \ge s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Now assume that $R \ne \emptyset$. If $|V(H)| \ge 2r + s$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $r + s \le |V(H)| \le 2r + s - 1$. By Lemma 2.3(6), we may assume $2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| \le 2r - 2$. By $2\ell + p + m = \lceil \frac{n}{2} \rceil + r + s - 1$ and $\lceil \frac{n}{2} \rceil - r \ge \lceil \frac{n}{3} \rceil$, we have $0 \ge 2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| - (2r - 2) \ge (\lceil \frac{n}{2} \rceil + r + s - 1) - (2r - 2) - (s + 1) + p - |R| \ge \lceil \frac{n}{3} \rceil + p - |R|$. By Lemma 2.3(4), $|R| \le \lfloor \frac{m}{3} \rfloor \le \lfloor \frac{n-1}{3} \rfloor$. Thus $\lceil \frac{n}{3} \rceil + p \le \lfloor \frac{n-1}{3} \rfloor$, a contradiction.

If $|V(H)| \le r + s - 1$, then $m \ge \lceil \frac{n}{3} \rceil + r$. By Lemma 2.3(5), $|N_C(x) \setminus R| \le 1$ for each $x \in V(H)$ with $d_H(x) = 0$. Since $p \le r - 1$, we have $\ell \ge \lceil \frac{r-p}{2} \rceil$. By $m \ge \lceil \frac{n}{3} \rceil + r$, $|R| \le \lfloor \frac{m}{3} \rfloor$ and $\lceil \frac{n}{3} \rceil \ge \frac{\lfloor \frac{2n}{3} \rfloor}{2} \ge \frac{r}{2}$, then $m - 2|R| \ge m - 2\lfloor \frac{m}{3} \rfloor \ge \lceil \frac{m}{3} \rceil \ge \lceil \frac{\lceil \frac{n}{3} \rceil + r}{3} \rceil \ge \lceil \frac{\frac{r}{2} + r}{3} \rceil = \lceil \frac{r}{2} \rceil$, and hence we can use $\lceil \frac{r-p}{2} \rceil$ edges of C to breakout $\lceil \frac{r-p}{2} \rceil$ edges of H and obtain a realization of π in which H contains at least $p + 2\lceil \frac{r-p}{2} \rceil \ge r$ isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \le |R| + r \le \lfloor \frac{m}{3} \rfloor + r \le \lceil \frac{n}{3} \rceil + r - 1$. Then $|G| - |S \cup N_C(S)| \ge (\lceil \frac{n}{2} \rceil + r + s - 1) - (\lceil \frac{n}{3} \rceil + 2r - 1) \ge s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. □

Proof of Theorem 1.6. Clearly, $r \ge \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + 1$ and $r + s \ge \lfloor \frac{2n}{3} \rfloor + 1$. By $(2^6) \nrightarrow (C_4, S_{1,4}), (C_4, S_{2,2})$ and $(C_5, S_{2,2})$, and $(2^8) \nrightarrow (C_6, S_{3,2})$, we have $r_{pot}(C_n, S_{r,s}) \ge \lceil \frac{n}{3} \rceil + 2r + s - 1$ for $(n, r, s) = (4, 1, 4), (5, 2, 2), (4, 2, 2), (6, 3, 2)$. Moreover, by Theorem 1.3, we also have $r_{pot}(C_n, S_{r,s}) \ge \lceil \frac{n}{3} \rceil + 2r + s - 1$ when $n \ge 4$, $r \ge 1$, and $s \ge 1$ is odd. Let $\pi = (d_1, \ldots, d_k)$ be a graphic sequence with $k = \lceil \frac{n}{3} \rceil + 2r + s - 1$. We now prove that $\pi \to (C_n, S_{r,s})$. If no realization of π contains a cycle, by Lemma 2.1, then $d_3 \le 1$. Let G be a realization of π. Then $|G| = \lceil \frac{n}{3} \rceil + 2r + s - 1 \ge 2r + 2$. Let $v_1, v_2 \in V(G)$ so that $d_G(v_1) = d_1$ and $d_G(v_2) = d_2$. Then in G, each vertex of $V(G) \setminus \{v_1, v_2\}$ has degree at most one. By $|V(G) \setminus \{v_1, v_2\}| \ge 2r$, we can choose an independent set $S \subseteq V(G) \setminus \{v_1, v_2\}$ of G with $|S| = r$. Then $|N_G(S)| \le r$. Since $|G| - |S \cup N_G(S)| \ge \lceil \frac{n}{3} \rceil + s - 1 \ge s$, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Suppose that there is a realization G of π containing a cycle $C = v_0v_1 \cdots v_{m-1}$ with $m \ge n$, and amongst all such realizations let m be minimum. If $m = n$ then we are done, so further assume that $m \ge n + 1$. Then C is induced by Lemma 2.2(1).

Assume first that $m = n + 1$. By Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 2}}$, where $C = C_{n+1}$. Since $2r + s - \lfloor \frac{2n}{3} \rfloor - 2 \ge r - 1$, v_1 in C along with $r - 1$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 2}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r + 2) = \lceil \frac{n}{3} \rceil + r + s - 3 \ge s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

If $m = n + 2$, by Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}}$, where $C = C_{n+2}$. If $r = 1$, then there are $|G| - 3 = \lceil \frac{n}{3} \rceil + s - 2 \ge s$ vertices which are not adjacent to v_1 in G, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r \ge 2$, then $2r + s - \lfloor \frac{2n}{3} \rfloor - 3 \ge r - 2$, and so $\{v_1, v_3\}$ along with $r - 2$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r + 3) = \lceil \frac{n}{3} \rceil + 2r + s - 1 - (r + 3) = \lceil \frac{n}{3} \rceil + r + s - 4 \ge s$ vertices which are not adjacent to each vertex of S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

If $m \ge n + 3$, then replace the induced C_m in G with a copy of $C_{m-3} \cup C_3$, contradicting the choice of m. Hence, we assume that every realization of π has circumference at most $n - 1$. Let G be a realization of π containing a longest cycle $C = v_1v_2 \cdots v_m$ with $m \le n - 1$ and suppose that G has the maximum circumference amongst all realizations of π. Let $H = G \setminus V(C)$. Then $|V(H)| = |G| - |V(C)| \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1) - (n - 1) \ge r$.

Claim 1 $\Delta(H) \le 1$.

Proof of Claim 1. The proof is similar to that of Claim 1 of Theorem 1.4. □

Claim 2 If $\Delta(H) = 0$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 2. Clearly, $V(H)$ is an independent set of G. By Lemma 2.3(7), $\pi \to (C_n, S_{r,s})$. □

Claim 3 If $\Delta(H) = 1$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 3. Let H contain $2\ell \ge 2$ vertices with degree one and p isolated vertices. If $p \ge r$, by Lemma 2.3(7), then $\pi \to (C_n, S_{r,s})$. Assume $p \le r - 1$. By Lemma 2.3(3), $N_C(u) = N_C(u')$ for any two distinct vertices $u, u' \in V(H)$ with $d_H(u) = d_H(u') = 1$. Denote $R = N_C(u)$, where $u \in V(H)$ with $d_H(u) = 1$, and let x_iy_i, $1 \le i \le \ell$ be the (disjoint) edges in H.

