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The Point Spectrum and Residual Spectrum of Upper Triangular Operator Matrices

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Abstract. The point and residual spectra of an operator are, respectively, split into 1,2-point spectrum and 1,2-residual spectrum, based on the denseness and closedness of its range. Let $\mathcal{H}, \mathcal{K}$ be infinite dimensional complex separable Hilbert spaces and write $M_X = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. For given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the sets $\bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{*,i}(M_X)$ $(* = p, r; i = 1, 2)$, are characterized. Moreover, we obtain some necessary and sufficient condition such that $\sigma_{*,i}(M_X) = \sigma_{*,i}(A) \cup \sigma_{*,i}(B)$ $(* = p, r; i = 1, 2)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

1. Introduction

We assume throughout that $\mathcal{H}$ and $\mathcal{K}$ are both complex separable infinite dimensional Hilbert spaces. If A is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$, we write $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and, if $\mathcal{H} = \mathcal{K}$, $A \in \mathcal{B}(\mathcal{H})$. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$ and simply by I if the underlying space is clear from the context. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are, respectively, used to denote the kernel and the range of A, and we write $n(A) := \dim \mathcal{N}(A)$ and $d(A) := \dim \mathcal{R}(A)^{\perp}$.

If there exists an operator $A_l^{-1} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $A_l^{-1}A = I_{\mathcal{H}}$ (resp. $AA_r^{-1} = I_{\mathcal{K}}$), then A is said to be left (resp. right) invertible. If there exists an operator $A^{-1} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $A^{-1}A = I_{\mathcal{H}}$ and $AA^{-1} = I_{\mathcal{K}}$, then we call it invertible. Obviously, A is invertible if and only if A is both left and right invertible. In the Hillbert space, we have the following well-known properties: (i) A is left invertible if and only if A is bounded below, and if and only if A is injective, i.e., $\mathcal{N}(A) = \{0\}$ and its range $\mathcal{R}(A)$ is closed; (ii) A is right invertible if and only if A is surjective, i.e., $\mathcal{R}(A) = \mathcal{K}$ (see [2]). According to the Fredholm alternative theorem, A is left (resp. right) invertible if and only if A^* is right (resp. left) invertible, where $(\cdot)^*$ denotes the adjoint operation.

Recall we say that the operator A^+ is the Moore-Penrose inverse of A in $\mathcal{B}(\mathcal{K}, \mathcal{H})$, if it solves the following system of operator equations

$$\begin{aligned} AA^+A = A, &\quad A^+AA^+ = A^+, \\ (AA^+)^* = AA^+, &\quad (A^+A)^* = A^+A. \end{aligned}$$

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Note that A is Moore-Penrose invertible if and only if its range $\mathcal{R}(A)$ is closed (see [1]).

Now, let $\mathcal{H} = \mathcal{K}$, i.e., $A \in \mathcal{B}(\mathcal{H})$. Then, the sets

$$\begin{aligned}
&\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not invertible}\},\\
&\sigma_p(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is noninjective}\},\\
&\sigma_r(A) = \left\{\lambda \in \mathbb{C} : A - \lambda \text{ is injective and } \overline{\mathcal{R}(A-\lambda)} \neq \mathcal{H}\right\},\\
&\sigma_c(A) = \left\{\lambda \in \mathbb{C} : A - \lambda \text{ is injective}, \overline{\mathcal{R}(A-\lambda)} = \mathcal{H} \text{ and } \mathcal{R}(A-\lambda) \neq \mathcal{H}\right\},\\
&\sigma_m(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Moore-Penrose invertible}\},\\
&\sigma_l(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not left invertible}\},\\
&\sigma_\delta(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not right invertible}\}.
\end{aligned}$$

are the spectrum, point spectrum, residual spectrum, continuous spectrum, Moore-Penrose spectrum, left spectrum and right spectrum of A, respectively. As usual, the resolvent set of A is defined by $\rho(A) = \mathbb{C}\backslash\sigma(A)$. For convenience, we write $\rho_m(A) = \mathbb{C}\backslash\sigma_m(A)$ and $\rho_l(A) = \mathbb{C}\backslash\sigma_l(A)$. In terms of the density and the closedness of $\mathcal{R}(A-\lambda)$, the point spectrum $\sigma_p(A)$ and the residual spectrum $\sigma_r(A)$ of A have the following subdivisions: $\sigma_p(A) = \sigma_{p,1}(A) \cup \sigma_{p,2}(A)$ (see [1, p. 89]) and $\sigma_r(A) = \sigma_{r,1}(A) \cup \sigma_{r,2}(A)$, where

$$\begin{aligned}
&\sigma_{p,1}(A) = \left\{\lambda \in \sigma_p(A) : \overline{\mathcal{R}(A-\lambda)} = \mathcal{H}\right\},\\
&\sigma_{p,2}(A) = \left\{\lambda \in \sigma_p(A) : \overline{\mathcal{R}(A-\lambda)} \neq \mathcal{H}\right\},\\
&\sigma_{r,1}(A) = \left\{\lambda \in \sigma_r(A) : \mathcal{R}(A-\lambda) \text{ is closed}\right\},\\
&\sigma_{r,2}(A) = \left\{\lambda \in \sigma_r(A) : \mathcal{R}(A-\lambda) \text{ is not closed}\right\}.
\end{aligned}$$

As we will see, the above subdivisions closely connect with the relevant space decomposition, and are useful in the research of spectral inclusion properties of operators.

