



Convex Optimization of Interval Valued Functions on Mixed Domains

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Abstract. In this paper, we study a class of convex type interval-valued functions on the domain of the product of closed subsets of real numbers. By considering LW order relation on the class of closed intervals, we proposed some optimal solutions. LW convexity concepts and generalized Hukuhara differentiability (viz. delta and nabla) for interval-valued functions yield the necessary and sufficient conditions for interval programming problem. In addition, we compare our results with the results given in the literature. These results may open a new avenue for modeling and solve a different type of optimization problems that involve both discrete and continuous variables at the same time.

1. Introduction

The uncertainty of parameters is predominantly modeled using fuzzy, stochastic, grey/inexact programming approaches and hybrid of all these approaches provide insight of these approaches with representative literature, specific advantages and limitations, see for example [2, 3, 6–9, 14, 15, 19]. Interval analysis is a particular case and it has relevant applications in the treatment of the uncertainty that appears in the modeling of some real-world problems [7, 10, 11, 13, 17, 20]. In this direction, recently Yadav et al. [20] presented interval-valued facility location model.

Discrete and continuous analysis and optimizations are closely related, yet they are usually treated separately. Convex optimization on mixed domains have been investigated by Adivar et al. [1]. Recently, Lupulescu in [12] develop calculus for interval-valued functions on time scales, using the concept of generalized Hukuhara difference provided by Stefanini and Bede [18].

A simultaneous presentation of two theories under the umbrella of time scales might provide a new perspective and easiness for modeling and solving optimization of interval-valued functions on general domain

In the main part of this paper, we develop a convex analysis for interval-valued functions on mixed domains and we consider new type of order relation to study theoretical and practical solution method for interval-valued objective functions considering LW order relationship between two closed intervals in \mathbb{R} . The results are illustrated with number of examples.

2. Definitions, notations and prerequisites

For a self-contained presentation of our study, we recall briefly the necessary background material.

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2.1. Calculus on time scale

Let \mathbb{T} be a time scale. As usual, for $t \in \mathbb{T} \subset \mathbb{R}$, $\sigma(t) := \inf \{s \in \mathbb{T} : t < s\}$, $\rho(t) := \sup \{s \in \mathbb{T} : t > s\}$, $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$ defined, which denote its forward jump operator, backward jump operator, forward graininess function and backward graininess function respectively.

A point $t \in \mathbb{T}$, is called right-scattered (right-dense), left-scattered (left-dense) and isolated (dense), if $\sigma(t) > t$ ($\sigma(t) = t$), $\rho(t) < t$ ($\rho(t) = t$), and $\sigma(t) > t > \rho(t)$ ($\sigma(t) = t = \rho(t)$) respectively.

Two sets \mathbb{T}_k and \mathbb{T}^k are derived from a time scale \mathbb{T} : if \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} - \{m\}$ and if \mathbb{T} has a left-scattered maximum M , then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}_k = \mathbb{T}$.

For a function $f : \mathbb{T} \rightarrow \mathbb{R}$, the delta derivative f^Δ is defined at a point $t \in \mathbb{T}^k$ by

$$f^\Delta(t) = \lim_{\substack{s \rightarrow t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

and nabla derivative f^∇ is defined at a point $t \in \mathbb{T}_k$ by

$$f^\nabla(t) = \lim_{\substack{s \rightarrow t \\ s \neq \rho(t)}} \frac{f(\rho(t)) - f(s)}{\rho(t) - s}.$$

Hereafter, we use the notation Λ^n to denote the product $\mathbb{T}_1 \times \mathbb{T}_2 \times \dots \times \mathbb{T}_n$ of the time scales.

A set S in Λ^n , is called convex in Λ^n , if $\sum_{i=1}^m \lambda_i x^i \in S$ for all $x^1, x^2, \dots, x^m \in S$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in [0, 1]$ such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i x^i \in \Lambda^n$.

As usual, for a convex set $S \subset \Lambda^n$, $ccl_{\Lambda^n}(S) = \overline{\text{conv}_{\mathbb{R}^n}(S)} \cap \Lambda^n$, $cint_{\Lambda^n}(S) = \widehat{\text{conv}_{\mathbb{R}^n}(S)} \cap \Lambda^n$ and $cbdy_{\Lambda^n}(S) = \partial \text{conv}_{\mathbb{R}^n}(S) \cap \Lambda^n$ defined, which denote the convex-closure, convex-interior and convex-boundary respectively, where $\overline{\text{conv}_{\mathbb{R}^n}(S)}$, $\widehat{\text{conv}_{\mathbb{R}^n}(S)}$, and $\partial \text{conv}_{\mathbb{R}^n}(S)$ indicates the closure, interior and boundary of convex hull $\text{conv}_{\mathbb{R}^n}(S)$ of S in \mathbb{R}^n , respectively.

For any convex set S in Λ^n , a function $f : S \rightarrow \mathbb{R}$ is said to be convex on time scales if $f(\sum_{i=1}^m \lambda_i x^i) \leq \sum_{i=1}^m \lambda_i f(x^i)$. Clearly, this generalizes the inequality

$$f(a + \lambda(b - a)) \leq f(a) + \lambda(f(b) - f(a))$$

for all $a, b \in S$ and $\lambda \in [0, 1]$ such that $a + \lambda(b - a) \in \Lambda^n$. However, the converse of this statement may not be true in multidimensional case (see e.g., [1, Example 7 and Remark 3]).