Firstly, suppose that $R = \emptyset$. If $|V(H)| \ge 2r + s$, then we can choose an independent set S of H with $|S| = r$. Moreover, by $|V(H)| - |S \cup N_H(S)| \ge 2r + s - 2r = s$, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r + s \le |V(H)| \le 2r + s - 1$, then $m \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1) - (2r + s - 1) \ge \lceil \frac{n}{3} \rceil$ and $|V(H)| = \lceil \frac{n}{3} \rceil + 2r + s - 1 - m$. For each $i = 1, \ldots, \min\{\ell, m\}$, we exchange the edges x_iy_i, v_iv_{i+1} for the nonedges $v_ix_i, v_{i+1}y_i$, then let H contain $2\ell'$ vertices with degree one and p' isolated vertices. If $\ell \le m$, then $\ell' = 0$, by $|V(H)| \ge r + s$, we have $\pi \to (C_n, S_{r,s})$. Assume $\ell \ge m + 1$. Then $p' \ge 2m$. If $p' \ge r$, by $|V(H)| - r \ge r + s - r = s$, then $\pi \to (C_n, S_{r,s})$. If $p' \le r - 1$, by $\ell' = \frac{|V(H)| - p'}{2} \ge \frac{\lceil \frac{n}{3} \rceil + 2r + s - 1 - m - p'}{2} \ge \frac{(2r - 2p') + p' - m}{2} \ge r - p'$, then p' isolated vertices in H along with $r - p'$ vertices from $r - p'$ edges in H forms an independent set S with $|S| = r$ in G. It follows from $|N_H(S)| = r - p' \le r - 2m$ and $|V(H)| - |N_H(S) \cup S| \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1 - m) - (2r - 2m) \ge s$ that $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $|V(H)| \le r + s - 1$, then $m \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1) - (r + s - 1) \ge \lceil \frac{r}{2} \rceil$. By Lemma 2.3(5), $d_C(x) = |N_C(x)| \le 1$ for each $x \in V(H)$ with $d_H(x) = 0$. For each $i = 1, \ldots, \min\{\ell, \lceil \frac{r}{2} \rceil\}$, we can exchange the edges x_iy_i, v_iv_{i+1} for the nonedges $v_ix_i, v_{i+1}y_i$, and obtain a realization of π in which H contains at least r isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \le r$. Then $|G| - |S \cup N_C(S)| \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1) - 2r \ge s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Now assume that $R \neq \emptyset$. If $|V(H)| \ge 2r + s$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $r + s \le |V(H)| \le 2r + s - 1$. By Lemma 2.3(6), we may assume $2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| \le 2r - 2$. By $2\ell + p + m = \lceil \frac{n}{3} \rceil + 2r + s - 1$, we have $0 \ge 2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| - (2r - 2) \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1) - (2r - 2) - (s + 1) + p - |R| \ge \lceil \frac{n}{3} \rceil + p - |R|$. By Lemma 2.3(4), $|R| \le \lfloor \frac{m}{3} \rfloor \le \lfloor \frac{n-1}{3} \rfloor$. Thus $\lceil \frac{n}{3} \rceil + p \le \lfloor \frac{n-1}{3} \rfloor$, a contradiction.

If $|V(H)| \le r + s - 1$, then $m \ge \lceil \frac{n}{3} \rceil + r$ and $r \le \lfloor \frac{2n}{3} \rfloor - 1$. By Lemma 2.3(5), $|N_C(x) \setminus R| \le 1$ for each $x \in V(H)$ with $d_H(x) = 0$. Since $p \le r - 1$, we have $\ell \ge \lceil \frac{r-p}{2} \rceil$. By $m \ge \lceil \frac{n}{3} \rceil + r$, $|R| \le \lfloor \frac{m}{3} \rfloor$ and $\lceil \frac{n}{3} \rceil \ge \frac{\lfloor \frac{2n}{3} \rfloor}{2} \ge \frac{r}{2}$, then $m - 2|R| \ge m - 2\lfloor \frac{m}{3} \rfloor \ge \lceil \frac{m}{3} \rceil \ge \lceil \frac{\lceil \frac{n}{3} \rceil + r}{3} \rceil \ge \lceil \frac{\frac{r}{2} + r}{3} \rceil = \lceil \frac{r}{2} \rceil$, and hence we can use $\lceil \frac{r-p}{2} \rceil$ edges of C to breakout $\lceil \frac{r-p}{2} \rceil$ edges of H and obtain a realization of π in which H contains at least $p + 2\lceil \frac{r-p}{2} \rceil \ge r$ isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \le |R| + r \le \lfloor \frac{m}{3} \rfloor + r \le \lceil \frac{n}{3} \rceil + r - 1$. Then $|G| - |S \cup N_C(S)| \ge (\lceil \frac{n}{3} \rceil + 2r + s - 1) - (\lceil \frac{n}{3} \rceil + 2r - 1) \ge s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. □

Proof of Theorem 1.7. Clearly, $r \ge \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + 1$ and $r + s \ge \lfloor \frac{2n}{3} \rfloor + 1$. From Theorem 1.3 we have $r_{pot}(C_n, S_{r,s}) \ge \lceil \frac{n}{3} \rceil + 2r + s - 2$. Let $\pi = (d_1, \ldots, d_k)$ be a graphic sequence with $k = \lceil \frac{n}{3} \rceil + 2r + s - 2$. We now prove that $\pi \to (C_n, S_{r,s})$. If no realization of π contains a cycle, by Lemma 2.1, then $d_3 \le 1$. Let G be a realization of π. Then $|G| = \lceil \frac{n}{3} \rceil + 2r + s - 2 \ge 2r + 2$. Let $v_1, v_2 \in V(G)$ so that $d_G(v_1) = d_1$ and $d_G(v_2) = d_2$. Then in G, each vertex of $V(G) \setminus \{v_1, v_2\}$ has degree at most one. By $|V(G) \setminus \{v_1, v_2\}| \ge 2r$, we can choose an independent set $S \subseteq V(G) \setminus \{v_1, v_2\}$ of G with $|S| = r$. Then $|N_G(S)| \le r$. Since $|G| - |S \cup N_G(S)| \ge \lceil \frac{n}{3} \rceil + s - 2 \ge s$, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Suppose that there is a realization G of π containing a cycle $C = v_0v_1 \cdots v_{m-1}$ with $m \ge n$, and amongst all such realizations let m be minimum. If $m = n$ then we are done, so further assume that $m \ge n + 1$. Then C is induced by Lemma 2.2(1).