For given diagonal entries $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the authors have extensively studied the upper triangular operator matrix

$$M_X = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$$

with an unknown operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. See, e.g., [3–19]. In [5, 6, 9, 10, 12, 14–18], the perturbations of different spectra (the spectra, left (right) spectra, point spectra, continuous spectra, residual spectra,$\cdots$) of M_X were discussed. In [14, 15], the sets

$$\bigcup_{X\in Inv(\mathcal{K},\mathcal{H})} \sigma_l(M_X) \quad \text{and} \quad \bigcup_{X\in Inv(\mathcal{K},\mathcal{H})} \sigma_{lw}(M_X)$$

were characterized, where $\sigma_{lw}(\cdot)$ and $Inv(\mathcal{K}, \mathcal{H})$ denote the left Weyl spectrum and the set of all invertible operators from $\mathcal{K}$ into $\mathcal{H}$. In [13], the set $\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_X)$ was given by

$$\begin{aligned}
\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_X) = [&\{\lambda \in \sigma_m(A) : d(A-\lambda) + d(B-\lambda) > 0\}\\
&\cup\{\lambda \in \mathbb{C} : n(B-\lambda) \leq d(A-\lambda), n(B-\lambda) < d(A-\lambda) + d(B-\lambda)\}\\
&\cup\{\lambda \in \mathbb{C} : n(B-\lambda) = d(A-\lambda) = \infty\}] \setminus \sigma_p(A).
\end{aligned} \tag{1}$$

In [7, 8, 10, 11, 19] the authors were interested by the following equality

$$\sigma_*(M_X) = \sigma_*(A) \cup \sigma_*(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K}, \mathcal{H}),$$

where $\sigma_* \in \{\sigma, \sigma_e, \sigma_w, \sigma_b\}$, $\sigma_e(\cdot), \sigma_w(\cdot)$ and $\sigma_b(\cdot)$ denote the essential spectrum, Weyl spectrum and Browder spectrum.

One aim of the present paper is to describe the sets

$$\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{p,1}(M_X),\ \bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{p,2}(M_X),\ \bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{r,1}(M_X),\ \bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{r,2}(M_X).$$

The other aim is to explore the relation between $\sigma_{*,i}(M_X)$ and $\sigma_{*,i}(A)\cup\sigma_{*,i}(B)$ $(* = p, r; i = 1, 2)$. As a byproduct, we also obtain some necessary and sufficient condition of

$$\sigma_{*,i}(M_X) = \sigma_{*,i}(A)\cup\sigma_{*,i}(B)(* = p, r; i = 1, 2) \quad \text{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

in terms of the spectral properties of two diagonal entries A and B in M_X.

2. Main Results

We first review some auxiliary lemmas, which are useful to prove the main results.

Lemma 2.1 (see [13, Lemma 2.3]). *Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with $\mathcal{R}(A)$ nonclosed. Then, there exists a closed subspace $\Omega \subset \overline{\mathcal{R}(A)}$ of $\mathcal{H}$ such that $\Omega\cap\mathcal{R}(A) = \{0\}$ and $\dim\Omega = \infty$.*

The following Lemmas are obvious, and their proofs are omitted here.

Lemma 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then, M_X is injective for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if A and B are both injective.*

Lemma 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\overline{\mathcal{R}(M_X)} = \mathcal{H}\oplus\mathcal{K} \text{ for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

if and only if $\overline{\mathcal{R}(A)} = \mathcal{H}$ and $\overline{\mathcal{R}(B)} = \mathcal{K}$.

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, then*

$$\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{p,1}(M_X) = \Delta_1\cup\Delta_2\cup\Delta_3,$$

where

$$\begin{aligned}
\Delta_1 &= (\sigma_{p,1}(B)\cap\sigma_m(B))\cup(\sigma_p(A)\cap\sigma_c(B)),\\
\Delta_2 &= (\sigma_{p,1}(A)\cap\rho(B))\cup(\sigma_{p,1}(B)\cap\rho(A))\\
&\quad\cup(\sigma_{p,1}(A)\cap\sigma_{p,1}(B))\cup(\sigma_{p,1}(B)\cap\sigma_c(A))\\
&\quad\cup\{\lambda\in\sigma_{p,1}(B)\cap\sigma_r(A) : n(B-\lambda) > d(A-\lambda)\}\\
&\quad\cup\{\lambda\in\sigma_{p,1}(B)\cap\sigma_{p,2}(A) : n(B-\lambda) \geq d(A-\lambda)\},\\
\Delta_3 &= \{\lambda\in\sigma_{p,1}(B) : n(B-\lambda) = \infty\}.
\end{aligned}$$

Proof. First, we prove that $\overset{3}{\underset{k=1}{\cup}}\Delta_k \subseteq \bigcup_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})}\sigma_{p,1}(M_X)$. Without loss of generality, we only prove the case when $\lambda = 0$ in what follows. Let $0 \in \Delta_1$. Since $\mathcal{R}(B)$ is not closed, then $\mathcal{R}(B^*)$ is not closed. By Lemma 2.1, there exists an infinite dimensional closed subspace $\Omega \subset \overline{\mathcal{R}(B^*)} = \mathcal{N}(B)^\perp$ such that $\Omega\cap\mathcal{R}(B^*) = \{0\}$. If $n(A^*) < \infty$, then there exist closed subspaces Ω_1 and Ω_2 of Ω such that $\dim\Omega_1 = n(A^*)$ and $\Omega = \Omega_1\oplus\Omega_2$. Define $X_0^* \in \mathcal{B}(\mathcal{H},\mathcal{K})$ by

$$X_0^* = \begin{pmatrix} 0 & X_1^* \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^*)^\perp \\ \mathcal{N}(A^*) \end{pmatrix} \to \begin{pmatrix} \Omega_1 \\ \Omega_2\oplus\Omega^\perp \end{pmatrix},$$

where $X_1^*: \Omega_1 \to \mathcal{R}(A)^\perp$ is a unitary operator. Then, $M_{X_0}^*$ can be written as

$$M_{X_0}^* = \begin{pmatrix} A_1^* & 0 & 0 \\ 0 & X_1^* & B_1^* \\ 0 & 0 & B_2^* \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^*)^\perp \\ \mathcal{N}(A^*) \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \Omega_1 \\ \Omega_2 \oplus \Omega^\perp \end{pmatrix}.$$