A point ζ in \mathbb{R}^n is called the subgradient of f at $\bar{x} \in S$ if

$$f(x) \geq f(\bar{x}) + \langle x - \bar{x}, \zeta \rangle, \text{ for all } x \in S,$$

where $\langle \cdot, \cdot \rangle$ denote dot product on \mathbb{R}^n . If for every point $\bar{x} \in cint_{\Lambda^n}(S)$, there exists a subgradient vector ζ such that $f(x) \geq f(\bar{x}) + \langle x - \bar{x}, \zeta \rangle$ for all $x \in S$, then f is convex on $cint_{\Lambda^n}(S)$.

Moreover, subgradient may not be unique for a convex function defined on arbitrary time scales (see e.g., [1, Theorem 13]).

For a function $f : \Lambda^n \rightarrow \mathbb{R}$, the partial delta derivative $\frac{\partial f(x)}{\Delta_i x_i}$ with respect to $x_i \in \mathbb{T}_i^k$ is defined by

$$\frac{\partial f(x)}{\Delta_i x_i} = \lim_{\substack{s_i \rightarrow x_i \\ s_i \neq \sigma_i(x_i)}} \frac{f(x_1, \dots, \sigma_i(x_i), \dots, x_n) - f(x_1, \dots, s_i, \dots, x_n)}{\sigma_i(x_i) - s_i}$$

and partial nabla derivative $\frac{\partial f(x)}{\nabla_i x_i}$ is defined at a point $x_i \in (\mathbb{T}_i)_k$ by

$$\frac{\partial f(x)}{\nabla_i x_i} = \lim_{\substack{s_i \rightarrow x_i \\ s_i \neq \rho_i(x_i)}} \frac{f(x_1, \dots, \rho_i(x_i), \dots, x_n) - f(x_1, \dots, s_i, \dots, x_n)}{\rho_i(x_i) - s_i}.$$

For a point $x \in \Lambda^n$, a frame $F_x := \cup_{i=1}^n B_i(x, h_i^\pm)$, where

$$B_i(x, h_i^\pm) = \left\{ se^i + \sum_{\substack{j=1 \\ j \neq i}}^n x_j e^j : s \in N_i^\pm(x, h_i^\pm) \right\}$$

and

$$N_i^+(x, h_i^+) = \begin{cases} \{x_i, \sigma_i(x_i)\} & \text{if } \mu_i(x_i) > 0 \\ [x_i, x_i + h_i^+] & \text{if } \mu_i(x_i) = 0 \end{cases}$$

$$N_i^-(x, h_i^-) = \begin{cases} \{x_i, \rho_i(x_i)\} & \text{if } \nu_i(x_i) > 0 \\ (x_i - h_i^-, x_i] & \text{if } \nu_i(x_i) = 0 \end{cases}$$

are the open balls and neighborhoods at point $x \in \Lambda^n$. For the existence of partial derivatives at a point $x \in S$ one has to assume that $F_x \subset S$, which also implies that $x \in \text{cint}_{\Lambda^n}(S)$.

Lemma 2.1. [1, Theorem 14] Let S be a nonempty convex set in Λ^n . Let $f : S \rightarrow \mathbb{R}$ has all the partial derivatives $\frac{\partial f(x)}{\Delta_i x_i}$ and $\frac{\partial f(x)}{\nabla_i x_i}$ at $\bar{x} \in \text{cint}_{\Lambda^n}(S)$ with $F_x \subset S$.

1. If f is convex on S , then there exist scalars $\lambda_i(\bar{x}) \in [0, 1]$, such that a vector

$$\zeta(\bar{x}) = \sum_{i=1}^n \left(\lambda_i(\bar{x}) \frac{\partial f(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i(\bar{x})) \frac{\partial f(x)}{\nabla_i x_i} \right) e^i$$

is a subgradient of f at $\bar{x} \in \text{cint}_{\Lambda^n}(S)$ with $F_x \subset S$.

2. Suppose that $\text{cint}_{\Lambda^n}(S) = \{\bar{x} \in S : F_{\bar{x}} \subset S\}$. Then f is convex on $\text{cint}_{\Lambda^n}(S)$ provided that $\zeta(\bar{x})$ is subgradient of f at \bar{x} .

Lemma 2.2. [1, Theorem 15] If a convex function $f : S \rightarrow \mathbb{R}$ has all the partial derivatives $\frac{\partial f(x)}{\Delta_i x_i}$ and $\frac{\partial f(x)}{\nabla_i x_i}$ at $\bar{x} \in \text{cint}_{\Lambda^n}(S)$ with $F_x \subset S$. Then $f(x) \geq f(\bar{x})$ for all $x \in S$ if and only, if there exist scalars $\lambda_i(\bar{x}) \in [0, 1]$, such that a vector

$$\zeta(\bar{x}) = \sum_{i=1}^n \left(\lambda_i(\bar{x}) \frac{\partial f(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i(\bar{x})) \frac{\partial f(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e^i$$

is zero subgradient for f at \bar{x} .

