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Numerical Algorithms for Solving the Least Squares Symmetric Problem of Matrix Equation $AXB + CXD = E$

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Abstract. This paper focuses on the least squares symmetric problem of matrix equation $AXB + CXD = E$. The explicit expressions for least squares symmetric solution with the least norm of matrix equation $AXB + CXD = E$ are derived. Numerical algorithms and numerical examples show the feasibility of our methods.

1. Introduction

Throughout this paper, the symbols $\mathbf{R}^n$, $\mathbf{R}^{m\times n}$, $\mathbf{SR}^{n\times n}$ and I_n denote the set of all real vectors with n coordinates, the set of all $m \times n$ real matrices, the set of all $n \times n$ real symmetric matrices, and the identity matrix of order n, respectively. e_j is the jth column of the identity matrix I_n. For $A \in \mathbf{R}^{m\times n}$, A^T and A^+ denote the transpose and the Moore–Penrose generalized inverse of matrix A, respectively. For $A = (a_{ij}) \in \mathbf{R}^{n\times n}$, the trace of matrix A is denoted by $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. We define the inner product: $\langle A, B\rangle=\mathrm{tr}(AB^T)$ for all $A, B \in \mathbf{R}^{m\times n}$. Then $\mathbf{R}^{m\times n}$ is a Hilbert inner product space and the norm of a matrix generated by this inner product is the matrix Frobenius norm denoted by $\|\cdot\|$. The notation $A \otimes B$ stands for the Kronecker product of A and B. For matrix $A \in \mathbf{R}^{m\times n}$, denote by the stretching function

$$\mathrm{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T, \tag{1}$$

where a_j is the j-th column of matrix A. For matrices A, B and C with appropriate dimension, we have the following results associated with the stretching function and Kronecker product:

$$\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B). \tag{2}$$

In this paper, we mainly discuss the following problem.
Problem I. Given $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times s}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{n\times s}$ and $E \in \mathbf{R}^{m\times s}$, let

$$H_L = \{X | X \in \mathbf{SR}^{n\times n}, \|AXB + CXD - E\| = \min_{X_0\in \mathbf{SR}^{n\times n}} \|AX_0B + CX_0D - E\|\}.$$

Find $X_H \in H_L$ such that

$$\|X_H\| = \min_{X\in H_L} \|X\|. \tag{3}$$

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The solution X_H of Problem I is called the least squares symmetric solution with the least norm of matrix equation

$$AXB + CXD = E. \tag{4}$$

Linear matrix equations play an important role in matrix theory and have extensive applications. For instance, the well-known discrete Lyapunov matrix equation $AXA^T - X + Q = 0$ and continuous Lyapunov matrix equation $AX + XA^T + Q = 0$ (Q is a symmetric matrix.) occur in many branches of control theory, such as stability analysis and optimal control. The Sylvester matrix equation $AX + XB = C$ has found huge applications in optimal control and neural networks [11,13, 26].

Direct and iterative methods for solving the matrix equations such as $AXB = C$, $AXB + CYD = E$ and the extended Sylvester matrix equation (4) have been widely investigated. See [3-6, 12-18, 21-29, 32-37], and references cited therein. For the extended Sylvester matrix equation (4), Mitra [20] and Tian [30] considered the solvability condition for complex and real matrix equation (4), respectively. Hernández and Gassó [11] obtained the explicit solution of matrix equation (4). Mansour [19] studied the solvability condition for matrix equation (4) in the operator algebra. Huang [10] obtained necessary and sufficient conditions for existence of a solution or a unique solution of quaternion matrix equation (4).

Matrix decomposition is an efficient tool to study the least squares problems of matrix equations. There are many important results about the matrix equations. For example, see [2, 7, 8, 9, 16, 17, 25, 36] for more details. However, to our knowledge, the understanding of the problem for finding the least squares constraint solutions of the matrix equation (4) has not yet reached a satisfactory level. The reason is that there seems to have some difficulties in finding the constrained solutions or least squares constrained solutions of matrix equation (4) by using these matrix decomposition methods mentioned above.