Clearly, A_1^* and B^* are injective, and so by $\Omega \cap \mathcal{R}(B^*) = \{0\}$, one can see that $M_{X_0}^*$ is injective. On the other hand, we obtain that $\overline{\mathcal{R}(M_{X_0}^*)} \neq \mathcal{H} \oplus \mathcal{K}$ from $0 \in \sigma_{p,1}(B) \cup \sigma_p(A)$. This implies that $\overline{\mathcal{R}(M_{X_0})} = \mathcal{H} \oplus \mathcal{K}$ and M_{X_0} is noninjective. Therefore $0 \in \sigma_{p,1}(M_{X_0})$. If $n(A^*) = \infty$, then one can define a unitary operator X_1^* from $\mathcal{N}(A^*)$ onto Ω. Taking

$$X_0^* = \begin{pmatrix} 0 & X_1^* \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^*)^\perp \\ \mathcal{N}(A^*) \end{pmatrix} \to \begin{pmatrix} \Omega \\ \Omega^\perp \end{pmatrix},$$

we know that $M_{X_0}^*$ is clearly injective and $\overline{\mathcal{R}(M_{X_0}^*)} \neq \mathcal{H} \oplus \mathcal{K}$. Hence $0 \in \sigma_{p,1}(M_{X_0})$.

Let $0 \in \Delta_2$. If $0 \in (\sigma_{p,1}(A) \cap \rho(B)) \cup (\sigma_{p,1}(B) \cap \rho(A)) \cup (\sigma_{p,1}(A) \cap \sigma_{p,1}(B))$, then by Lemma 2.3, $\overline{\mathcal{R}(M_X)} = \mathcal{H} \oplus \mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Since $0 \in \sigma_{p,1}(A)$ or $0 \in \sigma_{p,1}(B) \cap \rho(A)$, it follows that $0 \in \sigma_p(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Hence $0 \in \sigma_{p,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $0 \in \sigma_{p,1}(B) \cap \sigma_c(A)$, then define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by $X_0 = 0$. It is obvious that $0 \in \sigma_{p,1}(M_{X_0})$. If $0 \in \sigma_{p,1}(B) \cap \sigma_r(A)$ and $n(B) > d(A)$, then there exists a finite dimensional subspace Ω of $\mathcal{N}(B)$ such that $\dim \Omega = d(A)$ and $\mathcal{N}(B) = \Omega \oplus \Omega^\perp$. Define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by

$$X_0 = \begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \Omega \\ \Omega^\perp \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \end{pmatrix}, \tag{2}$$

where $X_1: \Omega \to \mathcal{R}(A)^\perp$ is a unitary operator. Then, M_{X_0} can be written as

$$M_{X_0} = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & X_1 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \Omega \\ \Omega^\perp \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix},$$

Clearly, we have $0 \in \sigma_{p,1}(M_{X_0})$. If $0 \in \sigma_{p,1}(B) \cap \sigma_{p,2}(A)$ and $n(B) \geq d(A)$, then define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ as in (2).

Now, let $0 \in \Delta_3$. Then one can define a unitary operator X_1 from $\mathcal{N}(B)$ onto $\mathcal{H}$. Taking $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$

$$X_0 = \begin{pmatrix} X_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \mathcal{H},$$

we have the operator matrix

$$M_{X_0} = \begin{pmatrix} A & X_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}.$$

From the relation

$$\begin{pmatrix} A & X_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix} \begin{pmatrix} I & -X_1^{-1}A & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & X_1 & 0 \\ 0 & 0 & B_1 \end{pmatrix}$$

and $0 \in \sigma_{p,1}(B)$, we obtain that $0 \in \sigma_{p,1}(M_{X_0})$.

For the opposite inclusion, it suffices to prove that $0 \notin \bigcup_{k=1}^{3} \Delta_k$ implies $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_{p,1}(M_X)$. Now we consider four cases.

Case 1: A and B are both injective. Obviously, M_X is injective for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ from Lemma 2.2 . Therefore $0 \notin \bigcup_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_{p,1}(M_X)$.

Case 2: $\overline{\mathcal{R}(B)} \neq \mathcal{K}$. Then $\overline{\mathcal{R}(M_X)} \neq \mathcal{H}\oplus\mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Therefore $0 \notin \bigcup_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_{p,1}(M_X)$.

Case 3: B is surjective and $n(B) < d(A)$. Indeed, M_X as an operator from $\mathcal{N}(A)^\perp \oplus \mathcal{N}(A) \oplus \mathcal{N}(B)^\perp \oplus \mathcal{N}(B)$ into $\overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp \oplus \mathcal{K}$ admits the following block representation

$$M_X = \begin{pmatrix} A_1 & 0 & X_1 & X_2 \\ 0 & 0 & X_3 & X_4 \\ 0 & 0 & B_1 & 0 \end{pmatrix}$$

for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, where $B_1 : \mathcal{N}(B)^\perp \to \mathcal{K}$ is invertible. Then there is an invertible operator

$$U = \begin{pmatrix} I & 0 & -X_1B_1^{-1} \\ 0 & I & -X_3B_1^{-1} \\ 0 & 0 & I \end{pmatrix} : \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix} \tag{3}$$

such that

$$UM_X = \begin{pmatrix} A_1 & 0 & 0 & X_2 \\ 0 & 0 & 0 & X_4 \\ 0 & 0 & B_1 & 0 \end{pmatrix}. \tag{4}$$

In view of $n(B) < d(A)$, we see that $\overline{\mathcal{R}(X_4)} \neq \mathcal{R}(A)^\perp$. It follows from (4) that $\overline{\mathcal{R}(M_X)} \neq \mathcal{H}\oplus\mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Then $0 \notin \bigcup_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_{p,1}(M_X)$.