2.2. Interval analysis on time scales

Let I_c be the set of all nonempty compact intervals of \mathbb{R} . As usual, for $A, B \in I_c$ such that $A = [a^-, a^+]$, $B = [b^-, b^+]$ and $\lambda \in \mathbb{R}$, $A + B = [a^- + b^-, a^+ + b^+]$,

$$\lambda A = \begin{cases} [\lambda a^-, \lambda a^+] & \text{if } \lambda \geq 0, \\ [\lambda a^+, \lambda a^-] & \text{if } \lambda < 0; \end{cases}$$

and $A \oplus_g B = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}]$ defined, which denote *Minkowski addition*, *scalar multiplication* and *generalized Hukuhara difference* of two intervals, respectively.

Also, let H denote the *Pompeiu-Hausdorff distance* between two compact sets, and in particular between two closed intervals $A, B \in I_c$ is as follows

$$H(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

An interval-valued function $F : \mathbb{T} \rightarrow I_c$ such that $F(t) = [f^-(t), f^+(t)]$ has a \mathbb{T} -limit $A \in I_c$ at $t_0 \in \mathbb{T}$ if for every $\varepsilon > 0$, there exist $\delta > 0$ such that $H(F(t) \ominus_g A, \{0\}) \leq \varepsilon$ for all $t \in U_{\mathbb{T}}(t_0, \delta) := (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$. If limit A exists then it must be unique and is denoted by $\lim_{t \rightarrow t_0} F(t)$. Moreover, if $\lim_{t \rightarrow t_0} F(t) \in I_c$ exists and $\lim_{t \rightarrow t_0} F(t) = F(t_0)$, then F is called *continuous at $t_0 \in \mathbb{T}$* .

For an interval-valued function $F : \mathbb{T} \rightarrow I_c$, the *delta gH-derivative* F^Δ , is defined at a point $t \in \mathbb{T}^k$ by

$$F^\Delta(t) = \lim_{\substack{s \rightarrow t \\ s \neq \sigma(t)}} \frac{F(\sigma(t)) \ominus_g F(s)}{\sigma(t) - s},$$

and the *nabla gH-derivative* F^∇ , is defined at a point $t \in \mathbb{T}_k$

$$F^\nabla(t) = \lim_{\substack{s \rightarrow t \\ s \neq \rho(t)}} \frac{F(\rho(t)) \ominus_g F(s)}{\rho(t) - s}.$$

provided limit exists.

If the real-valued functions f^- and f^+ are delta differentiable (resp., nabla differentiable) at $t_0 \in \mathbb{T}^k$ ($t_0 \in \mathbb{T}_k$), then interval-valued function F is delta *gH-differentiable* (resp., nabla *gH-differentiable*) and

$$F^\Delta(t_0) = \left[\min\{(f^-)^\Delta(t_0), (f^+)^\Delta(t_0)\}, \max\{(f^-)^\Delta(t_0), (f^+)^\Delta(t_0)\} \right]$$

$$\left(\text{resp., } F^\nabla(t_0) = \left[\min\{(f^-)^\nabla(t_0), (f^+)^\nabla(t_0)\}, \max\{(f^-)^\nabla(t_0), (f^+)^\nabla(t_0)\} \right], \right)$$

but the converse is not true. However, under the *l-monotonicity* (i.e., $t \mapsto \text{len}(F(t)) := (f^+ - f^-)(t)$ is monotone) converse condition holds provided that $F^\Delta(t_0)$ (resp., $F^\nabla(t_0)$) exists.

For a fixed $x^* = (x_1^*, x_2^*, \dots, x_n^*) \in \Lambda^n$, let $h_i : \mathbb{T}_i \rightarrow I_c$ such that $h_i(x_i) := F(x_1^*, x_2^*, \dots, x_{i-1}^*, x_i, x_{i+1}^*, \dots, x_n^*)$. If h_i is delta (resp., nabla) *gH-differentiable* at x_i^* , then we say that F has the *ith partial delta* (resp., *nabla*) *gH-derivative at x^** and denoted by $\frac{\partial F}{\Delta_i x_i}$ (resp., $\frac{\partial F}{\nabla_i x_i}$). Moreover, F is *continuously delta* (resp., *nabla*) *gH-differentiable at x^** , if all the partial delta (resp., nabla) *gH-derivatives* $\frac{\partial F}{\Delta_i x_i}$ (resp., $\frac{\partial F}{\nabla_i x_i}$) exists on some neighborhood of x^* and are continuous at x^* (in the sense of interval valued function).

Since I_c is not totally order set. To compare the images of interval-valued functions in the context of optimization problems, several partial order relations exist in I_c , which is summarized as below.

For $A, B \in I_c$, such that $A = [a^-, a^+]$, $B = [b^-, b^+]$, we say that:

1. $A \leq_{LU} B$ (or $A \leq_{LR} B$), if and only if $a^- \leq b^-$ and $a^+ \leq b^+$, $A <_{LU} B$ if $A \leq_{LU} B$ and $A \neq B$.
2. $A \leq_{LC} B$ if and only if $a^- \leq b^-$ and $m(A) \leq m(B)$, $A <_{LC} B$ if $A \leq_{LC} B$ and $A \neq B$, where $m(A) = \frac{a^- + a^+}{2}$.
3. $A \leq_{UC} B$ if and only if $a^+ \leq b^+$ and $m(A) \leq m(B)$, $A <_{UC} B$ if $A \leq_{UC} B$ and $A \neq B$.
4. $A \leq_{CW} B$ if and only if $m(A) \leq m(B)$ and $w(A) \leq w(B)$, $A <_{CW} B$ if $A \leq_{CW} B$ and $A \neq B$, where $w(A) = a^+ - a^-$.
5. $A \leq_{LW} B$ if and only if $a^- \leq b^-$ and $w(A) \leq w(B)$, $A <_{LW} B$ if $A \leq_{LW} B$ and $A \neq B$.
6. $A \leq_{UW} B$ if and only if $a^+ \leq b^+$ and $w(A) \leq w(B)$, $A <_{UW} B$ if $A \leq_{UW} B$ and $A \neq B$.

