



# Finite Spectrum of Sturm-Liouville Problems with Eigenparameter-Dependent Boundary Conditions on Time Scales

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**Abstract.** The spectral analysis of a class of Sturm-Liouville problems with eigenparameter-dependent boundary conditions on bounded time scales is investigated. By partitioning the bounded time scale such that the coefficients of Sturm-Liouville equation satisfy certain conditions on the adjacent subintervals, the finite eigenvalue results are obtained. The results show that the number of eigenvalues not only depend on the partition of the bounded time scale, but also depend on the eigenparameter-dependent boundary conditions. Both of the self-adjoint and non-self-adjoint cases are considered in this paper.

## 1. Introduction

According to the classical Sturm-Liouville theory [16], the spectrum of a regular or singular, self-adjoint Sturm-Liouville problem (SLP) is unbounded and therefore infinite. In 1964, Atkinson in his well known book [6] suggested that if the coefficients of SLP satisfy some special conditions, the problem may have finite eigenvalues. In 2001, Kong, Wu, and Zettl proved the rationality of Atkinson's judgment in [12] by analyzing on a certain class of SLP. They demonstrate that this class of SLP has a finite number of eigenvalues. Then a host of researchers got a slice of crucial achievements in the last decades or so, please see [3–5, 11, 13] and the references therein.

It is well known, the SLP with eigenparameter-dependent boundary conditions have been an important research topic in mathematical physics [3, 7, 8]. These problems appeared in some physical problems and engineering problems such as heat conduction problems and vibrating string problems and so on [2, 9]. Generally, the spectrum of a SLP with eigenparameter-dependent boundary conditions will be influenced by the eigenparameter which arise not only in the equation but also in the boundary conditions. Hence there are some different characters on these problems compared to those classical SLPs. For the studies of these problems here we refer to [2, 3, 7–9].

As an effective tool to unify both of the discrete and continuous systems, the concept of time scale was put forward by German mathematician Stefan Hilger in 1988. There are numerous studies about the problems on time scales. Especially, in recent years the eigenvalue problems of Sturm-Liouville equation on time scales have been a new research topic in mathematical physics. There are some important results

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have been obtained by a multitude of researchers. In 1999, Agarwal et al studied the SLP with separated boundary conditions under the condition  $p = 1$  in [1]. They proved the existence of eigenvalues and the number of the generalized zeros of eigenfunctions. In 2008, Kong considered the more general SLP and investigated the dependence of the eigenvalues on the boundary conditions in [10]. In 2010, Zhang et al studied the existence of eigenvalues about the SLP with coupled boundary conditions and showed the relationship between the number of eigenvalues and the boundary conditions in [17]. In 2013, Zhao generated the finite spectrum results of SLP to SLP on time scales in [18]. In 2016, Tuna in [14] studied the completeness theorem for the dissipative SLP on time scales.

In the present paper, we study the finite spectrum of Sturm-Liouville problems with eigenparameter-dependent boundary conditions on a bounded time scale and consider the following equation

$$-(px^\Delta)^\Delta + qx^\sigma = \lambda wx^\sigma \quad \text{on } \mathbb{T}, \tag{1}$$

satisfying

$$1/p, q, w \in C_{\text{prd}}(\mathbb{T}), \tag{2}$$

together with the separated eigenparameter-dependent boundary conditions of the form (3) and the coupled eigenparameter-dependent boundary conditions (see [4]) of the form (4) respectively.

The separated eigenparameter-dependent boundary conditions are

$$A_\lambda X(a) + B_\lambda X(b) = 0, \quad X = \begin{bmatrix} x \\ px^\Delta \end{bmatrix}, \tag{3}$$

where  $A_\lambda = \begin{bmatrix} \lambda\alpha'_1 + \alpha_1 & \lambda\alpha'_2 + \alpha_2 \\ 0 & 0 \end{bmatrix}$ ,  $B_\lambda = \begin{bmatrix} 0 & 0 \\ \lambda\beta'_1 + \beta_1 & \lambda\beta'_2 + \beta_2 \end{bmatrix}$  with  $\alpha_i, \alpha'_i, \beta_i, \beta'_i \in \mathbb{R}$ ,  $i = 1, 2$  satisfying  $\theta_1 = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha'_1 & \alpha'_2 \end{vmatrix} \neq 0$ ,  $\theta_2 = \begin{vmatrix} \beta_1 & \beta_2 \\ \beta'_1 & \beta'_2 \end{vmatrix} \neq 0$ . Here  $\lambda$  is the spectral parameter.

The coupled eigenparameter-dependent boundary conditions are

$$\hat{A}_\lambda X(a) + \hat{B}_\lambda X(b) = 0, \quad X = \begin{bmatrix} x \\ px^\Delta \end{bmatrix}, \tag{4}$$

where  $\hat{A}_\lambda = \begin{bmatrix} \lambda\alpha'_1 + \alpha_1 & \lambda\alpha'_2 + \alpha_2 \\ \lambda\beta'_1 + \beta_1 & \lambda\beta'_2 + \beta_2 \end{bmatrix}$ ,  $\hat{B}_\lambda = \begin{bmatrix} \lambda\alpha'_3 + \alpha_3 & \lambda\alpha'_4 + \alpha_4 \\ \lambda\beta'_3 + \beta_3 & \lambda\beta'_4 + \beta_4 \end{bmatrix}$  with  $\alpha_i, \alpha'_i, \beta_i, \beta'_i \in \mathbb{R}$ ,  $i = 1, 2, 3, 4$  satisfying

$$\det(\hat{A}_\lambda) \neq 0, \quad \det(\hat{B}_\lambda) \neq 0,$$

$$\text{rank} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} = 2, \quad \text{rank} \begin{bmatrix} \alpha'_1 & \alpha'_2 & \alpha'_3 & \alpha'_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \end{bmatrix} = 2,$$

$$\text{rank} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha'_1 & \alpha'_2 & \alpha'_3 & \alpha'_4 \end{bmatrix} = 2, \quad \text{rank} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \end{bmatrix} = 2.$$

Here  $\lambda$  is the spectral parameter.

Following the method of [5, 16], by partitioning the bounded time scale such that the coefficients of (1) satisfy certain conditions on adjacent subintervals, we construct a kind of SLP with eigenparameter-dependent boundary conditions on bounded time scales which has exactly finite number of eigenvalues. Here the problems include both of the self-adjoint and non-self-adjoint cases and we will consider the problems with separated and coupled eigenparameter-dependent boundary conditions respectively. As far as we know, much less is known for boundary value problems with coupled eigenparameter-dependent boundary conditions.