Case 4: A is injective, B is surjective and $n(B) = d(A) < \infty$. Then we have

$$M_X = \begin{pmatrix} A_1 & X_1 & X_2 \\ 0 & X_3 & X_4 \\ 0 & B_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B)^\perp \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{R}(A)} \\ \mathcal{R}(A)^\perp \\ \mathcal{K} \end{pmatrix}$$

for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, where $B_1 : \mathcal{N}(B)^\perp \to \mathcal{K}$ is invertible. We also have

$$U\begin{pmatrix} A_1 & X_1 & X_2 \\ 0 & X_3 & X_4 \\ 0 & B_1 & 0 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & X_2 \\ 0 & 0 & X_4 \\ 0 & B_1 & 0 \end{pmatrix},$$

where U as in (3). Note that $n(B) = d(A) < \infty$. If $X_4 : \mathcal{N}(B) \to \mathcal{R}(A)^\perp$ is noninjective, then $\overline{\mathcal{R}(X_4)} \neq \mathcal{R}(A)^\perp$, and hence $\overline{\mathcal{R}(M_X)} \neq \mathcal{H}\oplus\mathcal{K}$. If $X_4 : \mathcal{N}(B) \to \mathcal{R}(A)^\perp$ is injective, then M_X is injective. This implies that $0 \notin \bigcup_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_{p,1}(M_X)$.

Corollary 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{p,1}(M_X) \subseteq \sigma_{p,1}(A) \cup \sigma_{p,1}(B) \quad \textit{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

if and only if $\sigma_{p,2}(A) \cap \sigma_c(B) = \emptyset$.

Proof. Sufficiency. By Theorem 2.4, we have

$$\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{p,1}(M_X) \subseteq \sigma_{p,1}(A) \cup \sigma_{p,1}(B) \cup (\sigma_{p,2}(A) \cap \sigma_c(B)).$$

If $\sigma_{p,2}(A) \cap \sigma_c(B) = \emptyset$, then $\sigma_{p,1}(M_X) \subseteq \sigma_{p,1}(A) \cup \sigma_{p,1}(B)$ for every$X \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

Necessity. Assume to the contrary that there exists $\lambda_0 \in \mathbb{C}$, such that $\lambda_0 \in \sigma_{p,2}(A) \cap \sigma_c(B)$. By Theorem 2.4, $\lambda_0 \in \sigma_{p,1}(M_{X_0})$ for some $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$. This contradicts the assumption $\sigma_{p,1}(M_X) \subseteq \sigma_{p,1}(A) \cup \sigma_{p,1}(B)$ for every$X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, since $\lambda_0 \in \sigma_{p,2}(A) \cap \sigma_c(B)$.

Corollary 2.6. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{p,1}(M_X) = \sigma_{p,1}(A) \cup \sigma_{p,1}(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

if and only if $\sigma_{p,2}(A) \cap \sigma_c(B) = \emptyset$, and the following statements are fulfilled:

(i) *$\lambda \in \sigma_{p,1}(A)$ implies $\lambda \in \rho(B) \cup \sigma_{p,1}(B) \cup \sigma_c(B)$;*

(ii) *$\lambda \in \sigma_{p,1}(B)$ implies $\lambda \in \rho(A) \cup \sigma_{p,1}(A)$.*

Proof. Sufficiency. Assume that $\sigma_{p,2}(A) \cap \sigma_c(B) = \emptyset$. By Corollary 2.5, we get $\sigma_{p,1}(M_X) \subseteq \sigma_{p,1}(A) \cup \sigma_{p,1}(B)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Now, we prove the opposite inclusion. Suppose that $\lambda = 0$. If $0 \in \sigma_{p,1}(A)$, then $0 \in \rho(B) \cup \sigma_{p,1}(B) \cup \sigma_c(B)$, and hence $0 \in \sigma_{p,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ from the proof of Theorem 2.4. If $0 \in \sigma_{p,1}(B)$, then $0 \in \rho(A) \cup \sigma_{p,1}(A)$, and hence $0 \in \sigma_{p,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Therefore, $\sigma_{p,1}(A) \cup \sigma_{p,1}(B) \subseteq \sigma_{p,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

Necessity. Assume to the contrary that there exists $\lambda_0 \in \mathbb{C}$, such that one of the assertions (i) and (ii) fails to hold. There are three cases to consider.

Case 1: $\lambda_0 \in \sigma_{p,1}(A)$ and $\lambda_0 \in \sigma_{p,2}(B) \cup \sigma_r(B)$. Take $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$ by $X_0 = 0$. It is obvious that $\lambda_0 \notin \sigma_{p,1}(M_{X_0})$. This contradicts the assumption $\sigma_{p,1}(M_X) = \sigma_{p,1}(A) \cup \sigma_{p,1}(B)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, since $\lambda_0 \in \sigma_{p,1}(A) \cup \sigma_{p,1}(B)$.

Case 2: $\lambda_0 \in \sigma_{p,1}(B)$ and $\lambda_0 \in \sigma_{p,2}(A) \cup \sigma_r(A)$. Take $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$ by $X_0 = 0$. Then $\lambda_0 \in \sigma_{p,2}(M_{X_0})$, and hence $\lambda_0 \notin \sigma_{p,1}(M_{X_0})$.

Case 3: $\lambda_0 \in \sigma_{p,1}(B) \cap \sigma_c(A)$. By Lemma 2.1, there exists an infinite dimensional closed subspace $\Omega \subset \overline{\mathcal{R}(A-\lambda_0)}$ such that $\Omega \cap \mathcal{R}(A-\lambda_0) = \{0\}$. then we may further define a unitary operator X_1 from $\mathcal{N}(B-\lambda_0)$ to some closed subspace of Ω. Take $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$ by

$$X_0 = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0) \\ \mathcal{N}(B-\lambda_0)^{\perp} \end{pmatrix} \to \begin{pmatrix} \Omega \\ \Omega^{\perp} \end{pmatrix}. \tag{5}$$

Clearly, $M_{X_0} - \lambda_0$ is injective, and hence $\lambda_0 \notin \sigma_{p,1}(M_{X_0})$.