Let $\leq_* \in \mathbb{P} = \{\leq_{LU}, \leq_{LC}, \leq_{UC}, \leq_{CW}, \leq_{LW}, \leq_{UW}\}$ be a special set of partial orders on I_c . Then an interval-valued function $F : X \subset \mathbb{R}^n \rightarrow I_c$ is called \leq_* -convex at x^* , if $F(\lambda x^* + (1 - \lambda)x) \leq_* \lambda F(x^*) + (1 - \lambda)F(x)$, $\lambda \in (0, 1)$ and $x \in X$.

3. Convexity of interval-valued functions on Λ^n

Now we turn our attention to the convex functions and its properties on a convex set in Λ^n .

Definition 3.1. Let S be a convex set in Λ^n and $\leq_* \in \mathbb{P}$. Then $F : S \rightarrow I_c$ is said to be \leq_* -convex interval-valued function at $x_i^* \in S$ if and only if

$$F\left(\sum_{\substack{j=1 \\ i \neq j}}^m \lambda_j x_j + \lambda_i x_i^*\right) \leq_* \sum_{\substack{j=1 \\ i \neq j}}^m \lambda_j F(x_j) + \lambda_i F(x_i^*) \tag{1}$$

for all $x_j \in S, j = 1, 2, \dots, m$, and $\lambda_j \in [0, 1], j = 1, 2, \dots, m$, such that $\sum_{j=1}^m \lambda_j = 1$ and $\sum_{j=1}^m \lambda_j x_j \in \Lambda^n$.

Clearly, inequality (1), implies that, for a \leq_* -convex function $F : S \subset \Lambda^n \rightarrow I_c$, it follows that

$$F(\lambda_1 x_1^* + \lambda_2 x_2) \leq_* \lambda_1 F(x_1^*) + \lambda_2 F(x_2) \tag{2}$$

holds for all $x_1^*, x_2 \in S$, and $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 x_1^* + \lambda_2 x_2 \in \Lambda^n$. However, the converse of this statement may not be true. For example, examine the following example:

Example 3.2. For $\Lambda^n = \mathbb{Z} \times \mathbb{Z}; S = \{(-1, 0), (0, -1), (0, 0), (0, 1), (1, 0), (1, 1)\}$ and $\leq_* = \leq_{LU}$, let $F : S \rightarrow I_c$ as follows: $F(-1, 0) = F(0, -1) = F(0, 1) = F(1, 0) = [1, 2], F(0, 0) = [0, 2]$ and $F(1, 1) = [-3, 2]$. It is easy to see that F is LU -convex if and only if, f^- and f^+ are convex functions on time domain S . However, f^- is not convex on S because, in the case when $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, we have,

$$-1 + \frac{2}{3} < f^-\left(\frac{1}{3}(-1, 0) + \frac{1}{3}(0, -1) + \frac{1}{3}(1, 1)\right) = f^-(0, 0) = 0.$$

On the other hand, the inequality (2) is satisfied for all $x_1^*, x_2 \in S$, and $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 x_1^* + \lambda_2 x_2 \in \mathbb{Z} \times \mathbb{Z}$.

Corollary 3.3. Let S be a convex set in Λ^n . If the interval-valued function $F : \mathbb{R}^n \rightarrow I_c$ is \leq_* -convex on $conv_{\mathbb{R}^n}(S)$, then the restricted function $\hat{F} := F|_S$ is \leq_* -convex on S .

Lemma 3.4. Let $\mathbb{P}_1 := \{\leq_{LU}, \leq_{LC}, \leq_{UC}, \leq_{CW}, \leq_{UW}\}$. If $A \leq_{LW} B$, then $A \leq_* B$ for all $\leq_* \in \mathbb{P}_1$.

Proof. For $A, B \in I_c$, such that $A = [a^-, a^+], B = [b^-, b^+]$, we have $a^- \leq b^-$ and $a^+ - a^- \leq b^+ - b^-$. By adding these two inequalities, it follows that $a^+ \leq b^+$ and furthermore, $m(A) \leq m(B)$. Hence $A \leq_* B$, for all $\leq_* \in \mathbb{P}_1$. \square

Lemma 3.5. Let $\mathbb{P}_2 := \{\leq_{UC}, \leq_{UW}\}$. If $A \leq_{CW} B$, then $A \leq_* B$ for all $\leq_* \in \mathbb{P}_2$.

Proof. For $A, B \in I_c$, such that $A = [a^-, a^+], B = [b^-, b^+]$, we have $a^- + a^+ \leq b^- + b^+$ and $a^+ - a^- \leq b^+ - b^-$. By adding these two inequalities, it follows that $a^+ \leq b^+$. Hence $A \leq_* B$, for all $\leq_* \in \mathbb{P}_2$. \square

Lemma 3.6. Let $A, B, C \in I_c$. If $A \leq_{LW} B$ and $len(A) \geq len(C)$, then $A \Theta_g C \leq_{LW} B \Theta_g C$.