The paper is organized as follows: following the introduction in Section 1, some basic definitions about time scales and related lemmas are listed in Section 2. The main results regarding an analysis of eigenvalues of the considered problems and corresponding examples are given in Section 3.

**2. Preliminaries**

Before presenting the main results, in this section, we recall the following concepts related to time scales for the convenience of the reader and list some lemmas which are needed to prove our main theorems.

**Definition 2.1.** A time scale  $\mathbb{T}$  is a closed subset of  $\mathbb{R}$ . For  $t \in \mathbb{T}$  we define the forward-jump operator  $\sigma$  and the backward-jump operator  $\rho$  on  $\mathbb{T}$  by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where  $\inf \emptyset := \sup \mathbb{T}$  and  $\sup \emptyset := \inf \mathbb{T}$ .

**Definition 2.2.** A point  $t \in \mathbb{T}$  is called right-scattered, right-dense, left-scattered, and left-dense if  $\sigma(t) > t, \sigma(t) = t, \rho(t) < t$  and  $\rho(t) = t$  respectively. The graininess  $\mu : \mathbb{T} \rightarrow [0, \infty)$  is defined by

$$\mu(t) = \sigma(t) - t.$$

**Definition 2.3.** We put  $\mathbb{T}^\kappa = \mathbb{T}$  if  $\mathbb{T}$  is unbounded above and  $\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(\max \mathbb{T}), \max \mathbb{T}]$  otherwise. Let  $f : \mathbb{T} \rightarrow \mathbb{C}, \forall t \in \mathbb{T}^\kappa$ ,

$$f^\Delta(t) := \begin{cases} \lim_{s \rightarrow t} \frac{f(s) - f(t)}{s - t}, & \mu(t) = 0 \\ \frac{f^\sigma(t) - f(t)}{\mu(t)}, & \mu(t) > 0, \end{cases} \tag{5}$$

where  $f^\sigma(t) = f(\sigma(t))$ .

**Definition 2.4.** A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is rd-continuous provided it is continuous at all right-dense points of  $\mathbb{T}$  and its left-sided limit exists (finite) at left-dense points of  $\mathbb{T}$ . The set of all right-dense continuous functions on  $\mathbb{T}$  is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}).$$

Similarly, a function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is prd-continuous provided it is continuous at all points except finite right-dense points of  $\mathbb{T}$  and its left-sided limit exists (finite) at left-dense points of  $\mathbb{T}$ . The set of all these functions on  $\mathbb{T}$  is denoted by

$$C_{prd} = C_{prd}(\mathbb{T}).$$

If  $F^\Delta(t) = f(t)$ , for  $\forall t \in \mathbb{T}^\kappa$ ,

$$\int_a^b f(\tau) \Delta(\tau) = F(b) - F(a), \quad a, b \in \mathbb{T}.$$

**Definition 2.5.** Let  $[a, b]_{\mathbb{T}} = \{\forall t \in \mathbb{T}, a \leq t \leq b\}$ . The function  $x(\cdot, \lambda) : [a, b]_{\mathbb{T}} \rightarrow \mathbb{C}$  is the solution of equation (1) if and only if  $x(\cdot, \lambda) \in D_{\Delta\Delta}(\lambda)$ . And  $\forall t \in [a, b]_{\mathbb{T}}$ ,  $x(\cdot, \lambda)$  satisfy the equation (1), where  $D_{\Delta\Delta}(\lambda) := \{x(\cdot, \lambda) : [a, b]_{\mathbb{T}} \rightarrow \mathbb{C}, x(\cdot, \lambda) \in C_{prd}^1(\mathbb{T}), (px^\Delta)(\cdot, \lambda) \in C_{prd}^1(\mathbb{T})\}$ ,  $C_{prd}^1(\mathbb{T}) = \{f : f \in C_{prd}(\mathbb{T}), \text{ and } f \text{ is } \Delta\text{-derivable}\}$ .

**Lemma 2.6.** Equation (1) is equivalent to the following form

$$\begin{bmatrix} x \\ u \end{bmatrix}^\Delta = \begin{bmatrix} 0 & \frac{1}{p(t)} \\ q(t) - \lambda w(t) & 0 \end{bmatrix} \begin{bmatrix} x^\sigma \\ u \end{bmatrix}, \quad u(t) = p(t)x^\Delta(t)$$

or

$$X^\Delta = A(t)X, \tag{6}$$

where  $X = \begin{bmatrix} x \\ u \end{bmatrix}$ ,  $A(t) = \begin{bmatrix} 0 & \frac{1}{p(t)} \\ q(t) - \lambda w(t) & \frac{[q(t) - \lambda w(t)]\mu(t)}{p(t)} \end{bmatrix}$ .

*Proof.* It can be proved by direct calculation.  $\square$

**Lemma 2.7.**  $\forall t_0 \in [a, b]_{\mathbb{T}}$ ,  $A(t) \in C_{prd}$  is  $n \times n$  functional matrix, and  $\forall t \in [a, t_0]_{\mathbb{T}}$ ,  $I + \mu(t)A(t)$  is invertible matrix, then the initial value problem

$$X^\Delta = A(t)X, \quad X(t_0) = x_0, \quad x_0 \in \mathbb{C}^n,$$

has unique solution  $X \in C_{prd}^1$ . (Refer to part 2.2 in [15]) It can be assumed that  $\Phi(t, \lambda) = [\phi_{ij}(t, \lambda)]$ ,  $t \in [a, b]_{\mathbb{T}}$  is the fundamental matrix solution of equation (6) satisfied with the initial condition  $\Phi(a, \lambda) = I$ , where

$$\Phi(t, \lambda) = \begin{bmatrix} \theta(t, \lambda) & \varphi(t, \lambda) \\ (p\theta^\Delta)(t, \lambda) & (p\varphi^\Delta)(t, \lambda) \end{bmatrix}. \tag{7}$$