Theorem 2.7. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, then*

$$\begin{aligned}&\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{p,2}(M_X)\\ &= \sigma_{p,2}(A) \cup \sigma_{p,2}(B) \cup (\sigma_{p,1}(B) \cap \sigma_r(A)) \cup (\sigma_{p,1}(A) \cap \sigma_r(B)).\end{aligned}$$

Proof. Without loss of generality, we suppose that $\lambda = 0$. Let $0 \in \sigma_{p,2}(A) \cup \sigma_{p,2}(B) \cup (\sigma_{p,1}(B) \cap \sigma_r(A)) \cup (\sigma_{p,1}(A) \cap \sigma_r(B))$. Define $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$ by $X_0 = 0$. Clearly, we have $0 \in \sigma_{p,2}(M_{X_0})$.

Now, let $0 \notin \sigma_{p,2}(A) \cup \sigma_{p,2}(B) \cup (\sigma_p(B) \cap \sigma_r(A)) \cup (\sigma_p(A) \cap \sigma_r(B))$. Then we consider two cases:

Case 1: A and B are injective. By Lemma 2.2, M_X is injective for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Therefore $0 \notin \bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{p,2}(M_X)$.

Case 2: $\overline{\mathcal{R}(A)} = \mathcal{H}$ and $\overline{\mathcal{R}(B)} = \mathcal{K}$. By Lemma 2.3, $\overline{\mathcal{R}(M_X)} = \mathcal{H} \oplus \mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Hence $0 \notin \bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{p,2}(M_X)$.

Corollary 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{p,2}(M_X) \subseteq \sigma_{p,2}(A) \cup \sigma_{p,2}(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

if and only if $(\sigma_{p,1}(B) \cap \sigma_r(A)) \cup (\sigma_{p,1}(A) \cap \sigma_r(B)) = \emptyset$.

Proof. In the similar way as the proof of Corollary 2.5, using Theorem 2.7, we get the desired result.

Corollary 2.9. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{p,2}(M_X) = \sigma_{p,2}(A) \cup \sigma_{p,2}(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$$

if and only if $(\sigma_{p,1}(B) \cap \sigma_r(A)) \cup (\sigma_{p,1}(A) \cap \sigma_r(B)) = \emptyset$, and the following statements are fulfilled:

(i) $\lambda \in \sigma_{p,2}(A)$ *implies* $\lambda \in \sigma_{p,2}(B) \cup \sigma_r(B) \cup \rho(B) \cup \{\lambda \in \sigma_{p,1}(B) \cap \rho_m(B) : n(B-\lambda) < d(A-\lambda)\}$;

(ii) $\lambda \in \sigma_{p,2}(B)$ *implies* $\lambda \in \sigma_p(A) \cup \rho(A) \cup \{\lambda \in \sigma_{r,1}(A) : n(B-\lambda) > d(A-\lambda)\}$.

Proof. Sufficiency. Assume that $(\sigma_{p,1}(B) \cap \sigma_r(A)) \cup (\sigma_{p,1}(A) \cap \sigma_r(B)) = \emptyset$ and assertions (i) and (ii) hold. By Corollary 2.8, we get $\sigma_{p,2}(M_X) \subseteq \sigma_{p,2}(A) \cup \sigma_{p,2}(B)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Now, we prove the opposite inclusion. Suppose that $\lambda = 0$. If $0 \in (\sigma_{p,2}(A) \cap \rho(B)) \cup (\sigma_{p,2}(B) \cap \rho(A))$, then $0 \in \sigma_{p,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $0 \in \sigma_{p,2}(A)$ and $0 \in \sigma_{p,2}(B) \cup \sigma_r(B)$, then $0 \in \sigma_{p,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, since $\overline{\mathcal{R}(B)} \neq \mathcal{K}$. If $0 \in \sigma_{p,2}(A) \cap \sigma_{p,1}(B) \cap \rho_m(B)$ and $n(B) < d(A)$, then $\overline{\mathcal{R}(M_X)} \neq \mathcal{H} \oplus \mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ from the proof of *Case 3* of Theorem 2.4. Hence, $0 \in \sigma_{p,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. If $0 \in \sigma_{p,2}(B) \cap \sigma_{r,1}(A)$ and $n(B) > d(A)$, then $0 \in \sigma_{p,2}(B^*) \cap \sigma_{p,1}(A^*) \cap \rho_m(A^*)$ and $n(A^*) < d(B^*)$, and hence $0 \in \sigma_{p,2}(M_X^*)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ from the above discussion. Therefore, $\sigma_{p,2}(A) \cup \sigma_{p,2}(B) \subseteq \sigma_{p,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Necessity. Assume not, and let $\lambda_0 \in \mathbb{C}$, but one of the assertions (i) and (ii) fails to hold. There are four cases to consider.

Case 1: $\lambda_0 \in (\sigma_{p,2}(A) \cap \sigma_c(B)) \cup (\sigma_{p,2}(A) \cap \sigma_{p,1}(B) \cap \sigma_m(B))$. By Theorem 2.4, $\lambda_0 \in \sigma_{p,1}(M_{X_0})$ for some $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and hence $\lambda_0 \notin \sigma_{p,2}(M_{X_0})$. This contradicts the assumption $\sigma_{p,2}(M_X) = \sigma_{p,2}(A) \cup \sigma_{p,2}(B)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, since $\lambda_0 \in \sigma_{p,2}(A) \cup \sigma_{p,2}(B)$.

