Filomat 33:6 (2019), 1659–1666
https://doi.org/10.2298/FIL1906659X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Determinants Involving the Numbers of the Stirling-Type

Aimin Xu[a]

[a]*Institute of Mathematics, Zhejiang Wanli University, Ningbo 315100, China*

Abstract. By the LU factorization, we evaluate a determinant involving the complete symmetric functions. From a viewpoint of symmetric functions, some results for the evaluations of the determinants of the matrices consisting of the numbers of the Stirling-type are given.

1. Introduction

The q-Stirling numbers of the second kind [21] obey the recurrence

$$S_q(n,k) = q^{k-1}S_q(n-1,k-1) + [k]_q S_q(n-1,k),\ n,k \geq 1,$$

with $S_q(n,0) = 0$ and $S_q(0,k) = \delta_{k,0}$. In 2003, Ehrenborg [9] obtained a beautiful determinant identity involving the q-analogue of the Stirling numbers:

$$\det(S_q(s+i+j,s+j))_{0\leq i,j\leq n} = q^{\binom{s+n+1}{3}-\binom{s}{3}}\prod_{k=1}^{n}[s+k]_q^k, \tag{1}$$

where s and n be nonnegative integers. In his paper, Ehrenborg gave two nice proofs of this result, one bijective and one based upon factoring the matrix.

Let $\widetilde{S}_q(n,k) = q^{-\binom{k}{2}}S_q(n,k)$ be the q-Stirling numbers of the second kind due to Carlitz [4, 5, 27]. Recently, Cai, Ehrenborg and Readdy [3] proved the following identity by using RG-words:

$$\det(\widetilde{S}_q(s+i+j,s+j))_{0\leq i,j\leq n} = \prod_{k=1}^{n}[s+k]_q^k. \tag{2}$$

These interesting works motivate us to evaluate the determinants involving the numbers of the Stirling-type. The purpose of this paper is to use the method of LU factorization to evaluate a determinant involving the complete symmetric functions. By the connections between the complete symmetric functions and the Striling numbers and their generalizations, we obtain some evaluations of the determinants involving the numbers of the Stirling-type including the above results in a unified approach.

2010 *Mathematics Subject Classification*. Primary 05E05; Secondary 15A15, 15A23
Keywords. Complete symmetric function, Determinant, LU factorization, Stirling number
Received: 16 February 2018; Accepted: 29 April 2019
Communicated by Dragana Cvetković Ilić
Research supported by the Natural Science Foundation of Zhejiang Province (LY18A010001), the National Natural Science Foundation of China (11201430) and the Ningbo Natural Science Foundation (2017A610140).
Email address: xuaimin1009@hotmail.com (Aimin Xu)

2. A determinant involving complete symmetric functions

Let n be a positive integer. For $k = 0, 1, 2, \ldots$, the complete symmetric function of degree k in n variables $x_1, x_2, \ldots, x_n$ is the sum of all monomials of total degree k in the variables. Namely,

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

which can also be alternatively written as

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{\substack{i_1 + i_2 + \cdots + i_n = k \\ i_j \ge 0}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

In particular, the first few cases are

$$\begin{aligned} h_0(x_1, x_2, \ldots, x_n) &= 1, \\ h_1(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} x_i, \\ h_2(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i \le j \le n} x_i x_j, \\ h_3(x_1, x_2, \ldots, x_n) &= \sum_{1 \le i \le j \le k \le n} x_i x_j x_k. \end{aligned}$$

The complete symmetric functions can be characterized by the following identity:

$$\prod_{i=1}^{n} (1 - x_i t)^{-1} = \sum_{k \ge 0} h_k(x_1, x_2, \ldots, x_n) t^k.$$

It is easy to check that for $n > k \ge 1$ the following recurrence holds

$$h_{n-k}(x_1, x_2, \ldots, x_k) = h_{n-k}(x_1, x_2, \ldots, x_{k-1}) + x_k h_{n-k-1}(x_1, x_2, \ldots, x_k), \tag{3}$$

with initial conditions $h_0(x_1, x_2, \ldots, x_k) = 1$ and $h_{n-k}(x_1, x_2, \ldots, x_k) = 0$ if $n < k$.