In [37, 38], whether the least squares problems of matrix equations or inverse eigenvalue problems over some constrained matrix sets, the authors found it was efficient to change them into the unconstrained problems. They solved them by using the Kronecker product, vec-operator and Moore-Penrose generalized inverse. Recently Yuan and Liao [39] proposed a matrix-vector product method to solve the least squares Hermitian problem of the complex matrix equation (4). We think the idea is also suitable for Problem I. Our methods are to change the constrained problem of matrix equation (4) into a unconstrained problem of a system of real equations.

We now briefly survey the contents of our paper. In Section 2, we introduce a matrix-vector product of $\mathbf{R}^{m\times n}$ and discuss the structure of $AXB + CXD$ over $X \in \mathbf{SR}^{n\times n}$ by the product. In Sections 3 and 4, we provide two methods and two algorithms to study Problem I. In Section 5, we discuss Kronecker product method for solving Problem I. Finally, in Section 6, we give numerical examples to illustrate the theoretical results in this paper.

2. The structure of $AXB + CXD$ over $X \in \mathbf{SR}^{n\times n}$

The main aim of this section is to propose the structure of $AXB + CXD$ over $X \in \mathbf{SR}^{n\times n}$ for solving Problem I. We first introduce a matrix-vector product of $\mathbf{R}^{m\times n}$, which is similar to the matrix-vector product of $\mathbf{C}^{m\times n}$ in [39].

Definition 1. Let $x = (x_1, x_2, \dots, x_k)^T$, $y = (y_1, y_2, \dots, y_k)^T \in \mathbf{R}^k$ and $A = (A_1, A_2, \dots, A_k)$, $A_i \in \mathbf{R}^{m\times n}$, $(i = 1, 2, \dots, k)$, $B = (B_1, B_2, \dots, B_s) \in \mathbf{R}^{k\times s}$, $B_i \in \mathbf{R}^k$, $(i = 1, 2, \dots, s)$. Define

1. $A \odot x = x_1A_1 + x_2A_2 + \cdots + x_kA_k \in \mathbf{R}^{m\times n}$;
2. $A \odot (x, y) = (A \odot x, A \odot y)$;
3. $A \odot B = (A \odot B_1, A \odot B_2, \dots, A \odot B_s)$.

Obviously, $A \odot x$ is the linear combination of matrices $A_1, A_2, \dots, A_k$. If $A_i \in \mathbf{R}^m$ then $A \odot x = Ax$. The relationship of $A \odot x$ with Kronecker product is $A \odot x = A(x \otimes I_n)$.

Definition 2. For matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$, let $a_1 = (a_{11}, a_{21}, \dots, a_{n1})$, $a_2 = (a_{22}, a_{32}, \dots, a_{n2})$, $\dots$, $a_{n-1} = (a_{(n-1)(n-1)}, a_{n(n-1)})$, $a_n = a_{nn}$, and denote by $\mathrm{vec}_S(A)$ the following vector:

$$\mathrm{vec}_S(A) = (a_1, a_2, \dots, a_{n-1}, a_n)^T \in \mathbf{R}^{\frac{n(n+1)}{2}}. \tag{5}$$

For $i, j = 1, 2, \dots, n$, let $E_{ij} = (f_{st}) \in \mathbf{R}^{n\times n}$, where

$$f_{st} = \begin{cases} 1, & (s,t) = (i,j), \\ 0, & otherwise. \end{cases}$$

Let

$$K_S = (E_{11}, E_{21} + E_{12}, \dots, E_{n1} + E_{1n}, E_{22}, E_{32} + E_{23}, , E_{n2} + E_{2n}, \dots, E_{(n-1)(n-1)}, E_{n(n-1)} + E_{(n-1)n}, E_{nn}). \tag{6}$$

Note that $K_S \in \mathbf{R}^{n\times \frac{n^2(n+1)}{2}}$. We can get the following conclusions.