Case 2: $\lambda_0 \in \sigma_{p,2}(A) \cap \sigma_{p,1}(B) \cap \rho_m(B)$ and $n(B-\lambda_0) \geq d(A-\lambda_0)$. By Theorem 2.4, $\lambda_0 \in \sigma_{p,1}(M_{X_0})$ for some $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and hence $\lambda_0 \notin \sigma_{p,2}(M_{X_0})$.

Case 3: $\lambda_0 \in \sigma_{p,2}(B) \cap (\sigma_c(A) \cup \sigma_{r,2}(A)$. Use the operator X_0 defined as in (5). Then M_{X_0} is injective, and hence $\lambda_0 \notin \sigma_{p,2}(M_{X_0})$.

Case 4: $\lambda_0 \in \sigma_{p,2}(B) \cap \sigma_{r,1}(A)$ and $n(B-\lambda_0) \leq d(A-\lambda_0)$. then we may further define a unitary operator X_1 from $\mathcal{N}(B-\lambda_0)$ to some closed subspace of $\mathcal{R}(A-\lambda_0)^\perp$. Take $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by

$$X_0 = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B-\lambda_0) \\ \mathcal{N}(B-\lambda_0)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A-\lambda_0)^\perp \\ \mathcal{R}(A-\lambda_0) \end{pmatrix}.$$

Then M_{X_0} is injective, and hence $\lambda_0 \notin \sigma_{p,2}(M_{X_0})$.

Corollary 2.10. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, then*

$$\sigma_p(A) \cup \sigma_p(B) = \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{p,1}(M_X) \cup \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{p,2}(M_X).$$

Corollary 2.11. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then $\lambda \in \sigma_{p,1}(M_{X_1})$ and $\lambda \in \sigma_{p,2}(M_{X_2})$ for certain $X_1, X_2 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, if and only if one of the statements (a)–(f) is fulfilled:*

(a) $\lambda \in (\sigma_{p,1}(B) \cap \sigma_m(B) \cap \sigma_r(A))$;

(b) $\lambda \in (\sigma_{p,1}(B) \cap \sigma_m(B) \cap \sigma_{p,2}(A)) \cup (\sigma_{p,2}(A) \cap \sigma_c(B))$;

(c) $\lambda \in \sigma_{p,1}(B) \cap \sigma_r(A)$ *and* $n(B-\lambda) > d(A-\lambda)$;

(d) $\lambda \in \sigma_{p,1}(B) \cap \sigma_{p,2}(A)$ *and* $n(B-\lambda) \geq d(A-\lambda)$;

(e) $\lambda \in \sigma_{p,1}(B) \cap \sigma_r(A)$ *and* $n(B-\lambda) = \infty$;

(f) $\lambda \in \sigma_{p,1}(B) \cap \sigma_{p,2}(A)$ *and* $n(B-\lambda) = \infty$.

Proof. The result is immediately from Theorem 2.4 and Theorem 2.7.

Theorem 2.12. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, then*

$$\bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,1}(M_X) = \Delta_1 \cup \Delta_2 \cup \Delta_3,$$

where

$$\begin{aligned}
\Delta_1 &= (\sigma_{r,1}(A)\cap\rho(B))\cup(\sigma_{r,1}(B)\cap\rho(A))\cup(\sigma_{r,1}(A)\cap\sigma_{r,1}(B)),\\
\Delta_2 &= \{\lambda\in\sigma_{r,1}(A)\cap\sigma_{p,1}(B)\cap\rho_m(B): n(B-\lambda)<d(A-\lambda)<\infty\}\\
&\quad\cup\{\lambda\in\sigma_{r,1}(A)\cap\sigma_{p,2}(B)\cap\rho_m(B): n(B-\lambda)\le d(A-\lambda)<\infty\},\\
\Delta_3 &= \{\lambda\in\sigma_{r,1}(A): d(A-\lambda)=\infty\}.
\end{aligned}$$

Proof. First, we prove that $\bigcup_{k=1}^{3}\Delta_k\subseteq\bigcup_{X\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_{r,1}(M_X)$. Without loss of generality, we suppose that $\lambda=0$. Let $0\in\Delta_1$. If $0\in(\sigma_{r,1}(A)\cap\rho(B))\cup(\sigma_{r,1}(B)\cap\rho(A))$, then we clearly have $0\in\sigma_{r,1}(M_X)$ for every $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$. If $0\in\sigma_{r,1}(A)\cap\sigma_{r,1}(B)$, then M_X as an operator from $\mathcal{H}\oplus\mathcal{K}$ into $\mathcal{R}(A)\oplus\mathcal{R}(A)^\perp\oplus\mathcal{R}(B)\oplus\mathcal{R}(B)^\perp$ admits the following block representation

$$M_X=\begin{pmatrix}A_1 & X_1\\ 0 & X_2\\ 0 & B_1\\ 0 & 0\end{pmatrix}$$

for every $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$. Clearly, $A_1:\mathcal{H}\to\mathcal{R}(A)$ and $B_1:\mathcal{K}\to\mathcal{R}(B)$ are invertible. Thus there is an invertible operator

$$U=\begin{pmatrix}I & 0 & -X_1B_1^{-1} & 0\\ 0 & I & -X_2B_1^{-1} & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{pmatrix}:\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{R}(A)^\perp\\ \mathcal{R}(B)\\ \mathcal{R}(B)^\perp\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{R}(A)^\perp\\ \mathcal{R}(B)\\ \mathcal{R}(B)^\perp\end{pmatrix}$$

such that

$$UM_X=\begin{pmatrix}A_1 & 0\\ 0 & 0\\ 0 & B_1\\ 0 & 0\end{pmatrix}.$$

This shows that $0\in\sigma_{r,1}(M_X)$ for every $X\in\mathcal{B}(\mathcal{K},\mathcal{H})$.