It is well known that determinants of matrices whose entries are symmetric functions have a rich, very developed theory. In his famous book of enumerative combinatorics, Stanley [26, Theorem 2.7.1] obtained a very general determinant identity involving the complete symmetric functions by using the method of lattice path. As a useful and simple consequence, in this paper the following evaluation of a determinant involving the complete symmetric functions is recovered by using the LU factorization method.

Proposition 2.1. *Let s and n be nonnegative integers. Then the following identity holds:*

$$\det(h_i(x_0, x_1, \ldots, x_{s+j}))_{0 \le i,j \le n} = \prod_{k=1}^{n} x_{s+k}^{k}. \tag{4}$$

This proposition plays a central role in our paper. As an example, we let $\binom{n}{k}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ be the binomial coefficients and their q-analogues, respectively. Both of them are specializations of the complete symmetric functions [12], that is

$$\begin{aligned} h_k(\underbrace{1, 1, \ldots, 1}_{n}) &= \binom{n+k-1}{k}, \\ h_k(1, q, \ldots, q^{n-1}) &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q. \end{aligned}$$

Thus, the following identities are special cases of Proposition 2.1, namely,

$$\det\left(\binom{s+i+j}{s+j}\right)_{0\le i,j\le n}=1,$$
$$\det\left(\begin{bmatrix}s+i+j\\ s+j\end{bmatrix}_q\right)_{0\le i,j\le n}=q^{\frac{1}{6}n(n+1)(2n+1)+s\binom{n+1}{2}},$$

where s and n be the nonnegative integers.

As further applications of Proposition 2.1, some results for the evaluations of the determinants of the matrices consisting of the numbers of the Stirling-type including the Stirling numbers of the second kind, the r-Whitney numbers of the second kind and their generalizations will be given in the third section. Our main idea follows from Merca's interesting works [12, 16]. Recently, Merca [12, 16] obtained some convolution formulas for the complete and elementary symmetric functions. By the convolutions, he discovered and proved many combinatorial identities involving r-Whitney numbers, Stirling numbers, binomial coefficients, Bernoulli numbers and some of their generalizations.

In order to prove Proposition 2.1, we need the following lemmas.

Lemma 2.1. *Let p, q and t be nonnegative integers. There holds*

$$h_{p+q-t}(x_0,x_1,\ldots,x_t)=\sum_{k=\max\{0,t-q\}}^{\min\{t,p\}}h_{p-k}(x_0,x_1,\ldots,x_k)h_{q-t+k}(x_k,x_{k+1},\ldots,x_t). \tag{5}$$

Proof. First of all, we claim that if $n\ge 0$, then

$$x^n=\sum_{k=0}^{n}h_{n-k}(x_0,x_1,\ldots,x_k)\prod_{i=0}^{k-1}(x-x_i). \tag{6}$$

We proceed by induction on n. Obviously, the equality holds for $n=0,1$. Assume that the equality holds for n, and let us prove it for $n+1$.

$$\begin{aligned}
x^{n+1}&=x\sum_{k=0}^{n}h_{n-k}(x_0,x_1,\ldots,x_k)\prod_{i=0}^{k-1}(x-x_i)\\
&=\sum_{k=0}^{n}h_{n-k}(x_0,x_1,\ldots,x_k)\prod_{i=0}^{k}(x-x_i)+\sum_{k=0}^{n}x_kh_{n-k}(x_0,x_1,\ldots,x_k)\prod_{i=0}^{k-1}(x-x_i)\\
&=\sum_{k=1}^{n+1}h_{n+1-k}(x_0,x_1,\ldots,x_{k-1})\prod_{i=0}^{k-1}(x-x_i)+\sum_{k=0}^{n}x_kh_{n-k}(x_0,x_1,\ldots,x_k)\prod_{i=0}^{k-1}(x-x_i)\\
&=x_0h_n(x_0)+\sum_{k=1}^{n}h_{n+1-k}(x_0,x_1,\ldots,x_k)\prod_{i=0}^{k-1}(x-x_i)+h_0(x_0,x_1,\ldots,x_n)\prod_{i=0}^{n}(x-x_i).
\end{aligned}$$

The last equality holds because of (3). Since $h_{n+1}(x_0)=x_0h_n(x_0)$ and $h_0(x_0,x_1,\ldots,x_n)=h_0(x_0,x_1,\ldots,x_{n+1})=1$,