Lemma 3. Suppose $X \in \mathbf{R}^{n\times n}$, then

$$X \in \mathbf{SR}^{n\times n} \Longleftrightarrow X = K_S \odot \mathrm{vec}_S(X). \tag{7}$$

Theorem 4. Suppose $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times s}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{n\times s}$ and $X \in \mathbf{SR}^{n\times n}$. Let $A = (A_1, A_2, \dots, A_n)$, $A_i \in R^m$ is the ith column vector of matrix A, $C = (C_1, C_2, \dots, C_n)$, $C_i \in R^m$ is the ith column vector of matrix C,

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_n \end{pmatrix},$$

where $B_j \in \mathbf{R}^s$ is the jth row vector of matrix B, $D_j \in R^s$ is the jth row vector of matrix D. Then the following conclusion holds.

(i) $AE_{ij}B = A_iB_j$.

(ii) Let $F_{ij} \in \mathbf{R}^{m\times s}$ $(i, j = 1, 2, \dots, n, i \geq j)$, where

$$F_{ij} = \begin{cases} A_iB_j + C_iD_j, & i = j, \\ A_iB_j + A_jB_i + C_iD_j + C_jD_i, & i > j, \end{cases} \tag{8}$$

$$AXB + CXD = (F_{11}, F_{21}, \dots, F_{n1}, \\ F_{22}, F_{32}, \dots, F_{n2}, \dots, F_{(n-1)(n-1)}, F_{n(n-1)}, F_{nn}) \odot \mathrm{vec}_S(X).$$

Proof. (i) Obviously.

(ii) By (i) and Lemma 3, we can get

$$\begin{aligned}
& AXB + CXD \\
= \; & A(K_S \odot \mathrm{vec}_S(X))B + C(K_S \odot \mathrm{vec}_S(X))D \\
= \; & ((AK_S) \odot \mathrm{vec}_S(X))B + ((CK_S) \odot \mathrm{vec}_S(X))D \\
= \; & (AE_{11}B, A(E_{21} + E_{12})B, \dots, A(E_{n1} + E_{1n})B, \dots, AE_{(n-1)(n-1)}B, \\
& A(E_{n(n-1)} + E_{(n-1)n})B, AE_{nn}B) \odot \mathrm{vec}_S(X) + (CE_{11}D, C(E_{21} + E_{12})D, \dots, \\
& C(E_{n1} + E_{1n})D, \dots, CE_{(n-1)(n-1)}D, C(E_{n(n-1)} + E_{(n-1)n})D, CE_{nn}D) \odot \mathrm{vec}_S(X) \\
= \; & (AE_{11}B + CE_{11}D, A(E_{21} + E_{12})B + C(E_{21} + E_{12})D, \dots, \\
& A(E_{n1} + E_{1n})B + C(E_{n1} + E_{1n})D, \dots, AE_{(n-1)(n-1)}B + CE_{(n-1)(n-1)}D, \\
& A(E_{n(n-1)} + E_{(n-1)n})B + C(E_{n(n-1)} + E_{(n-1)n})D, AE_{nn}B + CE_{nn}D) \odot \mathrm{vec}_S(X) \\
= \; & (F_{11}, F_{21}, \dots, F_{n1}, F_{22}, \dots, F_{n2}, \dots, F_{(n-1)(n-1)}, F_{n(n-1)}, F_{nn}) \odot \mathrm{vec}_S(X). \qquad \blacksquare
\end{aligned}$$

3. Method I for the solution of Problem I

For $A \in \mathbf{R}^{m\times n}, B \in \mathbf{R}^{n\times s}, C \in \mathbf{R}^{m\times n}, D \in \mathbf{R}^{n\times s}, E \in \mathbf{R}^{m\times s}$, set

$$G = \begin{pmatrix} \langle F_{11}, F_{11}\rangle & \langle F_{11}, F_{21}\rangle & \cdots & \langle F_{11}, F_{nn}\rangle \\ \langle F_{21}, F_{11}\rangle & \langle F_{21}, F_{21}\rangle & \cdots & \langle F_{21}, F_{nn}\rangle \\ \vdots & \vdots & & \vdots \\ \langle F_{nn}, F_{11}\rangle & \langle F_{nn}, F_{21}\rangle & \cdots & \langle F_{nn}, F_{nn}\rangle \end{pmatrix}, \quad e = \begin{pmatrix} \langle F_{11}, E\rangle \\ \langle F_{21}, E\rangle \\ \vdots \\ \langle F_{nn}, E\rangle \end{pmatrix}. \tag{9}$$

where F_{ij} is in the form of (8). Obviously, $G \in \mathbf{R}^{\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}}$, $e \in \mathbf{R}^{\frac{n(n+1)}{2}}$. By Theorem 4, we can provide the method I for the solution of Problem I. The following lemma is also necessary to derive the results.