Assume first that $m = n + 1$. By Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}}$, where $C = C_{n+1}$. If $r = 1$, then $n = 4$, $|V(C)| = 5$ and $|G| = s + 2$ is even, implying that $|V(\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}})| \ge 1$, and $\overline{\pi}$ is potentially $S_{1,s}$-graphic. Assume $r \ge 2$ and $4 \le n \le 6$. If $r + s \ge \lfloor \frac{2n}{3} \rfloor + 2$, then $2r + s - \lfloor \frac{2n}{3} \rfloor - 3 \ge r - 1$, and hence v_1 in C along with $r - 1$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r + 2) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r + 2) = \lceil \frac{n}{3} \rceil + r + s - 4 \ge s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r \ge 3$, by $r + s \ge \lfloor \frac{2n}{3} \rfloor + 1$, then $2r + s - \lfloor \frac{2n}{3} \rfloor - 3 \ge r - 2$, and hence $\{v_1, v_3\}$ in C along with $r - 2$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r + 3) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r + 3) = \lceil \frac{n}{3} \rceil + r + s - 5 \ge s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. For the case of $r = 2$ and $r + s = \lfloor \frac{2n}{3} \rfloor + 1$, if $n = 4$, then $r + s = 3$ and $s = 1$, a contradiction; if $n = 5$, then $(n, r, s) = (5, 2, 2)$, a contradiction; if $n = 6$, then $r + s = 5$ and $s = 3$, a contradiction. Assume $r \ge 2$ and $n \ge 7$. By $2r + s - \lfloor \frac{2n}{3} \rfloor - 3 \ge r - 2$, $\{v_1, v_3\}$ in C along with $r - 2$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor - 3}}$ forms an independent set S with $|S| = r$ in G and there are

$|G|-(r+3) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r+3) = \lceil \frac{n}{3} \rceil + r + s - 5 \geq s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Suppose that $m = n+2$. By Lemma 2.2(2), we have $d_G(x) = 0$ for each vertex in $V(G) \setminus V(C)$, i.e., $G = C \cup \overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor -4}}$, where $C = C_{n+2}$. If $r = 1$, then $n = 4$, $|V(C)| = 6$ and $|G| = s+2$. In this case, if $s = 4$, then $(n, r, s) = (4, 1, 4)$, a contradiction; if $s \geq 6$, then $|V(\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor -4}})| \geq 1$, and $\overline{\pi}$ is potentially $S_{1,s}$-graphic. Assume $r \geq 2$ and $4 \leq n \leq 6$. If $r + s \geq \lfloor \frac{2n}{3} \rfloor + 3$, then $2r + s - \lfloor \frac{2n}{3} \rfloor - 4 \geq r - 1$, and hence v_1 in C along with $r-1$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor -4}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r+2) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r+2) = \lceil \frac{n}{3} \rceil + r + s - 4 \geq s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r + s = \lfloor \frac{2n}{3} \rfloor + 2$ and $r \geq 3$, then $2r + s - \lfloor \frac{2n}{3} \rfloor - 4 = r - 2$, and hence $\{v_1, v_3\}$ in C along with $r-2$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor -4}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r+3) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r+3) = \lceil \frac{n}{3} \rceil + r + s - 5 \geq s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. For the case of $r + s = \lfloor \frac{2n}{3} \rfloor + 2$ and $r = 2$, if $n = 4$, then $(n, r, s) = (4, 2, 2)$, a contradiction; if $n = 5$, then $r + s = 5$ and $s = 3$, a contradiction; if $n = 6$, then $r + s = 6$, $s = 4$ and $|G| = m = 8$, implying that $G = C_8$ and $\pi = (2^8)$. It is easy to see that $\pi \to (C_6, S_{2,4})$. For the case of $r + s = \lfloor \frac{2n}{3} \rfloor + 1$, by $|G| = \lceil \frac{n}{3} \rceil + 2r + s - 2 \geq m = n+2$, we have $r \geq 3$ and $r + s \geq 5$, implying that $(n, r, s) = (6, 3, 2)$, a contradiction. Assume $r \geq 2$ and $n \geq 7$. If $r = 2$, then there are $|G| - 5 = (\lceil \frac{n}{3} \rceil + s + 2) - 5 = \lceil \frac{n}{3} \rceil + s - 3 \geq s$ vertices which are not adjacent to v_1 and v_3, $\overline{\pi}$ is potentially $S_{2,s}$-graphic. If $r \geq 3$, then $2r + s - \lfloor \frac{2n}{3} \rfloor - 4 \geq r - 3$, and hence $\{v_1, v_3, v_5\}$ in C along with $r-3$ vertices in $\overline{K_{2r+s-\lfloor \frac{2n}{3} \rfloor -4}}$ forms an independent set S with $|S| = r$ in G and there are $|G| - (r+4) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r+4) = \lceil \frac{n}{3} \rceil + r + s - 6 \geq s$ vertices which are not adjacent to each vertex in S, $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

If $m \geq n+3$, then replace the induced C_m in G with a copy of $C_{m-3} \cup C_3$, contradicting the choice of m. Hence, we assume that every realization of π has circumference at most $n-1$. Let G be a realization of π containing a longest cycle $C = v_1 v_2 \cdots v_m$ with $m \leq n-1$ and suppose that G has the maximum circumference amongst all realizations of π. Let $H = G \setminus V(C)$. Then $|V(H)| = |G| - |V(C)| \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (n-1) \geq r$.

Claim 1 $\Delta(H) \leq 1$.

Proof of Claim 1. The proof is similar to that of Claim 1 of Theorem 1.4. □

Claim 2 If $\Delta(H) = 0$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 2. Clearly, $V(H)$ is an independent set of G. By Lemma 2.3(7), $\pi \to (C_n, S_{r,s})$. □

Claim 3 If $\Delta(H) = 1$, then $\pi \to (C_n, S_{r,s})$.

Proof of Claim 3. Let H contain $2\ell \geq 2$ vertices with degree one and p isolated vertices. If $p \geq r$, by Lemma 2.3(7), then $\pi \to (C_n, S_{r,s})$. Assume $p \leq r-1$. By Lemma 2.3(3), $N_C(u) = N_C(u')$ for any two distinct vertices $u, u' \in V(H)$ with $d_H(u) = d_H(u') = 1$. Denote $R = N_C(u)$, where $u \in V(H)$ with $d_H(u) = 1$, and let $x_i y_i$, $1 \leq i \leq \ell$ be the (disjoint) edges in H.

Firstly, suppose that $R = \emptyset$. If $|V(H)| \geq 2r + s$, then we can choose an independent set S of H with $|S| = r$. Moreover, by $|V(H)| - |S \cup N_H(S)| \geq 2r + s - 2r = s$, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $r + s \leq |V(H)| \leq 2r + s - 1$, then $m \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (2r + s - 1) \geq \lceil \frac{n}{3} \rceil - 1$ and $|V(H)| = \lceil \frac{n}{3} \rceil + 2r + s - 2 - m$. For each $i = 1, \ldots, \min\{\ell, m\}$, we exchange the edges $x_i y_i, v_i v_{i+1}$ for the nonedges $v_i x_i, v_{i+1} y_i$, then let H contain $2\ell'$ vertices with degree one and p' isolated vertices. If $\ell \leq m$, then $\ell' = 0$, by $|V(H)| \geq r + s$, we have $\pi \to (C_n, S_{r,s})$. Assume $\ell \geq m+1$. Then $p' \geq 2m$. If $p' \geq r$, by $|V(H)| - r \geq r + s - r = s$, then $\pi \to (C_n, S_{r,s})$. If $p' \leq r-1$, by $\ell' = \frac{|V(H)| - p'}{2} \geq \frac{\lceil \frac{n}{3} \rceil + 2r + s - 2 - m - p'}{2} \geq \frac{(2r - 2p') + p' - m}{2} \geq r - p'$, then p' isolated vertices in H along with $r - p'$ vertices from $r - p'$ edges in H forms an independent set S with $|S| = r$ in G. It follows from $|N_H(S)| = r - p' \leq r - 2m$ and $|V(H)| - |N_H(S) \cup S| \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2 - m) - (2r - 2m) \geq s$ that $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $|V(H)| \leq r + s - 1$, then $m \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (r + s - 1) = \lceil \frac{n}{3} \rceil + r - 1 \geq \lceil \frac{r}{2} \rceil$ and $r \leq \lfloor \frac{2n}{3} \rfloor$. By Lemma 2.3(5), $d_C(x) = |N_C(x)| \leq 1$ for each $x \in V(H)$ with $d_H(x) = 0$. For each $i = 1, \ldots, \min\{\ell, \lceil \frac{r}{2} \rceil\}$, we can exchange the edges $x_i y_i, v_i v_{i+1}$ for the nonedges $v_i x_i, v_{i+1} y_i$, and obtain a realization of π in which H contains at least r isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \leq r$. Then $|G| - |S \cup N_C(S)| \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - 2r \geq s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