*Proof.* See [15].  $\square$

**Lemma 2.8.** Let (2) hold and  $\Phi(t, \lambda) = [\phi_{ij}(t, \lambda)]$ ,  $t \in [a, b]_{\mathbb{T}}$  is the fundamental matrix solution of equation (6) satisfied with the initial condition  $\Phi(a, \lambda) = I$ , then  $\lambda \in \mathbb{C}$  is the eigenvalue of SLP (1), (3) if and only if the characteristic function  $\delta(\lambda) = 0$ , where

$$\begin{aligned} \delta(\lambda) &= \det[A_\lambda + B_\lambda \Phi(b, \lambda)] \\ &= \det(A_\lambda) + \det(B_\lambda) + c_{11}\phi_{11}(b, \lambda) + c_{12}\phi_{12}(b, \lambda) + c_{21}\phi_{21}(b, \lambda) + c_{22}\phi_{22}(b, \lambda), \end{aligned} \tag{8}$$

with

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

$$\begin{aligned} c_{11} &= -(\lambda\alpha'_2 + \alpha_2)(\lambda\beta'_1 + \beta_1), & c_{12} &= (\lambda\alpha'_1 + \alpha_1)(\lambda\beta'_1 + \beta_1), \\ c_{21} &= -(\lambda\alpha'_2 + \alpha_2)(\lambda\beta'_2 + \beta_2), & c_{22} &= (\lambda\alpha'_1 + \alpha_1)(\lambda\beta'_2 + \beta_2). \end{aligned}$$

*Proof.* The proof of the first part of this lemma see [2] and the second part comes from a straightforward computation.  $\square$

**Lemma 2.9.** Let (2) hold and  $\Phi(t, \lambda) = [\phi_{ij}(t, \lambda)]$ ,  $t \in [a, b]_{\mathbb{T}}$  is the fundamental matrix solution of equation (6) satisfied with the initial condition  $\Phi(a, \lambda) = I$ , then  $\lambda \in \mathbb{C}$  is the eigenvalue of SLP (1), (4) if and only if the characteristic function  $\hat{\delta}(\lambda) = 0$ , where

$$\begin{aligned} \hat{\delta}(\lambda) &= \det[\hat{A}_\lambda + \hat{B}_\lambda \Phi(b, \lambda)] \\ &= \det(\hat{A}_\lambda) + \det(\hat{B}_\lambda) + \hat{c}_{11}\phi_{11}(b, \lambda) + \hat{c}_{12}\phi_{12}(b, \lambda) + \hat{c}_{21}\phi_{21}(b, \lambda) + \hat{c}_{22}\phi_{22}(b, \lambda), \end{aligned} \tag{9}$$

with

$$\hat{C} = \begin{bmatrix} \hat{c}_{11} & \hat{c}_{12} \\ \hat{c}_{21} & \hat{c}_{22} \end{bmatrix},$$

$$\begin{aligned} \hat{c}_{11} &= (\lambda\alpha'_3 + \alpha_3)(\lambda\beta'_2 + \beta_2) - (\lambda\alpha'_2 + \alpha_2)(\lambda\beta'_3 + \beta_3), & \hat{c}_{12} &= (\lambda\alpha'_1 + \alpha_1)(\lambda\beta'_3 + \beta_3) - (\lambda\alpha'_3 + \alpha_3)(\lambda\beta'_1 + \beta_1), \\ \hat{c}_{21} &= (\lambda\alpha'_4 + \alpha_4)(\lambda\beta'_2 + \beta_2) - (\lambda\alpha'_2 + \alpha_2)(\lambda\beta'_4 + \beta_4), & \hat{c}_{22} &= (\lambda\alpha'_1 + \alpha_1)(\lambda\beta'_4 + \beta_4) - (\lambda\alpha'_4 + \alpha_4)(\lambda\beta'_1 + \beta_1). \end{aligned}$$

*Proof.* The proof is similar to the one for Lemma 2.8.  $\square$

**Definition 2.10.** The SLPs with eigenparameter-dependent boundary conditions (1), (3) and (1), (4) are said to be degenerate if in (8), (9) either  $\delta(\lambda), \hat{\delta}(\lambda) \equiv 0$  for all  $\lambda \in \mathbb{C}$  or  $\delta(\lambda), \hat{\delta}(\lambda) \neq 0$  for any  $\lambda \in \mathbb{C}$ .

**3. Finite spectrum of SLPs with eigenparameter-dependent boundary conditions on time scale**

Assume (1) is defined on  $\mathbb{T} = [a, b] \cup \{c\} \cup [d, e]$  with  $-\infty < a < b < c < d < e < +\infty$  and there exists a partition of the intervals of time scale  $\mathbb{T}$

$$a = a_0 < a_1 < a_2 < \dots < a_{2m} < a_{2m+1} = b, \quad d = b_0 < b_1 < b_2 < \dots < b_{2n} < b_{2n+1} = e, \tag{10}$$

for some positive integers  $m$  and  $n$ , such that

$$\begin{aligned} r = 0 \text{ on } [a_{2k}, a_{2k+1}], \quad \int_{a_{2k}}^{a_{2k+1}} w \neq 0, \quad k = 0, 1, \dots, m, \\ r = 0 \text{ on } [b_{2i}, b_{2i+1}], \quad \int_{b_{2i}}^{b_{2i+1}} w \neq 0, \quad i = 0, 1, \dots, n, \end{aligned} \tag{11}$$

and

$$\begin{aligned} q = w = 0 \text{ on } [a_{2k+1}, a_{2k+2}], \quad \int_{a_{2k+1}}^{a_{2k+2}} r \neq 0, \quad k = 0, 1, \dots, m-1, \\ q = w = 0 \text{ on } [b_{2i+1}, b_{2i+2}], \quad \int_{b_{2i+1}}^{b_{2i+2}} r \neq 0, \quad i = 0, 1, \dots, n-1. \end{aligned} \tag{12}$$

Given (10)-(12), it is easy to set the following notation:

$$\begin{aligned} r_k = \int_{a_{2k+1}}^{a_{2k+2}} r, \quad k = 0, 1, \dots, m-1, \quad q_k = \int_{a_{2k}}^{a_{2k+1}} q, \quad w_k = \int_{a_{2k}}^{a_{2k+1}} w, \quad k = 0, 1, \dots, m, \\ \tilde{r}_i = \int_{b_{2i+1}}^{b_{2i+2}} r, \quad i = 0, 1, \dots, n-1, \quad \tilde{q}_i = \int_{b_{2i}}^{b_{2i+1}} q, \quad \tilde{w}_i = \int_{b_{2i}}^{b_{2i+1}} w, \quad i = 0, 1, \dots, n. \end{aligned} \tag{13}$$