Lemma 5 [1]. (i) The matrix equation $Ax = b$, with $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, has a solution $x \in \mathbf{R}^n$ if and only if

$$AA^+b = b, \tag{10}$$

in this case it has the general solution

$$x = A^+b + (I_n - A^+A)y, \tag{11}$$

where $y \in \mathbf{R}^n$ is an arbitrary vector. The solution of the matrix equation $Ax = b$ with the least norm is $x = A^+b$.

(ii) The least squares solutions of the matrix equation $Ax = b$, with $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, can be represented as

$$x = A^+b + (I_n - A^+A)y, \tag{12}$$

where $y \in \mathbf{R}^n$ is an arbitrary vector. The least squares solution of the matrix equation $Ax = b$ with the least norm is $x = A^+b$.

Theorem 6. The set H_L of Problem I can be expressed as

$$H_L = \left\{ X \middle| X = K_S \odot [G^+e + (I_{\frac{n(n+1)}{2}} - G^+G)y] \right\}, \tag{13}$$

where $y \in \mathbf{R}^{\frac{n(n+1)}{2}}$ is an arbitrary vector. Problem I has a unique solution $X_H \in H_L$ in the form

$$X_H = K_S \odot (G^+e). \tag{14}$$

Proof. By Theorem 4, the least squares problem

$$\|AXB + CXD - E\| = \min$$

with respect to the symmetric matrix X is equivalent to

$$G\mathrm{vec}_S(X) = e.$$

By Lemma 5, it follows that

$$\mathrm{vec}_S(X) = G^+e + (I_{\frac{n(n+1)}{2}} - G^+G)y,$$
$$X = K_S \odot [G^+e + (I_{\frac{n(n+1)}{2}} - G^+G)y],$$

and the unique solution X_H is in the form

$$X_H = K_S \odot (G^+e).$$

∎

We next discuss the consistency of matrix eqution (4). Since the normal equation $G\mathrm{vec}_S(X) = e$ is not useful to the consistent condition, we propose the following result by Lemma 5 and Theorem 6. Clearly,

$$AXB + CXD = E \Longleftrightarrow N\mathrm{vec}_S(X) = \mathrm{vec}(E), \tag{15}$$

where

$$N = (\mathrm{vec}(F_{11}), \mathrm{vec}(F_{21}), \ldots, \mathrm{vec}(F_{n1}), \mathrm{vec}(F_{22}), \ldots, \mathrm{vec}(F_{n2}), \ldots,$$

$$\mathrm{vec}(F_{(n-1)(n-1)}), \mathrm{vec}(F_{n(n-1)}), \mathrm{vec}(F_{nn})) \in \mathbf{R}^{ms\times\frac{n(n+1)}{2}}. \tag{16}$$

Corollary 7. The matrix equation (4) has a solution $X \in \mathbf{SR}^{n\times n}$ if and only if

$$NN^{+}\mathrm{vec}(E) = \mathrm{vec}(E). \tag{17}$$

In this case, denote by H_E the solution set of (4). Then

$$H_E = \left\{X \,\middle|\, X = K_S \odot [G^{+}e + (I_{\frac{n(n+1)}{2}} - G^{+}G)y]\right\}, \tag{18}$$

where $y \in \mathbf{R}^{\frac{n(n+1)}{2}}$ is an arbitrary vector. The least norm problem

$$\|X_E\| = \min_{X\in H_E} \|X\|$$

has a unique solution $X_E \in H_E$ and X_E can be expressed as

$$X_E = K_S \odot (G^{+}e). \tag{19}$$

Furthermore, if (17) holds, then the matrix equation (4) has a unique solution $X \in H_E$ if and only if

$$\mathrm{rank}(N) = \frac{n(n+1)}{2}. \tag{20}$$

In this case,

$$H_E = \left\{X \,\middle|\, K_S \odot (G^{+}e)\right\}. \tag{21}$$

Algorithm 1 For Problem I

Input: A, B, C, D and E ($A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times s}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{n\times s}$ and $E \in \mathbf{R}^{m\times s}$).