Now assume that $R \neq \emptyset$. If $|V(H)| \geq 2r + s$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $r + s \leq |V(H)| \leq 2r + s - 1$. By Lemma 2.3(6), we may assume $2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| \leq 2r - 2$. By $2\ell + p + m = \lceil \frac{n}{3} \rceil + 2r + s - 2$ and

s is even, we have $0 \geq 2\ell - 2\lceil \frac{s}{2} \rceil + 2p + m - |R| - (2r-2) = (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (2r-2) - s + p - |R| \geq \lceil \frac{n}{3} \rceil + p - |R|$. By Lemma 2.3(4), $|R| \leq \lfloor \frac{m}{3} \rfloor \leq \lfloor \frac{n-1}{3} \rfloor$. Thus $\lceil \frac{n}{3} \rceil + p \leq \lfloor \frac{n-1}{3} \rfloor$, a contradiction.

If $|V(H)| \leq r + s - 1$, then $m = |G| - |V(H)| \geq \lceil \frac{n}{3} \rceil + r - 1$, and hence $r \leq \lfloor \frac{2n}{3} \rfloor$ (by $m \leq n-1$). By $|R| \leq \lfloor \frac{m}{3} \rfloor$ and Lemma 2.3(5), we have $m - 2|R| \geq m - 2\lfloor \frac{m}{3} \rfloor \geq \lceil \frac{m}{3} \rceil \geq \lceil \frac{\lceil \frac{n}{3} \rceil + r - 1}{3} \rceil \geq \lceil \frac{r}{2} \rceil$ and $|N_C(x) \setminus R| \leq 1$ for each $x \in V(H)$ with $d_H(x) = 0$. We now consider the following two cases according to the value of $|R| \leq \lfloor \frac{m}{3} \rfloor \leq \lfloor \frac{n-1}{3} \rfloor = \lceil \frac{n}{3} \rceil - 1$. If $|R| \leq \lceil \frac{n}{3} \rceil - 2$, by $\ell \geq \lceil \frac{r-p}{2} \rceil$, then we can use $\lceil \frac{r-p}{2} \rceil$ edges of C to breakout $\lceil \frac{r-p}{2} \rceil$ edges of H to obtain a realization of π in which H contains at least $p + 2\lceil \frac{r-p}{2} \rceil \geq r$ isolated vertices. Let S be the set of r isolated vertices in H. Clearly, $|N_C(S)| \leq |R| + r \leq \lfloor \frac{m}{3} \rfloor + r \leq \lceil \frac{n}{3} \rceil + r - 2$. This implies that $|G| - |S \cup N_C(S)| \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (\lceil \frac{n}{3} \rceil + 2r - 2) = s$, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $|R| = \lceil \frac{n}{3} \rceil - 1$, then $|R| = \lfloor \frac{m}{3} \rfloor = \lceil \frac{n}{3} \rceil - 1$. In this case, if $r = 1$, then $n = 4$ and $m = 3$. Since $|V(H)| = |G| - m = s - 1 \geq 1$ is odd, we have $p \geq 1 = r$, a contradiction. Assume $r \geq 2$. If $m \not\equiv 0 \pmod 3$, by $|R| = \lfloor \frac{m}{3} \rfloor$ and Lemma 2.3(4), C has three consecutive vertices, say v_1, v_2, v_3, so that $v_1, v_2, v_3 \notin R$; moreover, $v_4 \in R$ or $v_m \in R$. Without loss of generality, we let $v_4 \in R$. If $v_1v_3 \in E(G)$, then we exchange the edges x_1y_1, v_1v_3 for the nonedges v_1x_1, v_3y_1 to obtain a realization of π containing a cycle $v_1v_2v_3y_1v_4 \cdots v_mv_1$ of length $m+1$, a contradiction. If $v_1v_3 \notin E(G)$, then we first exchange the edges x_1y_1, v_1v_2, v_2v_3 for the nonedges v_2x_1, v_2y_1, v_1v_3 to obtain a realization of π containing a cycle $v_1v_3v_4 \cdots v_mv_1$ of length $m-1$, then by $(m-1) - 2|R| \geq \lceil \frac{r}{2} \rceil - 1$, we can use $\lceil \frac{r-p}{2} \rceil - 1$ edges of $v_1v_3v_4 \cdots v_mv_1$ to breakout $\lceil \frac{r-p}{2} \rceil - 1$ edges of H to obtain a realization of π in which H has $2 + p + 2(\lceil \frac{r-p}{2} \rceil - 1) \geq r$ isolated vertices. Let S be the set of r isolated vertices in H and $x_1, y_1 \in S$. Clearly, $|N_C(S)| \leq |R| + r - 1 = \lfloor \frac{m}{3} \rfloor + r - 1 = \lceil \frac{n}{3} \rceil + r - 2$, and so $|G| - |S \cup N_C(S)| \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (\lceil \frac{n}{3} \rceil + 2r - 2) = s$ and $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $m \equiv 0 \pmod 3$. If $p \geq 1$, we let $x \in V(H)$ with $d_H(x) = 0$. In this case, if $N_C(x) \setminus R \neq \emptyset$, we let $v \in N_C(x) \setminus R$, by $m \equiv 0 \pmod 3$, $|R| = \lfloor \frac{m}{3} \rfloor$ and Lemma 2.3(4), then $v = v_{i+1}$ or v_{i-1} for some $v_i \in R$. Without loss of generality, we let $v = v_{i+1}$, then exchange the edges $x_1y_1, v_{i+1}x$ for the nonedges $v_{i+1}x_1, xy_1$ to obtain a realization of π containing a cycle $v_1v_2 \cdots v_ix_1v_{i+1} \cdots v_mv_1$ of length $m+1$, a contradiction. Hence $N_C(x) \setminus R = \emptyset$. By $m - 2|R| \geq \lceil \frac{r}{2} \rceil$, we can use $\lceil \frac{r-p}{2} \rceil$ edges of C to breakout $\lceil \frac{r-p}{2} \rceil$ edges of H to obtain a realization of π in which H contains at least $p + 2\lceil \frac{r-p}{2} \rceil \geq r$ isolated vertices. Let S be the set of r isolated vertices in H and $x \in S$. Clearly, $|N_C(S)| \leq |R| + r - 1 = \lfloor \frac{m}{3} \rfloor + r - 1 = \lceil \frac{n}{3} \rceil + r - 2$, and so $|G| - |S \cup N_C(S)| \geq (\lceil \frac{n}{3} \rceil + 2r + s - 2) - (\lceil \frac{n}{3} \rceil + 2r - 2) = s$ and $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $p = 0$, by $m \equiv 0 \pmod 3$ and $|R| = \lfloor \frac{m}{3} \rfloor = \frac{m}{3}$, we have that $|G| - |R| = |V(C)| - |R| + |V(H)| = m - \frac{m}{3} + 2\ell = \frac{2m}{3} + 2\ell$ is even. On the other hand, by $|R| = \lceil \frac{n}{3} \rceil - 1$, we have that $|G| - |R| = \lceil \frac{n}{3} \rceil + 2r + s - 2 - (\lceil \frac{n}{3} \rceil - 1) = 2r + s - 1$ is odd, a contradiction. □