Proof. For $A, B, C \in I_c$ such that $A = [a^-, a^+], B = [b^-, b^+]$ and $C = [c^-, c^+]$, we have $a^- \leq b^-$ and $a^+ - a^- \leq b^+ - b^-$. Since $len(A) \geq len(C)$, moreover $len(B) \geq len(A) \geq len(C)$, it follows that $A \Theta_g C = [a^- - c^-, a^+ - c^+]$ and $B \Theta_g C = [b^- - c^-, b^+ - c^+]$. By using the fact $a^- \leq b^-$ and $a^+ - a^- \leq b^+ - b^-$ implies that $a^- - c^- \leq b^- - c^-$ and $a^+ - a^- - (c^+ - c^-) \leq b^+ - b^- - (c^+ - c^-)$. Hence, we obtain that $A \Theta_g C \leq_{LW} B \Theta_g C$. \square

The following corollaries are direct implications of Lemma 3.4 and 3.5.

Corollary 3.7. If $A \leq_{LU} B$, then $A \leq_{LC} B$ and $A \leq_{UC} B$.

Corollary 3.8. If $A \leq_{CW} B$, then $A \leq_{UC} B$ and $A \leq_{UW} B$.

Corollary 3.9. If $A \leq_{UW} B$, then $A \leq_{UC} B$.

However, the converse of above implications may not be true. To illustrate this we give the following examples:

Example 3.10. For $A = [1, 4]$, and $B = [3, 5]$, $A \leq_{LU} B$, but $A \not\leq_{CW} B$, $A \not\leq_{LW} B$ and $A \not\leq_{UW} B$.

If $A = [1, 4]$, and $B = [3, 3.5]$, then $A \leq_{LC} B$, but $A \not\leq_* B$ for all $\{\leq_{LU}, \leq_{LW}, \leq_{UC}, \leq_{CW}, \leq_{UW}\}$.

$[1, 2] \leq_{UC} [\frac{1}{2}, 4]$, but $[1, 2] \not\leq_{LU} [\frac{1}{2}, 4]$ and $[1, 2] \not\leq_{LC} [\frac{1}{2}, 4]$, furthermore, $[2, \frac{7}{2}] \leq_{UC} [3, 4]$, $[2, \frac{7}{2}] \not\leq_* [3, 4]$, for all $\{\leq_{LW}, \leq_{CW}, \leq_{UW}\}$.

Moreover, for $A = [1, 2]$, and $B = [\frac{1}{2}, 5]$, $A \leq_{CW} B$, $A \not\leq_{LU} B$, $A \not\leq_{LC} B$ and $A \not\leq_{LW} B$.

Finally, let $A = [3, 4]$, and $B = [\frac{1}{2}, 5]$, then $A \leq_{UW} B$, $A \not\leq_{LU} B$, $A \not\leq_{LC} B$, $A \not\leq_{LW} B$ and $A \not\leq_{CW} B$.

Remark 3.11. For Lemma 3.4 and 3.5, the related corollaries also holds for \leq_* -convex interval-valued functions as defined in Definition 3.1. Moreover, from Lemma 3.4, it follows that if $F : S \rightarrow I_c$ is \leq_{LW} -convex, then F is \leq_* -convex for all $\leq_* \in \mathbb{P}_1$, but converse may not true in some cases as mentioned in Example 3.10. However, class of \leq_{LW} -convex interval-valued functions covers all other convex function classes.

Let $F : [a, b]_{\mathbb{T}} \rightarrow I_c$ be a \leq_{LW} -convex, then the inequality (2) can be written as

$$F(\lambda x + (1 - \lambda)y) \leq_{LW} \lambda F(x) + (1 - \lambda)F(y) \tag{3}$$

for all $a \leq x < y \leq b$ and $\lambda \in [0, 1]$.

Now onward, let us assume that $F : S \subset \Lambda^n \rightarrow I_c$ is interval-valued \leq_{LW} -convex function, then Definition 3.1 implies that f^- and $lenF$ are real-valued convex functions on S , and vice versa.

Theorem 3.12. For a nonempty convex set S in Λ^n . Let $F : S \rightarrow I_c$ be an interval-valued function such that the partial derivatives $\frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}}$, $\frac{\partial f^+(x)}{\Delta_i x_i} \Big|_{x=\bar{x}}$, $\frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$ and $\frac{\partial f^+(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$, $i = 1, 2, \dots, n$, exist at any point $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in cint_{\Lambda^n}(S)$ satisfying $F_x \subset S$.