(a) Compute K_S by (6).
(b) Compute G and e by (9).
(c) Compute N by (16) and $\mathrm{vec}(E)$.
(d) If (17) and (20) hold, then calculate $X_E(X_E \in H_E)$ according to (19). Otherwise go to next step.
(e) If (17) holds, then calculate $X_E(X_E \in H_E)$ by (19). Otherwise go to next step.
(f) Calculate $X_H(X_H \in H_L)$ according to (14).

Output: The solutions X_H and X_E of Problem I.

4. Method II for the solution of Problem I

In the discussions of the consistency for matrix equation (4) in Section 3, we can provide the method II for the solution of Problem I by (15) and Lemma 5.

Theorem 8. The set H_L of Problem I is in the form

$$H_L = \left\{ X \,\middle|\, X = K_S \odot [N^+ \mathrm{vec}(E) + (I_{\frac{n(n+1)}{2}} - N^+ N)y] \right\}, \tag{22}$$

where $y \in \mathbf{R}^{\frac{n(n+1)}{2}}$ is an arbitrary vector. Problem I has a unique solution $X_H \in H_L$ in the form

$$X_H = K_S \odot [N^+ \mathrm{vec}(E)]. \tag{23}$$

Corollary 9. The matrix equation (4) has a solution $X \in \mathbf{SR}^{n\times n}$ if and only if (17) holds. In this case, denote by H_E the solution set of (4). Then

$$H_E = \left\{ X \,\middle|\, X = K_S \odot [N^+ \mathrm{vec}(E) + (I_{\frac{n(n+1)}{2}} - N^+ N)y] \right\}, \tag{24}$$

where $y \in \mathbf{R}^{\frac{n(n+1)}{2}}$ is an arbitrary vector. The least norm problem

$$\|X_E\| = \min_{X\in H_E} \|X\|$$

has a unique solution $X_E \in H_E$ and X_E can be expressed as

$$X_E = K_S \odot [N^+ \mathrm{vec}(E)]. \tag{25}$$

Furthermore, if (17) holds, then the matrix equation (4) has a unique solution $X \in H_E$ if and only if

$$\mathrm{rank}(N) = \frac{n(n+1)}{2}. \tag{26}$$

In this case,

$$H_E = \left\{ X \,\middle|\, X = K_S \odot [N^+ \mathrm{vec}(E)] \right\}. \tag{27}$$

Algorithm 2 For Problem I

Input: A, B, C, D and E ($A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times s}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{n\times s}$ and $E \in \mathbf{R}^{m\times s}$).

(a) Compute K_S by (6).
(b) Compute N by (16) and vec(E).
(c) If (17) and (26) hold, then calculate $X_E(X_H \in H_E)$ according to (25). Otherwise go to next step.
(d) If (17) holds, then calculate $X_E(X_E \in H_E)$ by (25). Otherwise go to next step.
(e) Calculate $X_H(X_H \in H_L)$ according to (23).

Output: The solutions X_H and X_E of Problem I.

5. Method III for the solution of Problem I

The method in [37, 38] is based on Kronecker product, Moore–Penrose generalized inverse and vec-operation. In this section, we briefly recall this method for solving Problem I. Given the matrices $A \in \mathbf{R}^{m\times n}$,

$B \in \mathbf{R}^{n\times s}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{n\times s}$ and $E \in \mathbf{R}^{m\times s}$, let $K \in R^{n^2\times \frac{n(n+1)}{2}}$ be of the following form

$$K = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\ 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n \end{bmatrix}. \tag{28}$$