Then we can state the iterative formula as follows:

**Lemma 3.1.** *Let (10)-(13) hold. Let  $\Phi(t, \lambda) = [\phi_{ij}(t, \lambda)]$  be the fundamental matrix solution of the equation (6) determined by the initial condition  $\Phi(a, \lambda) = I$  for each  $\lambda \in \mathbb{C}$ . Then we have that*

$$\begin{aligned} \Phi(a_1, \lambda) &= \begin{bmatrix} 1 & 0 \\ q_0 - \lambda w_0 & 1 \end{bmatrix}, \\ \Phi(a_3, \lambda) &= \begin{bmatrix} 1 + (q_0 - \lambda w_0)r_0 & r_0 \\ \phi_{21}(a_3, \lambda) & 1 + (q_1 - \lambda w_1)r_0 \end{bmatrix}, \end{aligned}$$

where  $\phi_{21}(a_3, \lambda) = (q_0 - \lambda w_0) + (q_1 - \lambda w_1) + (q_0 - \lambda w_0)(q_1 - \lambda w_1)r_0$ .

And in general, for  $1 \leq i \leq m$ ,

$$\Phi(a_{2i+1}, \lambda) = \begin{bmatrix} 1 & r_{i-1} \\ q_i - \lambda w_i & 1 + (q_i - \lambda w_i)r_{i-1} \end{bmatrix} \Phi(a_{2i-1}, \lambda).$$

*Proof.* Let  $u = px^\Delta, r = \frac{1}{p}$ , then we have  $x^\Delta = ru, u^\Delta = (q - \lambda w)x^\sigma$ . Observe from the system, that  $x^\sigma$  is constant on each subinterval where  $r$  is identically zero and  $u$  is constant on each subinterval where both  $q$  and  $w$  are identically zero. Consider the equation (1) and the time scale  $\mathbb{T} = [a, b] \cup \{c\} \cup [d, e]$ . The result follows from repeated applications of the system.  $\square$

**Lemma 3.2.** *Let (10)-(13) hold. Let  $\Psi(t, \lambda) = [\psi_{ij}(t, \lambda)]$  be the fundamental matrix solution of the equation (6) determined by the initial condition  $\Psi(d, \lambda) = I$  for each  $\lambda \in \mathbb{C}$ . Then we have that*

$$\Psi(b_1, \lambda) = \begin{bmatrix} 1 & 0 \\ \tilde{q}_0 - \lambda \tilde{w}_0 & 1 \end{bmatrix},$$

$$\Psi(b_3, \lambda) = \begin{bmatrix} 1 + (\tilde{q}_0 - \lambda\tilde{w}_0)\tilde{r}_0 & \tilde{r}_0 \\ \psi_{21}(b_3, \lambda) & 1 + (\tilde{q}_1 - \lambda\tilde{w}_1)\tilde{r}_0 \end{bmatrix},$$

where  $\psi_{21}(b_3, \lambda) = (\tilde{q}_0 - \lambda\tilde{w}_0) + (\tilde{q}_1 - \lambda\tilde{w}_1) + (\tilde{q}_0 - \lambda\tilde{w}_0)(\tilde{q}_1 - \lambda\tilde{w}_1)\tilde{r}_0$ .

And in general, for  $1 \leq j \leq n$ ,

$$\Psi(b_{2j+1}, \lambda) = \begin{bmatrix} 1 & \tilde{r}_{j-1} \\ \tilde{q}_j - \lambda\tilde{w}_j & 1 + (\tilde{q}_j - \lambda\tilde{w}_j)\tilde{r}_{j-1} \end{bmatrix} \Psi(b_{2j-1}, \lambda).$$

*Proof.* The proof is similar to the one for Lemma 3.1.  $\square$

**Lemma 3.3.** Let (10)-(13) hold.  $\Phi(t, \lambda), \Psi(t, \lambda)$  be defined as Lemma 3.1 and Lemma 3.2 respectively, then

$$\Phi(e, \lambda) = \Psi(e, \lambda)M(\lambda)\Phi(b, \lambda), \tag{14}$$

where  $M(\lambda) = M_2(\lambda)M_1(\lambda)$ , and

$$M_1(\lambda) = \begin{bmatrix} 1 & (c-b)r(b) \\ k_1(\lambda) & l_1(\lambda) \end{bmatrix}, \quad M_2(\lambda) = \begin{bmatrix} 1 & (d-c)r(c) \\ k_2(\lambda) & l_2(\lambda) \end{bmatrix},$$

$$k_1(\lambda) = [q(b) - \lambda w(b)](c-b), \quad l_1(\lambda) = 1 + [q(b) - \lambda w(b)](c-b)^2 r(b),$$

$$k_2(\lambda) = [q(c) - \lambda w(c)](d-c), \quad l_2(\lambda) = 1 + [q(c) - \lambda w(c)](d-c)^2 r(c).$$

*Proof.* From (1), (5), we know that

$$(px^\Delta)(b) = \frac{p(b)x(c) - p(b)x(b)}{c-b}, \quad (px^\Delta)^\Delta(b) = \frac{(px^\Delta)(c) - (px^\Delta)(b)}{c-b}, \tag{15}$$

$$(px^\Delta)^\Delta(b) = [q(b) - \lambda w(b)]x(c). \tag{16}$$

Calculate from (15), (16) that

$$X(c) = M_1(\lambda)X(b),$$

where

$$M_1(\lambda) = \begin{bmatrix} 1 & (c-b)r(b) \\ k_1(\lambda) & l_1(\lambda) \end{bmatrix},$$

$$k_1(\lambda) = [q(b) - \lambda w(b)](c-b), \quad l_1(\lambda) = 1 + [q(b) - \lambda w(b)](c-b)^2 r(b).$$

Similarly, we have

$$X(d) = M_2(\lambda)X(c),$$

where

$$M_2(\lambda) = \begin{bmatrix} 1 & (d-c)r(c) \\ k_2(\lambda) & l_2(\lambda) \end{bmatrix},$$

$$k_2(\lambda) = [q(c) - \lambda w(c)](d-c), \quad l_2(\lambda) = 1 + [q(c) - \lambda w(c)](d-c)^2 r(c).$$

Also, because

$$X(b) = \Phi(b, \lambda)X(a), \quad X(e) = \Psi(e, \lambda)X(d),$$

then  $X(e) = \Psi(e, \lambda)M(\lambda)\Phi(b, \lambda)X(a)$  and  $X(e) = \Phi(e, \lambda)X(a)$ .