If F is interval-valued \leq_{LW} -convex function on S , then there exist scalars $\lambda_i^-(\bar{x}), \lambda_i^l(\bar{x}) \in [0, 1]$, $i = 1, 2, \dots, n$, such that the vectors

$$\xi^-(\bar{x}) = \sum_{i=1}^n \left(\lambda_i^-(\bar{x}) \frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i^-(\bar{x})) \frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e_i \tag{4}$$

and

$$\xi^l(\bar{x}) = \sum_{i=1}^n \left(\lambda_i^l(\bar{x}) \frac{\partial lenF(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i^l(\bar{x})) \frac{\partial lenF(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e_i \tag{5}$$

are the subgradients of f^- and $lenF$ at any point $\bar{x} \in cint_{\Lambda^n}(S)$ satisfying $F_x \subset S$, i.e.,

$$f^-(x) \geq f^-(\bar{x}) + \xi^-(\bar{x})^T(x - \bar{x}) \text{ for all } x \in S \tag{6}$$

and

$$lenF(x) \geq lenF(\bar{x}) + \xi^l(\bar{x})^T(x - \bar{x}) \text{ for all } x \in S. \tag{7}$$

Proof. From \leq_{LW} convexity of F on S , it implies that f^- and $lenF$ are real-valued convex on S , then by using Lemma 2.1 there exist scalars $\lambda_i^-(\bar{x}), \lambda_i^l(\bar{x}) \in [0, 1]$, $i = 1, 2, \dots, n$, such that the vectors

$$\xi^-(\bar{x}) = \sum_{i=1}^n \left(\lambda_i^-(\bar{x}) \frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i^-(\bar{x})) \frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e_i \tag{8}$$

and

$$\xi^l(\bar{x}) = \sum_{i=1}^n \left(\lambda_i^l(\bar{x}) \frac{\partial lenF(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i^l(\bar{x})) \frac{\partial lenF(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e_i$$

are the subgradients of f^- and $lenF$ at any point $\bar{x} \in cint_{\Lambda^n}(S)$ satisfying $F_x \subset S$. This completes the proof. \square

Remark 3.13. For a LW-convex interval-valued function $F : S \rightarrow I_c$, such that $F(x) = [f^-(x), f^+(x)]$, the vectors

$$\text{grad}_{\Delta} f^-(\bar{x}) = \sum_{i=1}^n \left. \frac{\partial f^-(x)}{\Delta_i x_i} \right|_{x=\bar{x}} e_i \tag{9}$$

$$\text{grad}_{\nabla} f^-(\bar{x}) = \sum_{i=1}^n \left. \frac{\partial f^-(x)}{\nabla_i x_i} \right|_{x=\bar{x}} e_i \tag{10}$$

$$\text{grad}_{\Delta} \text{len}F(\bar{x}) = \sum_{i=1}^n \left. \frac{\partial \text{len}F(x)}{\Delta_i x_i} \right|_{x=\bar{x}} e_i \tag{11}$$

and

$$\text{grad}_{\nabla} \text{len}F(\bar{x}) = \sum_{i=1}^n \left. \frac{\partial \text{len}F(x)}{\nabla_i x_i} \right|_{x=\bar{x}} e_i \tag{12}$$

may not be subgradient of f^- and $\text{len}F$ at a point satisfying $F_x \subset S$. To see this, one may consider the following interval-valued function $F : \mathbb{Z} \times \mathbb{Z} \rightarrow I_c$ such that

$$F(x) = \begin{cases} \left[(x_1 - x_2 - 1/2)^2, (x_1 - x_2)^2 \right], & \text{if } x_1 < x_2 \\ \left[(x_1 - x_2)^2, (x_1 - x_2 - 1/2)^2 \right], & \text{if } x_1 \geq x_2. \end{cases}$$

It is easy to see that

$$\text{len}F(x) = \begin{cases} x_1 - x_2 - 1/4, & \text{if } x_1 < x_2 \\ x_2 - x_1 + 1/4, & \text{if } x_1 \geq x_2. \end{cases}$$

Therefore, F is LW-convex function. Moreover

$$\text{grad}_{\Delta} f^-(\bar{x})^T = \begin{cases} (2x_1 - 2x_2, 2x_2 - 2x_1 + 2), & \text{if } x_1 < x_2 \\ (2x_1 - 2x_2 + 1, 2x_2 - 2x_1 + 1), & \text{if } x_1 \leq x_2; \end{cases}$$

$$\text{grad}_{\nabla} f^-(\bar{x})^T = \begin{cases} (2x_1 - 2x_2 - 2, 2x_2 - 2x_1), & \text{if } x_1 < x_2 \\ (2x_1 - 2x_2 - 1, 2x_2 - 2x_1 - 1), & \text{if } x_1 \geq x_2; \end{cases}$$

and

$$\text{grad}_{\Delta} \text{len}F(\bar{x})^T = \text{grad}_{\nabla} \text{len}F(\bar{x})^T = \begin{cases} (1, -1), & \text{if } x_1 < x_2 \\ (-1, 1), & \text{if } x_1 \geq x_2. \end{cases}$$

It is easy to see that $\text{grad}_{\nabla} \text{len}F(\bar{x})$ is subgradient for $\text{len}F$ at the point $(0,0)$, however, neither $\text{grad}_{\Delta} f^-(\bar{x})$ nor $\text{grad}_{\nabla} f^-(\bar{x})$ is subgradient for f^- at origin. On the other hand, for $\lambda = 1$ and $\check{\lambda} = 0$, the vector

$$\begin{aligned} \xi^-(x_1, x_2) &= (\lambda(2x_1 - 2x_2) + (1 - \lambda)(2x_2 - 2x_1 + 2))e_1 \\ &\quad + (\check{\lambda}(2x_1 - 2x_2 - 2) + (1 - \check{\lambda})(2x_2 - 2x_1))e_2 \end{aligned}$$

is a subgradient of f^- at the point (x_1, x_2) with $x_1 = x_2$.