For a matrix $X = (x_{ij}) \in \mathbf{SR}^{n\times n}$, let $x_1 = (x_{11}, \sqrt{2}x_{21}, \ldots, \sqrt{2}x_{n1})$, $x_2 = (x_{22}, \sqrt{2}x_{32}, \ldots, \sqrt{2}x_{n2})$, $\ldots$, $x_{n-1} = (x_{(n-1)(n-1)}, \sqrt{2}x_{n(n-1)})$, $x_n = x_{nn}$, and denote by $\mathrm{vec}_K(X)$ the following vector:

$$\mathrm{vec}_K(X) = (x_1, x_2, \ldots, x_{n-1}, x_n)^T \in \mathbf{R}^{\frac{n(n+1)}{2}}. \tag{29}$$

Let $M = (B^T \otimes A + D^T \otimes C)K$. Thus the least squares problem

$$\|AXB + CXD - E\| = \min$$

with respect to the symmetric matrix X is equivalent to the following least squares unconstrained problem

$$\|M\mathrm{vec}_K(X) - \mathrm{vec}(E)\| = \min.$$

We summarize the conclusions and an algorithm as follows.

Theorem 10. The set H_L of Problem I can be expressed as

$$H_L = \left\{X \middle| \mathrm{vec}(X) = KM^+\mathrm{vec}(E) + K(I_{\frac{n(n+1)}{2}} - M^+M)y\right\}, \tag{30}$$

where $y \in \mathbf{R}^{\frac{n(n+1)}{2}}$ is an arbitrary vector. Problem I has a unique solution $X_H \in H_L$ in the form

$$\mathrm{vec}(X_H) = KM^+\mathrm{vec}(E). \tag{31}$$

Corollary 11. The matrix equation (4) has a solution $X \in \mathbf{SR}^{n\times n}$ if and only if

$$MM^+\mathrm{vec}(E) = \mathrm{vec}(E). \tag{32}$$

In this case, denote by H_E the solution set of the matrix equation (4), and H_E can be expressed as

$$H_E = \left\{X \middle| \mathrm{vec}(X) = KM^+\mathrm{vec}(E) + (I_{\frac{n(n+1)}{2}} - M^+M)y]\right\}, \tag{33}$$

where $y \in \mathbf{R}^{\frac{n(n+1)}{2}}$ is an arbitrary vector. The least norm problem

$$\|X_E\| = \min_{X\in H_E} \|X\|$$

has a unique solution $X_E \in H_E$ and X_E can be expressed as

$$\mathrm{vec}(X_E) = KM^+\mathrm{vec}(E). \tag{34}$$

Furthermore, if (32) holds, then the matrix equation (4) has a unique solution $X \in H_E$ if and only if

$$\mathrm{rank}(M) = \frac{n(n+1)}{2}. \tag{35}$$

In this case,

$$H_E = \left\{ X \,\middle|\, \mathrm{vec}(X) = KM^{+}\mathrm{vec}(E) \right\}. \tag{36}$$

Algorithm 3 For Problem I

Input: A, B, C, D and E. ($A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times s}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{n\times s}$ and $E \in \mathbf{R}^{m\times s}$)

(a) Compute K by (28).
(b) Compute $M = (B^T \otimes A + D^T \otimes C)K$ and vec(E).
(c) If (32) and (35) hold, then calculate $X_E(X_E \in H_E)$ by (34).
(d) If (32) holds, then calculate $X_E(X_E \in H_E)$ by (34). Otherwise go to next step.
(e) Calculate $X_H(X_H \in H_L)$ by (31).

Output: The solutions X_H and X_E of Problem I.

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6. Numerical Verification

Based on the discussions in Sections 2-5, we now give numerical examples to show the feasibility of Algorithms 1, 2, 3. At first, we compare Algorithms 1, 2, 3 and derive them in Table 1, where $X_H^{(i)}$ $i = 1, 2, 3$ are denoted the solution of Problem I computed by Algorithms 1, 2, 3, respectively.