(6) is true. Thus we have

$$\begin{aligned} x^{p+q} &= x^q \sum_{k=0}^{p} h_{p-k}(x_0, x_1, \ldots, x_k) \prod_{i=0}^{k-1}(x - x_i) \\ &= \sum_{k=0}^{p} h_{p-k}(x_0, x_1, \ldots, x_k) \prod_{i=0}^{k-1}(x - x_i) \sum_{j=0}^{q} h_{q-j}(x_k, x_{k+1}, \ldots, x_{k+j}) \prod_{i=0}^{j-1}(x - x_{k+i}) \\ &= \sum_{k=0}^{p} \sum_{j=0}^{q} h_{p-k}(x_0, x_1, \ldots, x_k) h_{q-j}(x_k, x_{k+1}, \ldots, x_{k+j}) \prod_{i=0}^{k+j-1}(x - x_i). \end{aligned}$$

Replacing $k + j$ by t yields

$$x^{p+q} = \sum_{t=0}^{p+q} \prod_{i=0}^{t-1}(x - x_i) \sum_{k=\max\{0,t-q\}}^{\min\{t,p\}} h_{p-k}(x_0, x_1, \ldots, x_k) h_{q-t+k}(x_k, x_{k+1}, \ldots, x_t). \tag{7}$$

On the other hand, it follows clearly from (6) that

$$x^{p+q} = \sum_{t=0}^{p+q} h_{p+q-t}(x_0, x_1, \ldots, x_t) \prod_{i=0}^{t-1}(x - x_i). \tag{8}$$

By (7) and (8) we have

$$\begin{aligned} &\sum_{t=0}^{p+q} h_{p+q-t}(x_0, x_1, \ldots, x_t) \prod_{i=0}^{t-1}(x - x_i) \\ &\quad = \sum_{t=0}^{p+q} \prod_{i=0}^{t-1}(x - x_i) \sum_{k=\max\{0,t-q\}}^{\min\{t,p\}} h_{p-k}(x_0, x_1, \ldots, x_k) h_{q-t+k}(x_k, x_{k+1}, \ldots, x_t). \end{aligned} \tag{9}$$

Equating the coefficients of $\prod_{i=0}^{t-1}(x - x_i)$ on both sides of (9) yields (5). □

By Lemma 2.1 we obtain the LU factorization of the matrix whose entries are the complete symmetric functions $h_i(x_0, x_1, \ldots, x_{s+j})$.

Lemma 2.2. *Let s and n be nonnegative integers. Then*

$$\begin{aligned} &\begin{pmatrix} h_0(x_0, x_1, \ldots, x_s) & h_0(x_0, x_1, \ldots, x_{s+1}) & \cdots & h_0(x_0, x_1, \ldots, x_{s+n}) \\ h_1(x_0, x_1, \ldots, x_s) & h_1(x_0, x_1, \ldots, x_{s+1}) & \cdots & h_1(x_0, x_1, \ldots, x_{s+n}) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(x_0, x_1, \ldots, x_s) & h_n(x_0, x_1, \ldots, x_{s+1}) & \cdots & h_n(x_0, x_1, \ldots, x_{s+n}) \end{pmatrix} \\ &= \begin{pmatrix} h_0(x_0, x_1, \ldots, x_s) & 0 & \cdots & 0 \\ h_1(x_0, x_1, \ldots, x_s) & h_0(x_0, x_1, \ldots, x_{s+1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_n(x_0, x_1, \ldots, x_s) & h_{n-1}(x_0, x_1, \ldots, x_{s+1}) & \cdots & h_0(x_0, x_1, \ldots, x_{s+n}) \end{pmatrix} \\ &\quad \times \begin{pmatrix} h_0(x_s) & h_0(x_s, x_{s+1}) & \cdots & h_0(x_s, x_{s+1}, \ldots, x_{s+n}) \\ 0 & h_1(x_{s+1}) & \cdots & h_1(x_{s+1}, x_{s+2}, \ldots, x_{s+n}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_n(x_{s+n}) \end{pmatrix}. \end{aligned} \tag{10}$$

Proof. By taking $p = s + i$, $q = j$ and $t = s + j$ in (5) we have

$$h_i(x_0, x_1, \ldots, x_{s+j}) = \sum_{k=s}^{s+\min\{i,j\}} h_{s+i-k}(x_0, x_1, \ldots, x_k) h_{k-s}(x_k, x_{k+1}, \ldots, x_{s+j}).$$