Proof of Theorem 1.8. (1) By $(2^4) \nrightarrow (C_3, S_{1,2})$, we have $r_{pot}(C_3, S_{1,2}) \geq 5$. Let $\pi = (d_1, \ldots, d_5)$ be a graphic sequence. If π is not potentially C_3-graphic, by Lemma 2.1, then $d_3 \leq 1$ or $\pi = (2^5)$ or $\pi = (2^4, 0)$, implying that $\overline{d_1} = 4 - d_5 \geq 2$, and so $\overline{\pi}$ is potentially $S_{1,2}$-graphic. Thus $r_{pot}(C_3, S_{1,2}) = 5$. By $(2^5) \nrightarrow (C_3, S_{1,3})$, we have $r_{pot}(C_3, S_{1,3}) \geq 6$. Let $\pi = (d_1, \ldots, d_6)$ be a graphic sequence. If π is not potentially C_3-graphic, by Lemma 2.1, then $d_3 \leq 1$ or $\pi = (2^5, 0)$ or $\pi = (2^4, 0^2)$, thus $\overline{d_1} = 5 - d_6 \geq 3$, and so $\overline{\pi}$ is potentially $S_{1,3}$-graphic. Hence $r_{pot}(C_3, S_{1,3}) = 6$. For $s \geq 4$, by Theorem 1.3, $r_{pot}(C_3, S_{1,s}) \geq s + 2$. Let $\pi = (d_1, \ldots, d_{s+2})$ be a graphic sequence with $s \geq 4$. If $\overline{\pi}$ is not potentially $S_{1,s}$-graphic, then $\overline{d_1} \leq s - 1$, and hence $d_{s+2} = s + 1 - \overline{d_1} \geq 2$, by $s + 2 \geq 6$ and Lemma 2.1, π is potentially C_3-graphic. Thus $r_{pot}(C_3, S_{1,s}) = s + 2$ for $s \geq 4$.

(2) Let $r \geq 2$, $s \geq 1$ be odd and $(r, s) \neq (2, 1)$. By Theorem 1.3, $r_{pot}(C_3, S_{r,s}) \geq 2r + s$. Let $\pi = (d_1, \ldots, d_{2r+s})$ be a graphic sequence. If $s = 1$, by Theorem 1.2(1), then $r_{pot}(C_3, S_{r,1}) = 2r + 1$. Assume $s \geq 3$. If π is not potentially C_3-graphic, by Lemma 2.1, then $d_3 \leq 1$ or $\pi = (2^4, 0^{2r+s-4})$ or $\pi = (2^5, 0^{2r+s-5})$. If $d_3 \leq 1$, then $\overline{d_{r+s}} \geq \overline{d_{2r+s-2}} = 2r + s - 1 - d_3 \geq 2r + s - 2$, by Lemma 2.4, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $\pi = (2^4, 0^{2r+s-4})$ or $\pi = (2^5, 0^{2r+s-5})$, by $2r + s - 4 \geq 2r + s - 5 \geq r$, then every realization of π contains at least r isolated vertices, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic.

(3) Let $r \geq 2$, $s \geq 2$ be even and $(r, s) \neq (2, 2)$. By Theorem 1.3, $r_{pot}(C_3, S_{r,s}) \geq 2r + s - 1$. Let $\pi = (d_1, \ldots, d_{2r+s-1})$ be a graphic sequence. If π is not potentially C_3-graphic, by Lemma 2.1, then $d_3 \leq 1$ or $\pi = (2^4, 0^{2r+s-5})$ or $\pi = (2^5, 0^{2r+s-6})$. If $\pi = (2^4, 0^{2r+s-5})$, by $r + s \geq 5$, then every realization of π contains at least $2r + s - 5 \geq r$ isolated vertices, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Assume $\pi = (2^5, 0^{2r+s-6})$. If $r + s \geq 6$, then every realization of π contains at least $2r + s - 6 \geq r$ isolated vertices, and so $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If

$r+s=5$, then $r=3$, $s=2$ and $\overline{\pi}=(6^2,4^5)$, by Lemma 2.4, $\overline{\pi}$ is potentially $S_{3,2}$-graphic. Assume $d_3\le 1$. Let G be a realization of π with vertex set $V(G)=\{v_1,v_2,\ldots,v_{2r+s-1}\}$ so that $d_G(v_i)=d_i$ for each i. If $d_1\ge 3$, let $v,v'\in N_G(v_1)\setminus\{v_2\}$, by $\Delta(G[V(G)\setminus\{v_1,v_2,v,v'\}])\le 1$ and $|G[V(G)\setminus\{v_1,v_2,v,v'\}]|=|G|-4=2r+s-5\ge 2(r-2)$, then we can find an independent set of $G[V(G)\setminus\{v_1,v_2,v,v'\}]$ with order at least $r-2$, thus $\{v,v'\}$ along with $r-2$ vertices in $V(G)\setminus\{v_1,v_2,v,v'\}$ forms an independent set S of G with $|S|=r$, and by $|S\cup N_G(S)|\le 2r-1$, there are at least $(2r+s-1)-(2r-1)=s$ vertices which are not adjacent to each vertex in S, implying that $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $d_1=d_2=2$, then $d_{2r+s-1}=0$ as $d_3\le 1$ and $\sum_{i=1}^{2r+s-1} d_i$ is even. Clearly, $\overline{d_1}=2r+s-2$, $\overline{d_{r+s-1}}\ge\overline{d_{2r+s-3}}\ge 2r+s-3$ and $\overline{d_{r+s}}\ge\overline{d_{2r+s-2}}=2r+s-4\ge r$. By Lemma 2.4, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $d_1=2$ and $d_2=1$, let $v,v'\in N_G(v_1)$, by $|G[V(G)\setminus\{v_1,v,v'\}]|=|G|-3=2r+s-4\ge 2(r-2)$, then $\{v,v'\}$ along with $r-2$ vertices in $V(G)\setminus\{v_1,v,v'\}$ forms an independent set S of G with $|S|=r$, and there are at least $(2r+s-1)-(2r-1)=s$ vertices which are not adjacent to each vertex in S, thus $\overline{\pi}$ is potentially $S_{r,s}$-graphic. If $d_1\le 1$, then $d_{2r+s-1}=0$ as $\sum_{i=1}^{2r+s-1} d_i$ is even and $2r+s-1$ is odd. Clearly, $\overline{d_1}=2r+s-2$ and $\overline{d_{2r+s-1}}\ge 2r+s-3$. By Lemma 2.4, $\overline{\pi}$ is potentially $S_{r,s}$-graphic. Thus $r_{pot}(C_3,S_{r,s})=2r+s-1$.