From  $\det(I + \mu(b)A(b)) \neq 0$ , and Lemma 2.7 it can be obtained that

$$\Phi(e) = \Psi(e, \lambda)M(\lambda)\Phi(b, \lambda). \quad \square$$

**Corollary 3.4.** Let  $M(\lambda) = \begin{bmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{bmatrix}$ , then for the fundamental matrix  $\Phi$  we have that

$$\phi_{11}(e, \lambda) = R\tilde{R} \prod_{i=0}^{m-1} (q_i - \lambda w_i) \prod_{j=1}^{n-1} (\tilde{q}_j - \lambda \tilde{w}_j) H(\lambda) + \tilde{\phi}_{11}(\lambda),$$

$$\phi_{12}(e, \lambda) = R\tilde{R} \prod_{i=1}^{m-1} (q_i - \lambda w_i) \prod_{j=1}^{n-1} (\tilde{q}_j - \lambda \tilde{w}_j) H(\lambda) + \tilde{\phi}_{12}(\lambda),$$

$$\phi_{21}(e, \lambda) = R\tilde{R} \prod_{i=0}^{m-1} (q_i - \lambda w_i) \prod_{j=1}^n (\tilde{q}_j - \lambda \tilde{w}_j) H(\lambda) + \tilde{\phi}_{21}(\lambda),$$

$$\phi_{22}(e, \lambda) = R\tilde{R} \prod_{i=1}^{m-1} (q_i - \lambda w_i) \prod_{j=1}^n (\tilde{q}_j - \lambda \tilde{w}_j) H(\lambda) + \tilde{\phi}_{22}(\lambda),$$

where  $H(\lambda) = m_{11}(\lambda)(\tilde{q}_0 - \lambda \tilde{w}_0) + m_{12}(\lambda)(q_m - \lambda w_m)(\tilde{q}_0 - \lambda \tilde{w}_0) + m_{21}(\lambda) + m_{22}(\lambda)(q_m - \lambda w_m)$ ,  $R = \prod_{i=0}^{m-1} r_i$ ,  $\tilde{R} = \prod_{i=0}^{n-1} \tilde{r}_i$ , and  $\tilde{\phi}_{ij}(\lambda) = o(R\tilde{R})$  as  $\min\{r_k, \tilde{r}_l : k = 0, \dots, m-1, l = 0, \dots, n-1\} \rightarrow \infty$  for fixed  $q, w$  and  $\lambda, i, j = 1, 2$ .

Now we can state our main results. Consider the SLP consisting of the equation (1) together with separated eigenparameter-dependent boundary conditions (3). Then we have the following theorem.

**Theorem 3.5.** Let  $m, n \in \mathbb{N}$ , but  $\tilde{w}_0 + w(c)(d - c) \neq 0$ , let (10)-(13) hold. Consider the SLP (1), (3). Then:

(1) If  $\alpha'_2\beta'_2 \neq 0$ , then the SLP with separated eigenparameter-dependent boundary conditions (1), (3) has exactly  $m + n + 5$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 4$ .

(2) If  $\alpha'_2\beta'_2 = 0$ , but  $\alpha'_2\beta'_1 w_0 + (\alpha_2\beta'_2 + \alpha'_2\beta_2)w_0\tilde{w}_n - \alpha'_1\beta'_2\tilde{w}_n \neq 0$ , then the SLP with separated eigenparameter-dependent boundary conditions (1), (3) has exactly  $m + n + 4$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 3$ .

(3) If  $\alpha'_2\beta'_2 = \alpha'_2\beta'_1 w_0 + (\alpha_2\beta'_2 + \alpha'_2\beta_2)w_0\tilde{w}_n - \alpha'_1\beta'_2\tilde{w}_n = 0$ , but  $(\alpha'_2\beta_1 + \alpha_2\beta'_1)w_0 - \alpha'_1\beta'_1 + \alpha_2\beta_2 w_0\tilde{w}_n - (\alpha'_1\beta_2 + \alpha_1\beta'_2)\tilde{w}_n \neq 0$ , then the SLP with separated eigenparameter-dependent boundary conditions (1), (3) has exactly  $m + n + 3$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 2$ .

(4) If  $\alpha'_2\beta'_2 = \alpha'_2\beta'_1 w_0 + (\alpha_2\beta'_2 + \alpha'_2\beta_2)w_0\tilde{w}_n - \alpha'_1\beta'_2\tilde{w}_n = (\alpha'_2\beta_1 + \alpha_2\beta'_1)w_0 - \alpha'_1\beta'_1 + \alpha_2\beta_2 w_0\tilde{w}_n - (\alpha'_1\beta_2 + \alpha_1\beta'_2)\tilde{w}_n = 0$ , but  $\alpha_2\beta_1 w_0 - (\alpha_1\beta'_1 + \alpha'_1\beta_1) - \alpha_1\beta_2\tilde{w}_n \neq 0$ , then the SLP with separated eigenparameter-dependent boundary conditions (1), (3) has exactly  $m + n + 2$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 1$ .

(5) If  $\alpha'_2\beta'_2 = \alpha'_2\beta'_1 w_0 + (\alpha_2\beta'_2 + \alpha'_2\beta_2)w_0\tilde{w}_n - \alpha'_1\beta'_2\tilde{w}_n = (\alpha'_2\beta_1 + \alpha_2\beta'_1)w_0 - \alpha'_1\beta'_1 + \alpha_2\beta_2 w_0\tilde{w}_n - (\alpha'_1\beta_2 + \alpha_1\beta'_2)\tilde{w}_n = \alpha_2\beta_1 w_0 - (\alpha_1\beta'_1 + \alpha'_1\beta_1) - \alpha_1\beta_2\tilde{w}_n = 0$ , but  $\alpha_1\beta_1 \neq 0$ , then the SLP with separated eigenparameter-dependent boundary conditions (1), (3) has exactly  $m + n + 1$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n$ .

(6) If none of the above conditions holds, then the SLP with separated eigenparameter dependent boundary conditions (1), (3) either has  $l$  eigenvalues for  $l \in \{1, 2, \dots, m + n\}$  or is degenerate.