Replacing k by $s + k$ yields

$$h_i(x_0, x_1, \ldots, x_{s+j}) = \sum_{k=0}^{\min\{i,j\}} h_{i-k}(x_0, x_1, \ldots, x_{s+k}) h_k(x_{s+k}, x_{s+k+1}, \ldots, x_{s+j}),$$

which is equivalent to (10) and this completes the proof. □

It is worth noting that the above lemma will not only serve as a tool to prove Proposition 2.1 but it is also interesting on its own.
The proof of Proposition 2.1. Since $h_0(x_0, x_1, \ldots, x_i) = 1$ for $i \geq 0$ and $h_k(x_{s+k}) = x_{s+k}^k$ for $k \geq 1$, by (10) we obtain Proposition 2.1 directly.

3. Determinants involving the numbers of the Stirling-type

According to Proposition 2.1 we here give some results for the evaluations of the determinants of the matrices whose entries are the numbers of the Stirling-type.

3.1. *Stirling numbers of the second kind and their q-analogue, (p, q)-analogue*

Let $S(n, k)$ the Stirling number of the second kind [7]. It is well known that the following relation holds:

$$S(n, k) = h_{n-k}(0, 1, \ldots, k).$$

Thus, according to Proposition 2.1, we have

Theorem 3.1. *Let s and n be nonnegative integers. Then*

$$\det\left(S(s + i + j, s + j)\right)_{0 \leq i,j \leq n} = \prod_{k=1}^{n} (s + k)^k. \tag{11}$$

It is not difficult to obtain the following relation:

$$\widetilde{S}_q(n, k) = h_{n-k}([0]_q, [1]_q, \ldots, [k]_q).$$

Thus, by Proposition 2.1 we recover the following determinant identity involving the q-Stirling numbers of the second kind $\widetilde{S}_q(s + i + j, s + j)$.

Theorem 3.2 ([9]). *Let s and n be nonnegative integers. Then*

$$\det(\widetilde{S}_q(s + i + j, s + j))_{0 \leq i,j \leq n} = \prod_{k=1}^{n} [s + k]_q^k. \tag{12}$$

Since $S_q(n, k) = q^{\binom{k}{2}} \widetilde{S}_q(n, k)$ and $\binom{s+i}{2} = \binom{s+i+1}{3} - \binom{s+i}{3}$, it is easy to obtain the following:

Theorem 3.3 ([3]). *Let s and n be nonnegative integers. Then*

$$\det(S_q(s + i + j, s + j))_{0 \leq i,j \leq n} = q^{\binom{s+n+1}{3} - \binom{s}{3}} \prod_{k=1}^{n} [s + k]_q^k. \tag{13}$$

Let $S_{p,q}(n,k)$ be the (p,q)-Stirling numbers of the second kind [27] which satisfy

$$S_{p,q}(n,k) = p^{k-1}S_{p,q}(n-1,k-1) + [k]_{p,q}S_{p,q}(n-1,k),$$

where $[k]_{p,q} = p^{k-1} + p^{k-2}q + \cdots + pq^{k-2} + q^{k-1}$. Similarly, we have obtain the following relation:

$$S_{p,q}(n,k) = p^{\binom{k}{2}}h_{n-k}([0]_{p,q}, [1]_{p,q}, \ldots, [k]_{p,q}).$$

Therefore, it is clear that Proposition 2.1 allows us to obtain a new identity for the (p,q)-Stirling numbers of the second kind.

Theorem 3.4. *Let s and n be nonnegative integers. Then*

$$\det(S_{p,q}(s+i+j, s+j))_{0\le i,j\le n} = p^{\binom{s+n+1}{3}-\binom{s}{3}} \prod_{k=1}^{n} [s+k]_{p,q}^{k}. \tag{14}$$