Let $0\in\Delta_2$. Since $n(B)\le d(A)$, then there exists a finite dimensional subspace Ω of $\mathcal{R}(A)^\perp$ such that $\dim\Omega=n(B)$ and $\mathcal{R}(A)^\perp=\Omega\oplus\Omega^\perp$. Define $X_0\in\mathcal{B}(\mathcal{K},\mathcal{H})$ by

$$X_0=\begin{pmatrix}0 & 0\\ X_1 & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{N}(B)\\ \mathcal{N}(B)^\perp\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \Omega\\ \Omega^\perp\end{pmatrix},$$

where $X_1:\mathcal{N}(B)\to\mathcal{R}(A)^\perp$ is a unitary operator. Then, M_{X_0} can be written as

$$M_{X_0}=\begin{pmatrix}A_1 & 0 & 0\\ 0 & X_1 & 0\\ 0 & 0 & 0\\ 0 & 0 & B_1\end{pmatrix}:\begin{pmatrix}\mathcal{H}\\ \mathcal{N}(B)\\ \mathcal{N}(B)^\perp\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \Omega\\ \Omega^\perp\\ \mathcal{K}\end{pmatrix}.$$

Clearly $A_1:\mathcal{H}\to\mathcal{R}(B)$ is invertible and $B_1:\mathcal{K}\to\mathcal{R}(B)$ is left invertible. It is easy to see that M_{X_0} is injective and $\mathcal{R}(M_{X_0})$ is closed. On the other hand, since $n(B)<d(A)$ or $0\in\sigma_{p,2}(B)$, it follows that $\overline{\mathcal{R}(M_{X_0})}\neq\mathcal{H}\oplus\mathcal{K}$. Hence $0\in\sigma_{r,1}(M_{X_0})$.

Let $0\in\Delta_3$. Then one can define a unitary operator X_1 from $\mathcal{K}$ onto $\mathcal{R}(A)^\perp$. Taking

$$X_0=\begin{pmatrix}0\\ X_1\end{pmatrix}:\mathcal{K}\to\begin{pmatrix}\overline{\mathcal{R}(A)}\\ \mathcal{R}(A)^\perp\end{pmatrix},$$

it is easy to check that $0 \in \sigma_{r,1}(M_{X_0})$.

For the opposite inclusion, it suffices to prove that $0 \notin \bigcup_{k=1}^{3} \Delta_k$ implies $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,1}(M_X)$. Now we consider three cases.

Case 1: A is not left invertible. Obviously, M_X is not left invertible for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Therefore $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,1}(M_X)$.

Case 2: A is left invertible, $\mathcal{R}(B)$ is not closed and $d(A-\lambda) < \infty$. Then for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, M_X as an operator from $\mathcal{H} \oplus \mathcal{K}$ into $\mathcal{R}(A) \oplus \mathcal{R}(A)^{\perp} \oplus \mathcal{K}$ admits the following block representation

$$M_X = \begin{pmatrix} A_1 & X_1 \\ 0 & X_2 \\ 0 & B \end{pmatrix},$$

where $A_1 : \mathcal{H} \to \mathcal{R}(A)$ is invertible. So, we obtain

$$M_X \begin{pmatrix} I & -A_1^{-1}X_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & X_2 \\ 0 & B \end{pmatrix}.$$

Observe that X_2 is a finite rank operator. Therefore $0 \in \sigma_m(B)$ leads to $0 \in \sigma_m(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, and hence $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,1}(M_X)$.

Case 3: A is left invertible, $\mathcal{R}(B)$ is closed and $n(B) > d(A)$. Then, M_X admits the following block representation

$$M_X = \begin{pmatrix} A_1 & X_1 & X_2 \\ 0 & X_3 & X_4 \\ 0 & B_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^{\perp} \\ \mathcal{K} \end{pmatrix} \tag{6}$$

for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Clearly, $A_1 : \mathcal{H} \to \mathcal{R}(A)$ is invertible. Thus the invertible operator $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{N}(B) \oplus \mathcal{N}(B)^{\perp})$ given by

$$V = \begin{pmatrix} I & -A_1^{-1}X_1 & -A_1^{-1}X_2 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{7}$$

is such that

$$M_X V = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & X_3 & X_4 \\ 0 & B_1 & 0 \end{pmatrix}. \tag{8}$$

From $n(B) > d(A)$, X_4 is noninjective, and hence M_X is noninjective for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Therefore, $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,1}(M_X)$.

Corollary 2.13. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{r,1}(M_X) \subseteq \sigma_{r,1}(A) \cup \sigma_{r,1}(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H}).$$

Corollary 2.14. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{r,1}(M_X) = \sigma_{r,1}(A) \cup \sigma_{r,1}(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

if and only if the following statements are fulfilled:

(i) $\lambda \in \sigma_{r,1}(A)$ *implies* $\lambda \in \rho(B) \cup \sigma_{r,1}(B)$;

(ii) $\lambda \in \sigma_{r,1}(B)$ *implies* $\lambda \in \rho(A) \cup \sigma_{r,1}(A)$.

Proof. Sufficiency. By Corollary 2.13, we only need to prove $\sigma_{r,1}(A) \cup \sigma_{r,1}(B) \subseteq \sigma_{r,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Suppose that $\lambda = 0$. If $0 \in (\sigma_{r,1}(A) \cap \rho(B)) \cup (\sigma_{r,1}(B) \cap \rho(A)) \cup (\sigma_{r,1}(A) \cap \sigma_{r,1}(B))$, then $0 \in \sigma_{r,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ from the proof of Theorem 2.12. Therefore, $\sigma_{r,1}(A) \cup \sigma_{r,1}(B) \subseteq \sigma_{r,1}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Necessity. Assume not, and let $\lambda_0 \in \mathbb{C}$, but one of the assertions (i) and (ii) fails to hold. Take $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by $X_0 = 0$. Then $\lambda_0 \notin \sigma_{r,1}(M_{X_0})$. This contradicts the assumption $\sigma_{r,1}(M_X) = \sigma_{r,1}(A) \cup \sigma_{r,1}(B)$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, since $\lambda_0 \in \sigma_{r,1}(A) \cup \sigma_{r,1}(B)$.