*Proof.* We will prove the case (1) only, and the other cases can be proved in the same way, hence is omitted here. From Lemma 2.8 and the time scale  $\mathbb{T} = [a, b] \cup \{c\} \cup [d, e]$  with  $-\infty < a < b < c < d < e < +\infty$  we know that

$$\delta(\lambda) = \det(A_\lambda) + \det(B_\lambda) + c_{11}\phi_{11}(e, \lambda) + c_{12}\phi_{12}(e, \lambda) + c_{21}\phi_{21}(e, \lambda) + c_{22}\phi_{22}(e, \lambda),$$

note from Corollary 3.4 that the degree of  $\phi_{11}(e, \lambda), \phi_{12}(e, \lambda), \phi_{21}(e, \lambda)$ , and  $\phi_{22}(e, \lambda)$  in  $\lambda$  are  $m + n + 2, m + n + 1, m + n + 3$ , and  $m + n + 2$ , respectively. Thus when  $\alpha'_2\beta'_2 \neq 0$ , it can be concluded from Corollary 3.4 that the degree of the characteristic polynomial function  $\delta(\lambda)$  is  $m + n + 5$ , hence from Fundamental Theorem of Algebra we know that  $\delta(\lambda)$  has exactly  $m + n + 5$  roots, thus  $m + n + 5$  eigenvalues for SLP (1), (3) by Lemma 2.8.  $\square$

Consider the SLP consisting of the equation (1) together with coupled eigenparameter dependent boundary conditions (4). Then we have the following theorem.

**Theorem 3.6.** Let  $m, n \in \mathbb{N}$ , but  $\bar{w}_0 + w(c)(d - c) \neq 0$ , let (10)-(13) hold. Consider the SLP (1), (4). Then:

(1) If  $\alpha'_4\beta'_2 - \alpha'_2\beta'_4 \neq 0$ , then the SLP with coupled eigenparameter-dependent boundary conditions (1), (4) has exactly  $m + n + 5$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 4$ .

(2) If  $\alpha'_4\beta'_2 - \alpha'_2\beta'_4 = 0$ , but  $(\alpha'_3\beta'_2 - \alpha'_2\beta'_3)w_0 + (\alpha_4\beta'_2 + \alpha'_4\beta_2 - \alpha_2\beta'_4 - \alpha'_2\beta_4)w_0\bar{w}_n + (\alpha'_1\beta'_4 - \alpha'_4\beta'_1)\bar{w}_n \neq 0$ , then the SLP with coupled eigenparameter-dependent boundary conditions (1), (4) has exactly  $m + n + 4$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 3$ .

(3) If  $\alpha'_4\beta'_2 - \alpha'_2\beta'_4 = (\alpha'_3\beta'_2 - \alpha'_2\beta'_3)w_0 + (\alpha_4\beta'_2 + \alpha'_4\beta_2 - \alpha_2\beta'_4 - \alpha'_2\beta_4)w_0\bar{w}_n + (\alpha'_1\beta'_4 - \alpha'_4\beta'_1)\bar{w}_n = 0$ , but  $(\alpha_3\beta'_2 + \alpha'_3\beta_2 - \alpha_2\beta'_3 - \alpha'_2\beta_3)w_0 + (\alpha'_1\beta'_3 - \alpha'_3\beta'_1) + (\alpha_4\beta_2 - \alpha_2\beta_4)w_0\bar{w}_n + (\alpha_1\beta'_4 + \alpha'_1\beta_4 - \alpha_4\beta'_1 - \alpha'_4\beta_1)\bar{w}_n \neq 0$ , then the SLP with coupled eigenparameter-dependent boundary conditions (1), (4) has exactly  $m + n + 3$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 2$ .

(4) If  $\alpha'_4\beta'_2 - \alpha'_2\beta'_4 = (\alpha'_3\beta'_2 - \alpha'_2\beta'_3)w_0 + (\alpha_4\beta'_2 + \alpha'_4\beta_2 - \alpha_2\beta'_4 - \alpha'_2\beta_4)w_0\bar{w}_n + (\alpha'_1\beta'_4 - \alpha'_4\beta'_1)\bar{w}_n = (\alpha_3\beta'_2 + \alpha'_3\beta_2 - \alpha_2\beta'_3 - \alpha'_2\beta_3)w_0 + (\alpha'_1\beta'_3 - \alpha'_3\beta'_1) + (\alpha_4\beta_2 - \alpha_2\beta_4)w_0\bar{w}_n + (\alpha_1\beta'_4 + \alpha'_1\beta_4 - \alpha_4\beta'_1 - \alpha'_4\beta_1)\bar{w}_n = 0$ , but  $(\alpha_3\beta_2 - \alpha_2\beta_3)w_0 + (\alpha_1\beta'_3 + \alpha'_1\beta_3 - \alpha_3\beta'_1 - \alpha'_3\beta_1) + (\alpha_1\beta_4 - \alpha_4\beta_1)\bar{w}_n \neq 0$ , then the SLP with coupled eigenparameter dependent boundary conditions (1), (4) has exactly  $m + n + 2$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n + 1$ .

(5) If  $\alpha'_4\beta'_2 - \alpha'_2\beta'_4 = (\alpha'_3\beta'_2 - \alpha'_2\beta'_3)w_0 + (\alpha_4\beta'_2 + \alpha'_4\beta_2 - \alpha_2\beta'_4 - \alpha'_2\beta_4)w_0\bar{w}_n + (\alpha'_1\beta'_4 - \alpha'_4\beta'_1)\bar{w}_n = (\alpha_3\beta'_2 + \alpha'_3\beta_2 - \alpha_2\beta'_3 - \alpha'_2\beta_3)w_0 + (\alpha'_1\beta'_3 - \alpha'_3\beta'_1) + (\alpha_4\beta_2 - \alpha_2\beta_4)w_0\bar{w}_n + (\alpha_1\beta'_4 + \alpha'_1\beta_4 - \alpha_4\beta'_1 - \alpha'_4\beta_1)\bar{w}_n = (\alpha_3\beta_2 - \alpha_2\beta_3)w_0 + (\alpha_1\beta'_3 + \alpha'_1\beta_3 - \alpha_3\beta'_1 - \alpha'_3\beta_1) + (\alpha_1\beta_4 - \alpha_4\beta_1)\bar{w}_n = 0$ , but  $\alpha_1\beta_3 - \alpha_3\beta_1 \neq 0$ , then the SLP with coupled eigenparameter-dependent boundary conditions (1), (4) has exactly  $m + n + 1$  eigenvalues  $\lambda_j, j = 0, 1, \dots, m + n$ .