Table 1. the comparation of Algorithms 1, 2 and 3

	Algorithm 1	Algorithm 2	Algorithm 3
linear equation	$G\mathrm{vec}_S(X) = e$	$N\mathrm{vec}_S(X) = \mathrm{vec}(E)$	$M\mathrm{vec}_K(X) = \mathrm{vec}(E)$
consistent condition	$NN^{+}\mathrm{vec}(E) = \mathrm{vec}(E)$	$NN^{+}\mathrm{vec}(E) = \mathrm{vec}(E)$	$MM^{+}\mathrm{vec}(E) = \mathrm{vec}(E)$
X_H	$X_H^{(1)} = K_S \odot [G^{+}e]$	$X_H^{(2)} = K_S \odot [N^{+}\mathrm{vec}(E)]$	$\mathrm{vec}(X_H^{(3)}) = KM^{+}\mathrm{vec}(E)$
coefficiency matrix	$G \in \mathbf{R}^{\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}}$	$N \in \mathbf{R}^{ms\times\frac{n(n+1)}{2}}$	$M \in \mathbf{R}^{ms\times\frac{n(n+1)}{2}}$
size of K, K_S	$K_S \in \mathbf{R}^{n\times\frac{n^2(n+1)}{2}}$	$K_S \in \mathbf{R}^{n\times\frac{n^2(n+1)}{2}}$	$K \in \mathbf{R}^{n^2\times\frac{n(n+1)}{2}}$

We now report our numerical examples. When the consistent conditions for matrix equation (4) hold, Examples 1, 2 consider the numerical solutions of Problem I for $X \in \mathbf{SR}^{n\times n}$, respectively. In Example 3, by using a matrix to perturb the matrix E of Example 2, we can obtain the inconsistent matrix equation (4). Thus we can analyze the least squares solution with the least norm for matrix equation (4). For demonstration purpose and avoiding the matrices with large norm to interrupt the solutions of Problem I, we only consider the cases of small $n = 8$ and take the coefficient matrices in Examples 1, 2, 3. We also consider the cases of $n = 20, 40, 60$ and take the random matrices in Examples 4, 5.

Example 1. Let $m = 7, n = 8, s = 10$. Take

$$A = (\mathrm{Hankel}(1:m), -\mathrm{ones}(m, (n-m))),$$

$$C = (-\mathrm{Toeplitz}(1:m), \mathrm{ones}(m, (n-m))),$$

$$B = (\mathrm{Toeplitz}(1:n), \mathrm{zeros}(n, (s-n))), D = (\mathrm{Hankel}(1:n), -\mathrm{ones}(n, (s-n))),$$

$X = \mathrm{Hadamard}(n)$, $E = AXB + CXD$, where Toeplitz$(1:n)$, Hankel$(1:n)$ denote the Toeplitz matrix and Hankel matrix whose first rows are $(1, 2, \ldots, n)$, respectively, Hadamard(n) denotes the Hadamard matrix with order n, and ones(m, n), zeros(m, n) denote the $m \times n$ matrices whose all elments are one and zero, respectively. Obviously, X is a symmetric matrix. From Algorithms 1, 2, 3, we can obtain the consistent matrix equations (4), $M\mathrm{vec}_S(X) = \mathrm{vec}(E)$, and $N\mathrm{vec}_S(X) = \mathrm{vec}(E)$. By using matlab 7.7, we obtain rank$(M) = 36$, rank$(N) = 36$, $\|NN^{+}e - e\| = 2.5580\times10^{-12}$, $\|MM^{+}e - e\| = 1.4974\times10^{-12}$. We can see the matrix equation (4) has a unique solution $X_E \in H_E$, and get $\|X_E^{(1)} - X\| = 1.8280\times10^{-11}$, $\|X_E^{(2)} - X\| = 6.6084\times10^{-14}$, $\|X_E^{(3)} - X\| = 6.4843\times10^{-14}$, and $\|X_E^{(1)} - X_E^{(2)}\| = 1.8237\times10^{-11}$. Note that $G\mathrm{vec}_S(X) = e$ is consistent, we can also get $\|GG^{+}e - e\| = 6.7066\times10^{-8}$.