(4) By Theorem 1.2, $r_{pot}(C_3,S_{2,1})=6$. By $(2^5)\nrightarrow(C_3,S_{2,2})$, we have $r_{pot}(C_3,S_{2,2})\ge 6$. Let $\pi=(d_1,\ldots,d_6)$ be a graphic sequence. If π is not potentially C_3-graphic, by Lemma 2.1, then $d_3\le 1$ or $\pi=(2^5,0)$ or $\pi=(2^4,0^2)$, implying that $\overline{d_1}=5-d_6\ge 4$ and $\overline{d_4}=5-d_3\ge 3$. By Lemma 2.4, $\overline{\pi}$ is potentially $S_{2,2}$-graphic. Thus $r_{pot}(C_3,S_{2,2})=6$. □

Proof of Theorem 1.9. (1) If $s\le\lceil\frac{n}{2}\rceil-1$ and $r+s\le\lceil\frac{2n}{3}\rceil+\frac{-1+(-1)^s}{2}$, by Theorem 1.3, then $r_{pot}(P_n,S_{r,s})\ge n+r-1$. Moreover, $s\le\lceil\frac{n}{2}\rceil-1\le\lfloor\frac{n}{2}\rfloor$ and $r+s\le\lceil\frac{2n}{3}\rceil+\frac{-1+(-1)^s}{2}\le\lfloor\frac{2n}{3}\rfloor+\frac{-1+(-1)^s}{2}+1$. If $r+s\le\lfloor\frac{2n}{3}\rfloor$, by Theorem 1.4, then $r_{pot}(C_n,S_{r,s})=n+r-1$. If $r+s=\lfloor\frac{2n}{3}\rfloor+1$, then s is even and $\lceil\frac{2n}{3}\rceil=\lfloor\frac{2n}{3}\rfloor+1$, by Theorem 1.7, we also have $r_{pot}(C_n,S_{r,s})=\lceil\frac{n}{3}\rceil+2r+s-2=\lceil\frac{n}{3}\rceil+\lfloor\frac{2n}{3}\rfloor+r-1=n+r-1$. It follows from $r_{pot}(P_n,S_{r,s})\le r_{pot}(C_n,S_{r,s})$ that $r_{pot}(P_n,S_{r,s})=n+r-1$.

(2) If $s\ge\lceil\frac{n}{2}\rceil$ and $r\le\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil+\frac{-1+(-1)^s}{2}$, by $\alpha(P_n)=\lceil\frac{n}{2}\rceil$ and Theorem 1.3, then $r_{pot}(P_n,S_{r,s})\ge\lfloor\frac{n}{2}\rfloor+r+s-1$. Moreover, $s\ge\lceil\frac{n}{2}\rceil\ge\lfloor\frac{n}{2}\rfloor$ and $r\le\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil+\frac{-1+(-1)^s}{2}\le\lfloor\frac{2n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor+\frac{-1+(-1)^s}{2}+1$. If $r\le\lfloor\frac{2n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor$, by Theorem 1.5, then $r_{pot}(C_n,S_{r,s})=\lceil\frac{n}{2}\rceil+r+s-1$. If $r=\lfloor\frac{2n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor+1$, then s is even and $\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil=\lfloor\frac{2n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor+1$, by Theorem 1.7, we also have $r_{pot}(C_n,S_{r,s})=\lceil\frac{n}{3}\rceil+2r+s-2=\lceil\frac{n}{3}\rceil+\lfloor\frac{2n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor+r+s-1=\lceil\frac{n}{2}\rceil+r+s-1$. We now consider the following two cases in terms of the parity of n. If n is even, by $\lceil\frac{n}{2}\rceil=\lfloor\frac{n}{2}\rfloor$, then $r_{pot}(P_n,S_{r,s})\le r_{pot}(C_n,S_{r,s})=\lfloor\frac{n}{2}\rfloor+r+s-1$. Thus $r_{pot}(P_n,S_{r,s})=\lfloor\frac{n}{2}\rfloor+r+s-1$. Assume that n is odd. If $\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil+\frac{-1+(-1)^s}{2}\le\lfloor\frac{2(n-1)}{3}\rfloor-\lfloor\frac{n-1}{2}\rfloor$, by $s\ge\lceil\frac{n}{2}\rceil\ge\lfloor\frac{n-1}{2}\rfloor$ and Theorem 1.5, we have $r_{pot}(C_{n-1},S_{r,s})=\lceil\frac{n-1}{2}\rceil+r+s-1$. If $\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil+\frac{-1+(-1)^s}{2}\ge\lfloor\frac{2(n-1)}{3}\rfloor-\lfloor\frac{n-1}{2}\rfloor+1$, then s is even and $r=\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil=\lfloor\frac{2(n-1)}{3}\rfloor-\lfloor\frac{n-1}{2}\rfloor+1$, by $s\ge\lceil\frac{n}{2}\rceil\ge\lfloor\frac{n}{2}\rfloor$ and Theorem 1.7, we also have $r_{pot}(C_{n-1},S_{r,s})=\lceil\frac{n-1}{3}\rceil+2r+s-2=\lceil\frac{n-1}{3}\rceil+\lfloor\frac{2(n-1)}{3}\rfloor-\lfloor\frac{n-1}{2}\rfloor+r+s-1=\lceil\frac{n-1}{2}\rceil+r+s-1$. Let $\pi=(d_1,\ldots,d_k)$ be a graphic sequence with $k=\lfloor\frac{n}{2}\rfloor+r+s-1$. It follows from $\lfloor\frac{n}{2}\rfloor=\lceil\frac{n-1}{2}\rceil$ that $k=r_{pot}(C_{n-1},S_{r,s})$. Assume that $\overline{\pi}$ is not potentially $S_{r,s}$-graphic. Then π has a realization G containing C_{n-1}. If there exists one edge between $V(G)\setminus V(C_{n-1})$ and $V(C_{n-1})$, then π is potentially P_n-graphic. Assume that there is no edge between $V(G)\setminus V(C_{n-1})$ and $V(C_{n-1})$. If there exists one edge $xy\in E(G\setminus V(C_{n-1}))$, let v,v' be two consecutive vertices on C_{n-1}, then exchange the edges vv',xy with the non-edges $vx,v'y$, we obtain a realization of π which contains P_n. If there is no edge in $G\setminus V(C_{n-1})$, by $\lfloor\frac{n}{2}\rfloor+r+s-1-(n-1)\ge r$, then $\overline{G}$ contains $S_{r,s}$, that is, $\overline{\pi}$ is potentially $S_{r,s}$-graphic, a contradiction. Hence $r_{pot}(P_n,S_{r,s})=\lfloor\frac{n}{2}\rfloor+r+s-1$.

(3) If $s\le\lceil\frac{n}{2}\rceil-1$ and $r+s\ge\lceil\frac{2n}{3}\rceil+\frac{-1+(-1)^s}{2}+1$ or if $s\ge\lceil\frac{n}{2}\rceil$ and $r\ge\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil+\frac{-1+(-1)^s}{2}+1$, by $\alpha^{(1)}(P_n)=\lceil\frac{2n}{3}\rceil$ and Theorem 1.3, then $r_{pot}(P_n,S_{r,s})\ge\lfloor\frac{n}{3}\rfloor+2r+s+\frac{-3+(-1)^{s-1}}{2}$. Moreover, $r+s\ge\lceil\frac{2n}{3}\rceil+\frac{-1+(-1)^s}{2}+1$ and $r\ge\lceil\frac{2n}{3}\rceil-\lceil\frac{n}{2}\rceil+\frac{-1+(-1)^s}{2}+1$. Assume $(n,r,s)\ne(6,3,2)$.