(6) If none of the above conditions holds, then the SLP with coupled eigenparameter dependent boundary conditions (1), (4) either has  $l$  eigenvalues for  $l \in \{1, 2, \dots, m + n\}$  or is degenerate.

*Proof.* The proof is similar to the one for Theorem 3.5.  $\square$

**Corollary 3.7.** Assume that equation (1) is defined on time scale  $\mathbb{T}$ . Here  $\mathbb{T} = \{e_1, \dots, e_m\} \cup S, S = \cup_{i=1}^n [a_i, b_i], b_i < a_{i+1}, i = 1, 2, \dots, n - 1, -\infty < e_j < +\infty, m, n < +\infty$ . Assume that  $a$  is the lower bound of  $\mathbb{T}$  and  $b$  is the upper bound of  $\mathbb{T}$ . The SLP consisting of equation (1) with eigenparameter-dependent boundary conditions (3) and (4) respectively. We have partition of every interval  $[a_i, b_i]$  similar to (10). And the coefficients are defined as (11)-(13). Then the SLP has exactly finite eigenvalues.

To illustrate our main results two examples are given as follows:

**Example 1.** Consider the SLP with separated eigenparameter-dependent boundary conditions

$$\begin{cases} -(px^\Delta)^\Delta + qx^\sigma = \lambda wx^\sigma, & t \in \mathbb{T} = [-4, -1] \cup \{0\} \cup [1, 4], \\ \lambda x(-4) + (px^\Delta)(-4) = 0, \\ 2x(4) + (\lambda - 1)(px^\Delta)(4) = 0. \end{cases} \tag{17}$$

Let  $m = 1$  and  $n = 1$ , and  $p, q, w$  are piecewise constant functions defined as follows:

$$r(t) = \frac{1}{p(t)} = \begin{cases} 0, & t \in [-4, -3) \\ 1, & t \in [-3, -2) \\ 0, & t \in [-2, -1) \\ 1, & t = -1 \\ 1, & t = 0 \\ 0, & t \in [1, 2) \\ 1, & t \in [2, 3) \\ 0, & t \in [3, 4) \\ 1, & t = 4, \end{cases} \quad q(t) = \begin{cases} 1, & t \in [-4, -3) \\ 0, & t \in [-3, -2) \\ 1, & t \in [-2, -1) \\ 0, & t = 0 \\ 1, & t \in [1, 2) \\ 0, & t \in [2, 3) \\ 1, & t \in [3, 4], \end{cases} \quad w(t) = \begin{cases} 1, & t \in [-4, -3) \\ 0, & t \in [-3, -2) \\ 1, & t \in [-2, -1) \\ 0, & t = 0 \\ 1, & t \in [1, 2) \\ 0, & t \in [2, 3) \\ 1, & t \in [3, 4]. \end{cases} \tag{18}$$

From the conditions given we know that

$$A_\lambda = \begin{bmatrix} \lambda & 1 \\ 0 & 0 \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} 0 & 0 \\ 2 & \lambda - 1 \end{bmatrix},$$

and

$$\theta_1 = -1 \neq 0, \quad \theta_2 = 2 \neq 0.$$

Then we can deduce that the characteristic function

$$\begin{aligned} \delta(\lambda) &= -2\phi_{11} + 2\lambda\phi_{12} + (1 - \lambda)\phi_{21} + \lambda(\lambda - 1)\phi_{22} \\ &= 2\lambda^6 - 26\lambda^5 + 125\lambda^4 - 271\lambda^3 + 251\lambda^2 - 62\lambda - 13. \end{aligned}$$

Accordingly, the SLP with separated eigenparameter-dependent boundary conditions (17), (18) has exactly  $m + n + 4 = 6$  eigenvalues

$$\lambda_0 = -0.130464, \quad \lambda_1 = 0.714767, \quad \lambda_2 = 1.62582, \quad \lambda_3 = 2.58332, \quad \lambda_4 = 3.61264, \quad \lambda_5 = 4.59392.$$

The graph of the characteristic function is displayed in Figure 1.

**Example 2.** Consider the SLP with coupled eigenparameter-dependent boundary conditions

$$\begin{cases} -(px^\Delta)^\Delta + qx^\sigma = \lambda wx^\sigma, & t \in \mathbb{T} = [-3, 0] \cup \{\frac{1}{2}\} \cup [1, 4], \\ \lambda x(-3) + (px^\Delta)(-3) + (2\lambda + 1)x(4) + 3\lambda(px^\Delta)(4) = 0, \\ -x(-3) + (2\lambda + 1)(px^\Delta)(-3) + (\lambda + 2)x(4) + (px^\Delta)(4) = 0. \end{cases} \quad (19)$$

Let  $m = 1$  and  $n = 1$ , and  $p, q, w$  are piecewise constant functions defined as follows:

$$r(t) = \frac{1}{p(t)} = \begin{cases} 0, & t \in [-3, -2) \\ 4, & t \in [-2, -1) \\ 0, & t \in [-1, 0) \\ 1, & t = 0 \\ 1, & t = \frac{1}{2} \\ 0, & t \in [1, 2) \\ 2, & t \in [2, 3) \\ 0, & t \in [3, 4) \\ 1, & t = 4, \end{cases} \quad q(t) = \begin{cases} \frac{1}{2}, & t \in [-3, -2) \\ 0, & t \in [-2, -1) \\ 1, & t \in [-1, 0] \\ 1, & t = \frac{1}{2} \\ 3, & t \in [1, 2) \\ 0, & t \in [2, 3) \\ 1, & t \in [3, 4], \end{cases} \quad w(t) = \begin{cases} 1, & t \in [-3, -2) \\ 0, & t \in [-2, -1) \\ 3, & t \in [-1, 0] \\ \frac{1}{2}, & t = \frac{1}{2} \\ 1, & t \in [1, 2) \\ 0, & t \in [2, 3) \\ 2, & t \in [3, 4]. \end{cases} \quad (20)$$