Note that if the point $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in S$ mentioned in above theorem is a point having dense components, i.e., $\sigma_i(\bar{x}_i) = \rho_i(\bar{x}_i) = 0$ for all $i = 1, 2, \dots, n$, then

$$\left. \frac{\partial f^-(x)}{\Delta_i x_i} \right|_{x=\bar{x}} = \left. \frac{\partial f^-(x)}{\nabla_i x_i} \right|_{x=\bar{x}} = \left. \frac{\partial f^-(x)}{\partial x_i} \right|_{x=\bar{x}} \tag{13}$$

and

$$\frac{\partial f^+(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} = \frac{\partial f^+(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} = \frac{\partial f^+(x)}{\partial x_i} \Big|_{x=\bar{x}}. \tag{14}$$

From (13) and (14), it follows that

$$\frac{\partial F(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} = \frac{\partial F(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} = \frac{\partial F(x)}{\partial x_i} \Big|_{x=\bar{x}}, \tag{15}$$

where

$$\frac{\partial F(x)}{\partial x_i} = \left[\min \left\{ \frac{\partial f^-(x)}{\partial x_i}, \frac{\partial f^+(x)}{\partial x_i} \right\}, \max \left\{ \frac{\partial f^-(x)}{\partial x_i}, \frac{\partial f^+(x)}{\partial x_i} \right\} \right]$$

is called gradient of F in classical sense [14].

Corollary 3.14. Let S be a nonempty convex set in \mathbb{R}^n and $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in \text{cint}_{\mathbb{R}^n}(S)$ a point satisfying $\sigma_i(\bar{x}_i) = \rho_i(\bar{x}_i) = 0$ for all $i = 1, 2, \dots, n$. Let $F : S \rightarrow I_c$ be an interval-valued function such that the partial derivatives $\frac{\partial f^-(x)}{\partial x_i} \Big|_{x=\bar{x}}$ and $\frac{\partial f^+(x)}{\partial x_i} \Big|_{x=\bar{x}}$, exist for all $i = 1, 2, \dots, n$. If F is \leq_{LW} -convex on S , then

$$\text{grad} f^-(\bar{x}) = \sum_{i=1}^n \frac{\partial f^-(x)}{\partial x_i} \Big|_{x=\bar{x}} e_i$$

and

$$\text{grad} \text{len} F(\bar{x}) = \sum_{i=1}^n \frac{\partial \text{len} F(x)}{\partial x_i} \Big|_{x=\bar{x}} e_i$$

are the gradients of f^- and $\text{len} F$ at any point $\bar{x} \in \text{cint}_{\Lambda^n}(S)$.

4. Optimality of interval-valued functions

For the given function $F : \Lambda^n \rightarrow I_c$ and convex set $S \subset \Lambda^n$, consider the following interval-valued optimization problem

$$\text{(IVP): Minimize } F \text{ subject to } x \in S, \tag{16}$$

where S denotes the feasible set of primal problem (IVP).

Definition 4.1. A feasible point $x^* \in S$ is called LW (resp., strongly LW) optimal solution of (IVP), if there exist no $x \in S$ such that $F(x) <_{LW} F(x^*)$ (resp., $F(x) \leq_{LW} F(x^*)$).

Let us consider two corresponding scalar problems for (IVP) as follows

$$\text{(LIVP): Minimize } f^- \text{ subject to } x \in S \tag{17}$$

and

$$\text{(WIVP): Minimize } \text{len} F \text{ subject to } x \in S. \tag{18}$$

From the definition of optimal solution, it is easy to obtain the following result.

Lemma 4.2. If \bar{x} is an optimal solution of problems (17) and (18) simultaneously, then \bar{x} is LW solution of (16).

Remark 4.3. The converse of Lemma 4.2 is not true in general. For example, consider the following interval-valued function $F : [0, 1]_{\mathbb{T}} \rightarrow I_c$ such that $F(x) = [x^2, x^2 + 1]$. Clearly, $\bar{x} = 0$ is the unique LW optimal solution. On the other hand $\text{len}F \equiv 1$, and it does not have unique optimal solution. However, problem (16) cannot be equivalent to problems (17) and (18). Based on Lemma 4.2, problems (17) and (18) can just be regarded as the auxiliary problems for the problem (16).

Theorem 4.4. Let S be a convex set in Λ^n and $F : S \rightarrow I_c$ is a LW-convex interval-valued function such that the partial derivatives $\frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}'}$, $\frac{\partial f^+(x)}{\Delta_i x_i} \Big|_{x=\bar{x}'}$, $\frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$ and $\frac{\partial f^+(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$, $i = 1, 2, \dots, n$, exist at any point $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in \text{cint}_{\Lambda^n}(S)$ satisfying $F_x \subset S$. Then \bar{x} is an optimal solution of problems (17) and (18) if and only if there exist scalars $\lambda_i^-(\bar{x}), \lambda_i^l(\bar{x}) \in [0, 1], i = 1, 2, \dots, n$, such that the vectors

$$\xi^-(\bar{x}) = \sum_{i=1}^n \left(\lambda_i^-(\bar{x}) \frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i^-(\bar{x})) \frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e_i \tag{19a}$$

and

$$\xi^l(\bar{x}) = \sum_{i=1}^n \left(\lambda_i^l(\bar{x}) \frac{\partial \text{len}F(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} + (1 - \lambda_i^l(\bar{x})) \frac{\partial \text{len}F(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \right) e_i \tag{20}$$

are the zero subgradient for f^- and $\text{len}F$ respectively at \bar{x} .