3.2. Legendre-Stirling numbers and Jacobi-Stirling numbers of the second kind

Let $JS_n^{(k)}(z)$ be the Jacobi-Stirling numbers of the second kind [2, 10, 15, 22]. It was shown that the $JS_n^{(k)}(z)$ are determined by the recurrence

$$JS_n^{(k)}(z) = JS_{n-1}^{(k-1)}(z) + k(k+z)JS_{n-1}^{(k)}(z),$$

with initial conditions

$$JS_n^{(0)}(z) = JS_0^{(k)}(z) = 0, \quad JS_0^{(0)}(z) = 1.$$

Equivalently, they are determined by the identity

$$x^n = \sum_{k=0}^{n} JS_n^{(k)}(z) \prod_{i=0}^{k-1} (x - i(i+z)).$$

Mongelli [23] has shown that the Jacobi-Stirling numbers of the second kind are specializations of the complete symmetric functions, that is

$$JS_n^{(k)}(z) = h_{n-k}(0, 1+z, \ldots, k(k+z)).$$

Using the above relation, we have

Theorem 3.5. *Let s and n be nonnegative integers. Then*

$$\det\left(JS_{s+i+j}^{(s+j)}(z)\right)_{0\le i,j\le n} = \prod_{k=1}^{n} (s+k)^k (s+k+z)^k. \tag{15}$$

In particular, when $z = 1$, the Jacobi-Stirling numbers of the second kind $JS_n^{(k)}(z)$ reduce to the Legendre-Stirling numbers of the second kind [1] denoted by $PS_n^{(k)}$. Therefore, it is natural that we have the following theorem.

Theorem 3.6. *Let s and n be nonnegative integers. Then*

$$\det\left(PS_{s+i+j}^{(s+j)}\right)_{0\le i,j\le n} = (s+n+1)^n \prod_{k=1}^{n} (s+k)^{2k-1}. \tag{16}$$

3.3. r-Whitney numbers of the second kind and their (p, q)-analogue

Let $W(n,k;m,r)$ be the r-Whitney numbers of the second kind [6, 8, 13, 14, 17–20, 28]. Then we have the following identity [16]:

$$W(n,k;m,r) = h_{n-k}(r, m+r, \ldots, mk+r).$$

This relation leads to a determinant identity involving the r-Whitney numbers of the second kind due to Xu and Zhou [28].

Theorem 3.7 ([28]). *Let s and n be nonnegative integers. Then*

$$\det\left(W(s+i+j, s+j; m, r)\right)_{0\le i,j\le n} = \prod_{k=1}^{n}(m(s+k)+r)^k. \tag{17}$$

In particular, when $r = 1$, we can easily obtain the evaluations of the similar determinants involving the Whitney numbers of second kind.

Let $W_{p,q}(n,k;m,r)$ be a (p,q)-generalization of the r-Whitney numbers of the second kind [11, 24, 25]. It was shown in [11] that the $W_{p,q}(n,k;m,r)$ are determined by the identity

$$(mx+[r]_p)^n = \sum_{k=0}^{n} m^k W_{p,q}(n,k;m,r)[x]_q^k,$$

where the falling factorial

$$[x]_q^i = \begin{cases} x(x-[1]_q)\cdots(x-[i-1]_q), & i \ge 1, \\ 1, & i = 0. \end{cases}$$

Equivalently, they are determined by the generating function

$$\sum_{n\ge k} W_{p,q}(n,k;m,r)x^n = \frac{x^k}{(1-([r]_p+m[0]_q)x)\cdots(1-([r]_p+m[k]_q)x)}, \quad k \ge 0,$$

which leads to an explicit formula for the $W_{p,q}(n,k;m,r)$:

$$W_{p,q}(n,k;m,r) = \frac{1}{m^k[k]_q!}\sum_{j=0}^{k}(-1)^{k-j}q^{-\binom{j}{2}-j(k-j)}\begin{bmatrix}k\\j\end{bmatrix}_q (m[j]_q+[r]_p)^n.$$

It is easy to obtain the following relation:

$$W_{p,q}(n,k;m,r) = h_{n-k}([r]_p, m[1]_q+[r]_p, \ldots, m[k]_q+[r]_p). \tag{18}$$

By (18) and Proposition 2.1, a determinant identity involving the (p,q)-generalization of the r-Whitney numbers of the second kind can be derived.

Theorem 3.8. *Let s and n be nonnegative integers. Then*

$$\det(W_{p,q}(s+i+j, s+j; m, r))_{0\le i,j\le n} = \prod_{k=1}^{n}(m[s+k]_q+[r]_p)^k. \tag{19}$$

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