If $s\le\lceil\frac{n}{2}\rceil-1$ $(\le\lfloor\frac{n}{2}\rfloor)$ and $r+s\ge\lfloor\frac{2n}{3}\rfloor+1$ or if $s\ge\lceil\frac{n}{2}\rceil$ $(\ge\lfloor\frac{n}{2}\rfloor)$ and $r\ge\lfloor\frac{2n}{3}\rfloor-\lfloor\frac{n}{2}\rfloor+1$, by Theorems 1.6 and 1.7, then $r_{pot}(C_n,S_{r,s})=\lceil\frac{n}{3}\rceil+2r+s+\frac{-3+(-1)^{s-1}}{2}$, implying that $r_{pot}(P_n,S_{r,s})\le r_{pot}(C_n,S_{r,s})=\lceil\frac{n}{3}\rceil+2r+s+\frac{-3+(-1)^{s-1}}{2}$. If $n\equiv 0 \pmod 3$, by $\lceil\frac{n}{3}\rceil=\lfloor\frac{n}{3}\rfloor$, then $r_{pot}(P_n,S_{r,s})=\lfloor\frac{n}{3}\rfloor+2r+s+\frac{-3+(-1)^{s-1}}{2}$. Assume $n\not\equiv 0 \pmod 3$. Let $\pi=(d_1,\ldots,d_k)$ be a graphic sequence with $k=\lfloor\frac{n}{3}\rfloor+2r+s+\frac{-3+(-1)^{s-1}}{2}$. Clearly, $r+s\ge\lfloor\frac{2n}{3}\rfloor+1$ and

$r \geq \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + 1$. By $k \geq \lceil \frac{n-2}{3} \rceil + 2r + s + \frac{-3+(-1)^{s-1}}{2}$, $k \geq \lfloor \frac{n}{3} \rfloor + (\lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + 1) + r + s + \frac{-3+(-1)^{s-1}}{2} \geq \lceil \frac{n}{2} \rceil + r + s + \frac{-3+(-1)^{s-1}}{2} \geq \lceil \frac{n-2}{2} \rceil + r + s - 1$ and $k \geq \lfloor \frac{n}{3} \rfloor + (\lfloor \frac{2n}{3} \rfloor + 1) + r + \frac{-3+(-1)^{s-1}}{2} \geq n + r + \frac{-3+(-1)^{s-1}}{2} \geq (n-2) + r - 1$, we have $k \geq \max\{\lceil \frac{n-2}{3} \rceil + 2r + s + \frac{-3+(-1)^{s-1}}{2}, \lceil \frac{n-2}{2} \rceil + r + s - 1, (n-2) + r - 1\} = r_{pot}(C_{n-2}, S_{r,s})$ (by Theorem 1.4–1.7). Assume that $\overline{\pi}$ is not potentially $S_{r,s}$-graphic. Then π has a realization G containing C_{n-2}. Let $H = G \setminus V(C_{n-2})$. It follows from $r+s \geq \lceil \frac{2n}{3} \rceil + \frac{-1+(-1)^s}{2} + 1$ that $|H| = |G| - |V(C_{n-2})| = \lfloor \frac{n}{3} \rfloor + 2r + s + \frac{-3+(-1)^{s-1}}{2} - (n-2) \geq \lfloor \frac{n}{3} \rfloor + (\lceil \frac{2n}{3} \rceil + \frac{-1+(-1)^s}{2} + 1) + r + \frac{-3+(-1)^{s-1}}{2} - (n-2) \geq r + 1 \geq 2$. If $\Delta(H) = 0$, we let $S \subseteq V(H)$ with $|S| = r$. If $N_C(S) \leq \lfloor \frac{n}{2} \rfloor - 1$, by $r \geq \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + \frac{-1+(-1)^s}{2} + 1$, then $|G| - |S \cup N_C(S)| \geq \lfloor \frac{n}{3} \rfloor + 2r + s + \frac{-3+(-1)^{s-1}}{2} - (\lfloor \frac{n}{2} \rfloor + r - 1) \geq \lfloor \frac{n}{3} \rfloor + (\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + \frac{-1+(-1)^s}{2} + 1) + r + s + \frac{-3+(-1)^{s-1}}{2} - (\lfloor \frac{n}{2} \rfloor + r - 1) = s$, and $\overline{\pi}$ is potentially $S_{r,s}$-graphic, a contradiction. Hence $|N_C(S)| \geq \lfloor \frac{n}{2} \rfloor$ $(= \lfloor \frac{n-2}{2} \rfloor + 1)$. This implies that there are two consecutive vertices (say v_1, v_2) on C_{n-2} and two vertices $x, x' \in S$ so that $v_1x, v_2x' \in E(G)$. If $x \neq x'$, then π is potentially P_n-graphic. Assume $x = x'$. If there is one vertex $y \in V(H) \setminus \{x\}$ and one vertex $v \in V(C_{n-2})$ so that $vy \in E(G)$, then π is potentially P_n-graphic; if $d_C(y) = 0$ for each $y \in V(H) \setminus \{x\}$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic, a contradiction. If $\Delta(H) \geq 1$, let $xy \in E(H)$, then either there exists one edge between $\{x, y\}$ and $V(C_{n-2})$ (and so π is potentially P_n-graphic), or we take $vv' \in E(C_{n-2})$ and then exchange the edges vv', xy with the non-edges $vx, v'y$ to obtain a realization of π which contains P_n. Hence $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{3} \rfloor + 2r + s + \frac{-3+(-1)^{s-1}}{2}$.

If $s \leq \lceil \frac{n}{2} \rceil - 1$ $(\leq \lfloor \frac{n}{2} \rfloor)$ and $\lceil \frac{2n}{3} \rceil + \frac{-1+(-1)^s}{2} + 1 \leq r + s \leq \lfloor \frac{2n}{3} \rfloor$, then s is odd and $r + s = \lceil \frac{2n}{3} \rceil = \lfloor \frac{2n}{3} \rfloor$. Then $r_{pot}(P_n, S_{r,s}) \geq \lfloor \frac{n}{3} \rfloor + 2r + s - 1 = n + r - 1$. It follows from Theorem 1.4 that $r_{pot}(P_n, S_{r,s}) \leq r_{pot}(C_n, S_{r,s}) = n + r - 1$. Hence $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{3} \rfloor + 2r + s - 1$.

Assume $s \geq \lceil \frac{n}{2} \rceil$ $(\geq \lfloor \frac{n}{2} \rfloor)$ and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + \frac{-1+(-1)^s}{2} + 1 \leq r \leq \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$. Then we only have the following three cases: s is odd and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil \leq r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$; s is odd and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil = r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - 1$; s is even and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + 1 = r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$.