Proof. Convexity of F implies that f^- and $\text{len}F$ are real-valued convex on S . Since partial derivatives $\frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}'}$, $\frac{\partial f^+(x)}{\Delta_i x_i} \Big|_{x=\bar{x}'}$, $\frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$ and $\frac{\partial f^+(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$, $i = 1, 2, \dots, n$, exist at any point $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \in \text{cint}_{\Lambda^n}(S)$ satisfying $F_x \subset S$, it follows that, the vectors $\xi^-(\bar{x})$ and $\xi^l(\bar{x})$ are well define.

From Lemma 2.2, it implies that \bar{x} is an optimal solution of problems (17) and (18) if and only if $\xi^-(\bar{x})$ and $\xi^l(\bar{x})$ are the zero subgradient for f^- and $\text{len}F$ respectively at \bar{x} . This completes the proof. \square

Remark 4.5. Theorem 4.4, implies that for all $i = 1, 2, \dots, n$

$$\frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \leq 0 \leq \frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} \tag{21}$$

and

$$\frac{\partial \text{len}F(x)}{\nabla_i x_i} \Big|_{x=\bar{x}} \leq 0 \leq \frac{\partial \text{len}F(x)}{\Delta_i x_i} \Big|_{x=\bar{x}} \tag{22}$$

turns into necessary conditions. By using these conditions, one may find the critical points which are candidates to be the optimal solution to the problem (16).

If F becomes real-valued functions, i.e., $f^-(x) = f^+(x)$ for all $x \in S$ and $\bar{x} \in S$ is a point such that $\sigma_i(\bar{x}_i) = \rho_i(\bar{x}_i) = 0$ for all $i = 1, 2, \dots, n$, then the conditions (21) and (22) turn into a sufficient conditions guaranteeing optimality of \bar{x} . However in the case when $\sigma_i(\bar{x}_i) = \rho_i(\bar{x}_i) = 0$ is not true for any $i = 1, 2, \dots, n$, the conditions (21) and (22) are only necessary conditions for the optimality of \bar{x} .

By assuming the additional assumption, conditions (21) and (22) turn into the necessary and sufficient conditions for the optimality of $\bar{x} \in S$.

Theorem 4.6. Let S be a convex set in Λ^n and $F : S \rightarrow I_c$ is a LW-convex interval-valued function such that the partial derivatives $\frac{\partial f^-(x)}{\Delta_i x_i} \Big|_{x=\bar{x}'}$, $\frac{\partial f^+(x)}{\Delta_i x_i} \Big|_{x=\bar{x}'}$, $\frac{\partial f^-(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$ and $\frac{\partial f^+(x)}{\nabla_i x_i} \Big|_{x=\bar{x}}$, $i = 1, 2, \dots, n$, exist at any point $\bar{x} \in \text{cint}_{\Lambda^n}(S)$ satisfying $F_{\bar{x}} \subset S$. Suppose that the gradients defined by (9), (10), (11) and (12) are subgradients for f^- and $\text{len}F$ at $\bar{x} \in S$. Then \bar{x} is an optimal solution of problems (17) and (18) if and only if the conditions (21) and (22) hold.

Example 4.7. Let $\Lambda^2 = \mathbb{Z} \times \mathbb{Z}$ and $F : \Lambda^2 \rightarrow I_c$ be defined by

$$F(x_1, x_2) = \left[\left(x_1 - \frac{1}{3}\right)^2 + \left(x_2 - \frac{1}{4}\right)^2, x_1^2 + x_2^2 \right], \text{ for } x_1 \geq -x_2.$$

It is easy to see that the vectors

$$\text{grad}_{\Delta} f^{-}(x_1, x_2)^T = \left(2x_1 + \frac{1}{3}, 2x_2 + \frac{1}{2}\right)$$

$$\text{grad}_{\nabla} f^{-}(x_1, x_2)^T = \left(2x_1 - \frac{5}{3}, 2x_2 - \frac{3}{2}\right)$$

and

$$\text{grad}_{\Delta} \text{len}F(x_1, x_2)^T = \text{grad}_{\nabla} \text{len}F(x_1, x_2)^T = \left(\frac{2}{3}, \frac{1}{2}\right)$$

are subgradients for f^{-} and $\text{len}F$ at the point $(0, 0)$. Since the inequality (21) holds at the point $(0, 0)$, the optimal solution to the problem (17) is $(0, 0)$ whenever $S = \{(x_1, x_2) \in \Lambda^2 : x_1 \geq -x_2\}$. On the other hand, inequality (22) does not hold at the point $(0, 0)$, the optimal solution to the problem (18) is not $(0, 0)$. It implies by Lemma 4.2 that $(0, 0)$ is not LW optimal solution of (16).

Conclusion: In this paper, we have introduced convex type interval-valued functions on the domain of the product of closed subsets of real numbers. Comparison between different partial orders are presented. We obtained necessary and Sufficient optimal conditions for interval-programming problems. In addition, we compared our results with the results given in the literature, therefore the optimality conditions obtained are applicable to a wider range of functions. As result, many classical optimization results (when you have non interval-valued functions) are particular instances of the ones presented here. Moreover future research is oriented to consider the multiobjective programming problem for interval-valued convex functions on mixed as well as discrete domains.

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