From the conditions given we know that

$$\hat{A}_\lambda = \begin{bmatrix} \lambda & 1 \\ -1 & 2\lambda + 1 \end{bmatrix}, \quad \hat{B}_\lambda = \begin{bmatrix} 2\lambda + 1 & 3\lambda \\ \lambda + 2 & 1 \end{bmatrix},$$

and

$$\det(\hat{A}_\lambda) \neq 0, \quad \det(\hat{B}_\lambda) \neq 0,$$

$$\text{rank} \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 1 & 2 & 1 \end{bmatrix} = 2, \quad \text{rank} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix} = 2,$$

$$\text{rank} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 3 \end{bmatrix} = 2, \quad \text{rank} \begin{bmatrix} -1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} = 2.$$

Then we can deduce that the characteristic function

$$\begin{aligned} \delta(\lambda) &= \det(\hat{A}_\lambda) + \det(\hat{B}_\lambda) + (4\lambda^2 + 3\lambda - 1)\phi_{11} + (\lambda^2 + 4\lambda + 1)\phi_{12} + (6\lambda^2 + 3\lambda - 1)\phi_{21} + 4\lambda\phi_{22} \\ &= \frac{1}{64}(-8640\lambda^7 + 88272\lambda^6 - 264504\lambda^5 + 215030\lambda^4 + 187565\lambda^3 - 355140\lambda^2 + 134867\lambda - 4850). \end{aligned}$$

Hence the SLP with coupled eigenparameter-dependent boundary conditions (19), (20) has exactly  $m+n+5 = 7$  eigenvalues

$$\begin{aligned} \lambda_0 &= -1.06332, \quad \lambda_1 = 0.0401026, \quad \lambda_2 = 0.597517, \quad \lambda_3 = 1.16765 - 0.336124i, \\ \lambda_4 &= 1.16765 + 0.336124i, \quad \lambda_5 = 2.6273, \quad \lambda_6 = 5.67976. \end{aligned}$$

The graph of the characteristic function is displayed in Figures 2 and 3.

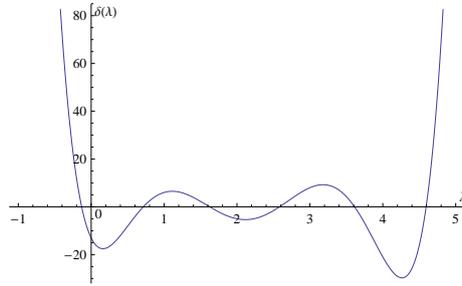


Figure 1: Characteristic Function in Example 1

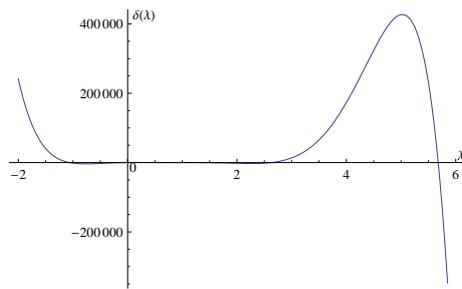


Figure 2: Characteristic Function in Example 2

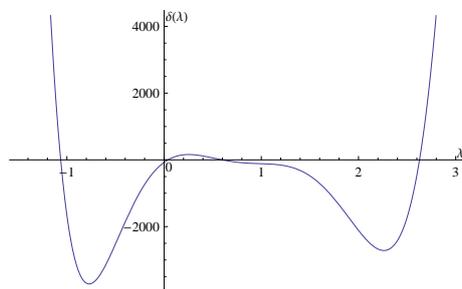


Figure 3: Characteristic Function in Example 2

## References

- [1] R. Agarwal, M. Bohner, P. Wong, Sturm-Liouville eigenvalue problems on time scales, *Appl. Math. Comput.* 99 (1999) 153–166.
- [2] Z. Akdoğan, M. Demirci, O. Mukhtarov, Green function of discontinuous boundary-value problem with transmission conditions, *Math. Meth. Appl. Sci.* 30 (2007) 1719–1738.
- [3] J.J. Ao, J. Sun, Matrix representations of Sturm-Liouville problems with eigenparameter-dependent boundary conditions, *Linear Algebra Appl.* 438 (2013) 2359–2365.
- [4] J.J. Ao, J. Sun, Matrix representations of Sturm-Liouville problems with coupled eigenparameter-dependent boundary conditions, *Appl. Math. Comput.* 244 (2014) 142–148.
- [5] J.J. Ao, J. Sun, M.Z. Zhang, The finite spectrum of Sturm-Liouville problems with transmission conditions, *Appl. Math. Comput.* 218 (2011) 1166–1173.
- [6] F. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York/London, 1964.
- [7] P. Binding, P. Browne, B. Watson, Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, *J. Comp. Appl. Math.* 148 (2002) 147–168.
- [8] P. Binding, P. Browne, B. Watson, Inverse spectral problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions, *Proc. London Math. Soc.* 62 (2004) 161–182.
- [9] C. Fulton, S. Pruess, Numerical methods for a singular eigenvalue problem with eigenparameter in the boundary conditions, *J. Math. Anal. Appl.* 71 (1979) 431–462.
- [10] Q. Kong, Sturm-Liouville eigenvalue problems on time scales with separated boundary conditions, *Results Math.* 52 (2008) 111–121.
- [11] Q. Kong, H. Volkmer, A. Zettl, Matrix representations of Sturm-Liouville problems with finite spectrum, *Results Math.* 54 (2009) 103–116.
- [12] Q. Kong, H. Wu, A. Zettl, Sturm-Liouville problems with finite spectrum, *J. Math. Anal. Appl.* 263 (2001) 748–762.
- [13] Q. Kong, A. Zettl, Inverse Sturm-Liouville problems with finite spectrum, *J. Math. Anal. Appl.* 386 (2012) 1–9.
- [14] H. Tuna, Completeness theorem for the dissipative Sturm-Liouville operator on bounded time scales, *Indian J. Pure Appl. Math.* 47 (2016) 535–544.
- [15] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.
- [16] A. Zettl, Sturm-Liouville Theory, in *Mathematical Surveys and Monographs*, 121, Amer. Math. Soc., Providence RI, 2005.
- [17] C. Zhang, D. Yang, Eigenvalues of second-order linear equations with coupled boundary conditions on time scales, *J. Appl. Math. Comput.* 33 (2010) 1–21.
- [18] N. Zhao, The finite spectrum of Sturm-Liouville problems on time scales, *Journal of Shandong University (Natural Science)* 48 (2013) 96–102. (in Chinese)