If s is odd and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil \leq r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$, by Theorem 1.5, then $r_{pot}(P_n, S_{r,s}) \leq r_{pot}(C_n, S_{r,s}) = \lceil \frac{n}{2} \rceil + r + s - 1 = \lceil \frac{n}{3} \rceil + 2r + s - 1$. If $n \equiv 0 \pmod 3$, then $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{3} \rfloor + 2r + s - 1$. If $n \not\equiv 0 \pmod 3$, let $\pi = (d_1, \ldots, d_k)$ be a graphic sequence with $k = \lfloor \frac{n}{3} \rfloor + 2r + s - 1$. It is easy to check that $k \geq \max\{\lceil \frac{n-2}{3} \rceil + 2r + s - 1, \lceil \frac{n-2}{2} \rceil + r + s - 1, (n-2) + r - 1\} = r_{pot}(C_{n-2}, S_{r,s})$. Assume that $\overline{\pi}$ is not potentially $S_{r,s}$-graphic. Then π has a realization G containing C_{n-2}. Let $H = G \setminus V(C_{n-2})$. If $\Delta(H) = 0$, we have $|H| = |G| - |V(C_{n-2})| \geq r + 1$, let $S \subseteq V(H)$ with $|S| = r$. If $N_C(S) \leq \lfloor \frac{n}{2} \rfloor - 1$, by $|G| - |S \cup N_C(S)| \geq s$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic, a contradiction. Hence $|N_C(S)| \geq \lfloor \frac{n}{2} \rfloor$. Then there are two consecutive vertices (say v_1, v_2) on C_{n-2} and two vertices $x, x' \in S$ so that $v_1x, v_2x' \in E(G)$. If $x \neq x'$, then π is potentially P_n-graphic. If $x = x'$, and if there is one vertex $y \in V(H) \setminus \{x\}$ and one vertex $v \in V(C_{n-2})$ so that $vy \in E(G)$, then π is potentially P_n-graphic; if $d_C(y) = 0$ for each $y \in V(H) \setminus \{x\}$, then $\overline{\pi}$ is potentially $S_{r,s}$-graphic, a contradiction. If $\Delta(H) \geq 1$, let $xy \in E(H)$, then either there exists one edge between $\{x, y\}$ and $V(C_{n-2})$ (and so π is potentially P_n-graphic), or we take $vv' \in E(C_{n-2})$ and then exchange the edges vv', xy with the non-edges $vx, v'y$ to obtain a realization of π which contains P_n. Hence $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{3} \rfloor + 2r + s - 1$.

If s is odd and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil = r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - 1$ or if s is even and $\lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + 1 = r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$, then $n \equiv 3 \pmod 6$ and $r = \lceil \frac{2n}{3} \rceil - \lceil \frac{n}{2} \rceil + \frac{-1+(-1)^s}{2} + 1 = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor + \frac{-1+(-1)^s}{2}$, and hence

$$\begin{aligned} r_{pot}(P_n, S_{r,s}) &\geq \lfloor \tfrac{n}{3} \rfloor + 2r + s + \tfrac{-3+(-1)^{s-1}}{2} \\ &= \lfloor \tfrac{n}{3} \rfloor + r + s + (\lceil \tfrac{2n}{3} \rceil - \lceil \tfrac{n}{2} \rceil + \tfrac{-1+(-1)^s}{2} + 1) + \tfrac{-3+(-1)^{s-1}}{2} \qquad (*) \\ &= \lfloor \tfrac{n}{2} \rfloor + r + s - 1. \end{aligned}$$

Now by $s \geq \lceil \frac{n}{2} \rceil \geq \lfloor \frac{n}{2} \rfloor$, $r \leq \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor$ and Theorem 1.5, we have $\lfloor \frac{n}{2} \rfloor + r + s - 1 \leq r_{pot}(P_n, S_{r,s}) \leq r_{pot}(C_n, S_{r,s}) = \lceil \frac{n}{2} \rceil + r + s - 1$. If s is odd, by $s \geq \lceil \frac{n}{2} \rceil \geq \lfloor \frac{n-1}{2} \rfloor$, $r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor - 1 \leq \lfloor \frac{2(n-1)}{3} \rfloor - \lfloor \frac{n-1}{2} \rfloor$ and Theorem 1.5, we have $r_{pot}(C_{n-1}, S_{r,s}) = \lceil \frac{n-1}{2} \rceil + r + s - 1$; if s is even, by $s \geq \lceil \frac{n}{2} \rceil \geq \lfloor \frac{n-1}{2} \rfloor$, $r = \lfloor \frac{2n}{3} \rfloor - \lfloor \frac{n}{2} \rfloor = \lfloor \frac{2(n-1)}{3} \rfloor - \lfloor \frac{n-1}{2} \rfloor + 1$ and Theorem 1.7, we also have $r_{pot}(C_{n-1}, S_{r,s}) = \lceil \frac{n-1}{3} \rceil + 2r + s - 2 = \lceil \frac{n-1}{3} \rceil + \lfloor \frac{2(n-1)}{3} \rfloor - \lfloor \frac{n-1}{2} \rfloor + r + s - 1 = \lceil \frac{n-1}{2} \rceil + r + s - 1$. Let $\pi = (d_1, \ldots, d_k)$ be a graphic sequence with $k = \lfloor \frac{n}{2} \rfloor + r + s - 1$. It follows from $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n-1}{2} \rceil$ that $k = r_{pot}(C_{n-1}, S_{r,s})$. Assume that $\overline{\pi}$ is not potentially $S_{r,s}$-graphic. Then π has a realization G containing C_{n-1}. If there exists one edge between $V(G) \setminus V(C_{n-1})$ and $V(C_{n-1})$, then π is potentially P_n-graphic. Assume that there is no edge between $V(G) \setminus V(C_{n-1})$ and $V(C_{n-1})$. If there exists one edge $xy \in E(G \setminus V(C_{n-1}))$, let v, v' be two consecutive vertices on C_{n-1}, then exchange the edges vv', xy with the non-edges $vx, v'y$, we obtain a realization of π which contains P_n. If there is no edge in $G \setminus V(C_{n-1})$, by $\lfloor \frac{n}{2} \rfloor + r + s - 1 - (n-1) \geq r$, then $\overline{G}$ contains $S_{r,s}$, that is, $\overline{\pi}$ is potentially $S_{r,s}$-graphic, a contradiction. Hence $r_{pot}(P_n, S_{r,s}) \leq \lfloor \frac{n}{2} \rfloor + r + s - 1$. It follows from $(*)$ that $r_{pot}(P_n, S_{r,s}) = \lfloor \frac{n}{3} \rfloor + 2r + s + \frac{-3+(-1)^{s-1}}{2}$.

If $(n, r, s) = (6, 3, 2)$, then $r_{pot}(P_6, S_{3,2}) \geq 8$. Let $\pi = (d_1, \ldots, d_8)$ be a graphic sequence. Assume that $\overline{\pi}$ is not potentially $S_{3,2}$-graphic. By Theorem 1.7, we have $r_{pot}(C_5, S_{3,2}) = 8$, and so π has a realization G containing C_5. If there exists one edge between $V(G) \setminus V(C_5)$ and $V(C_5)$, then π is potentially P_6-graphic. Assume that there is no edge between $V(G) \setminus V(C_5)$ and $V(C_5)$. If there exists one edge $xy \in E(G \setminus V(C_5))$, let v, v' be two consecutive vertices on C_5, then exchange the edges vv', xy with the non-edges $vx, v'y$, we obtain a realization of π which contains P_6. If there is no edge in $G \setminus V(C_5)$, by $|G| - 5 = 3$, then $\overline{G}$ contains $S_{3,2}$, that is, $\overline{\pi}$ is potentially $S_{3,2}$-graphic, a contradiction. □

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