Example 2. Suppose $A, B, C, D\ X, E$ are the same as in Example 1. Let $m = 5, n = 8, s = 10$. From Algorithms 1, 2, 3, we can obtain the consistent matrix equations (4), $M\mathrm{vec}_S(X) = \mathrm{vec}(E)$ and $N\mathrm{vec}_S(X) = \mathrm{vec}(E)$. By using matlab 7.7, we obtain $\mathrm{rank}(M) = 33$, $\mathrm{rank}(N) = 33$, $\|NN^+e - e\| = 1.6128\times10^{-12}$, $\|MM^+e - e\| = 2.6489\times10^{-12}$. According to Algorithm 1, 2, 3, we can see the matrix equation (4) has infinite solutions and a unique solution $X_E \in H_E$, and we can get $\|X_E^{(1)} - X\| = 2.8425$, $\|X_E^{(2)} - X\| = 2.8425$, $\|X_E^{(3)} - X\| = 2.8284$, and $\|X_E^{(1)} - X_E^{(2)}\| = 6.8996\times10^{-11}$. Note that $G\mathrm{vec}_S(X) = e$ is consistent, we can also get $\|GG^+e - e\| = 1.3241\times10^{-7}$.
Example 3. Suppose $A, B, C, D\ X$ are the same as in Example 1, $E = AXB + CXD + \mathrm{ones}(m, s)$. Let $m = 5, n = 8, s = 10$. From Algorithms 1, 2, 3, we can obtain the inconsistent matrix equations (4), $M\mathrm{vec}_S(X) = \mathrm{vec}(E)$ and $N\mathrm{vec}_S(X) = \mathrm{vec}(E)$. By using matlab 7.7, we obtain $\mathrm{rank}(M) = 33$, $\mathrm{rank}(N) = 33$, $\|NN^+e - e\| = 1.1430$, $\|MM^+e - e\| = 1.1430$. According to Algorithms 1, 2, 3, we can see the matrix equation (4) has a unique solution $X_H \in H_L$, and we can get $\|X_H^{(1)} - X\| = 2.7752$, $\|X_H^{(2)} - X\| = 2.7752$, $\|X_H^{(3)} - X\| = 2.8937$, and $\|X_H^{(1)} - X_H^{(2)}\| = 7.2709\times10^{-10}$. Note that $G\mathrm{vec}_S(X) = e$ is consistent, we can also get $\|GG^+e - e\| = 1.3094\times10^{-7}$.
Example 4. Take

$$A = \mathrm{rand(m, n)}, B = \mathrm{rand(n, s)}, C = \mathrm{rand(m, n)}, D = \mathrm{rand(n, s)}.$$

Let $X = \mathrm{hadamard(n)}$, $E = AXB + CXD$. The numerical results are listed in Table 2.

Table 2. Numerical results for Example 4.

(m, n, s)	rank(N)	$\|NN^+e - e\|$	$\|MM^+e - e\|$	$\|X_H^{(1)} - X\|$	$\|X_H^{(3)} - X\|$
(15, 20, 30)	210	3.9388×10^{-6}	1.0965×10^{-10}	6.5320×10^{-9}	1.6664×10^{-11}
(30, 40, 50)	820	1.7264×10^{-9}	1.9869×10^{-9}	1.9869×10^{-7}	1.7493×10^{-10}
(50, 60, 70)	1830	1.3029×10^{-8}	1.0133×10^{-8}	6.7614×10^{-10}	8.4034×10^{-7}

Example 5. Take

$$A = \mathrm{rand(m, n)}, B = \mathrm{rand(n, s)}, C = \mathrm{rand(m, n)}, D = \mathrm{rand(n, s)},$$

where rand(m, n) denotes the $m\times n$ random matrix. Let $X = \mathrm{hadamard(n)}$, $E = AXB + CXD + 1000\mathrm{ones}(m, s)$. The numerical results are listed in Table 3.

Table 3. Numerical results for Example 5.

(m, n, s)	rank(N)	$\|NN^+e - e\|$	$\|MM^+e - e\|$	$\|X_H^{(3)} - X\|$	$\|X_H^{(1)} - X_H^{(3)}\|$
(15, 20, 30)	210	1.4892×10^{3}	1.4892×10^{3}	1.1615×10^{3}	8.8607×10^{-9}
(30, 40, 50)	820	1.5342×10^{3}	1.5342×10^{3}	814.2038	1.5352×10^{-7}
(50, 60, 70)	1830	1.7561×10^{3}	2.2974×10^{3}	612.5633	7.6993×10^{-7}

Examples 1-5 are used to show the feasibility of Algorithms 1, 2, 3.

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