Theorem 2.15. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, then*

$$\bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X) = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4,$$

where

$$\begin{aligned}
\Delta_1 &= \sigma_{r,2}(A) \cup (\sigma_c(A) \cap \sigma_{p,2}(B)) \cup (\sigma_c(A) \cap \sigma_r(B)),\\
\Delta_2 &= (\rho(A) \cap \sigma_{r,2}(B)) \cup (\sigma_{r,1}(A) \cap \sigma_c(B)) \cup (\sigma_{r,1}(A) \cap \sigma_{r,2}(B)),\\
\Delta_3 &= \{\lambda \in \sigma_{r,1}(A) \cap \sigma_m(B) : d(A - \lambda) = \infty\}\\
&\quad \cup \{\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B) \cap \sigma_m(B) : n(B - \lambda) < d(A - \lambda) < \infty\}\\
&\quad \cup \{\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,2}(B) \cap \sigma_m(B) : n(B - \lambda) \le d(A - \lambda) < \infty\},\\
\Delta_4 &= \{\lambda \in \sigma_{r,1}(A) \cap \rho_m(B) : n(B - \lambda) = d(A - \lambda) = \infty\}.
\end{aligned}$$

Proof. First, we prove that $\bigcup_{k=1}^{4} \Delta_k \subseteq \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X)$. We suppose that $\lambda = 0$. Let $0 \in \Delta_1$. Then, by Lemma 2.1, there exists an infinite dimensional closed subspace $\Omega \subset \overline{\mathcal{R}(A)}$ such that $\Omega \cap \mathcal{R}(A) = \{0\}$. If $0 \in (\sigma_{r,2}(A) \cap \sigma_p(B)) \cup (\sigma_c(A) \cap \sigma_{p,2}(B))$, then we may further define a unitary operator X_1 from $\mathcal{N}(B)$ to some closed subspace of Ω. Taking

$$X_0 = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \Omega \\ \Omega^\perp \end{pmatrix},$$

we have the operator matrix

$$M_{X_0} = \begin{pmatrix} A_1 & X_1 & 0 \\ A_2 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \Omega \\ \Omega^\perp \\ \mathcal{K} \end{pmatrix}.$$

Clearly, X_1 and B_1 are injective, and so by $\Omega \cap \mathcal{R}(A) = \{0\}$, one can see that M_{X_0} is injective. On the other hand, from $0 \in \sigma_{r,2}(A) \cup \sigma_{p,2}(B)$, we have that $\overline{\mathcal{R}(M_{X_0})} \neq \mathcal{H} \oplus \mathcal{K}$. Now $0 \in \sigma_m(M_{X_0})$ follows from the fact that $0 \in \sigma_m(A)$. Therefore $0 \in \sigma_{r,2}(M_{X_0})$. If $0 \in (\sigma_{r,2}(A) \setminus \sigma_p(B)) \cup (\sigma_c(A) \cap \sigma_r(B))$, then define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by $X_0 = 0$. Clearly, $0 \in \sigma_{r,2}(M_{X_0})$.

Let $0 \in \Delta_2$. Define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by $X_0 = 0$. Clearly, $0 \in \sigma_{r,2}(M_{X_0})$.

Let $0 \in \Delta_3$. If $n(B) < \infty$, then there exists a closed subspace Ω of $\mathcal{R}(A)^\perp$ such that $\dim \Omega = n(B)$ and $\mathcal{R}(A)^\perp = \Omega \oplus \Omega^\perp$. Define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ by

$$X_0 = \begin{pmatrix} 0 & 0 \\ X_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \Omega \\ \Omega^\perp \end{pmatrix},$$

where $X_1 : \Omega \to \mathcal{R}(A)^\perp$ is a unitary operator. Then, M_{X_0} can be written as

$$M_{X_0} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & X_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \Omega \\ \Omega^\perp \\ \mathcal{K} \end{pmatrix}.$$

Clearly, M_{X_0} is injective. Since $n(B) < d(A)$ or $0 \in \sigma_{p,2}(B)$, it follows that $\overline{\mathcal{R}(M_{X_0})} \neq \mathcal{H} \oplus \mathcal{K}$. Note that $0 \in \sigma_m(B)$, then $0 \in \sigma_m(M_{X_0})$. Therefore $0 \in \sigma_{r,2}(M_{X_0})$. If $n(B) = d(A) = \infty$, then we may further define a unitary operator X_1 from $\mathcal{N}(B)$ onto $\mathcal{R}(A)^\perp$. Taking $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$

$$X_0 = \begin{pmatrix} 0 & 0 \\ X_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{pmatrix}, \tag{9}$$

we can verify that $0 \in \sigma_{r,2}(M_{X_0})$.

Let $0 \in \Delta_4$. Since $n(B) = d(A) = \infty$, then there is an operator $X_1 : \mathcal{N}(B) \to \mathcal{R}(A)^\perp$ such that $\mathcal{N}(X_1) = 0$, $\mathcal{R}(X_1) \neq \overline{\mathcal{R}(X_1)} = \mathcal{R}(A)^\perp$. Define $X_0 \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ as in (9). It is easy to check that $0 \in \sigma_{r,2}(M_{X_0})$.

For the opposite inclusion, it suffices to prove that $0 \notin \bigcup_{k=1}^{4} \Delta_k$ implies $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X)$. Now we consider four cases.

Case 1: $0 \in \sigma_p(A)$ or $\overline{\mathcal{R}(A)} = \mathcal{H}$ and $\overline{\mathcal{R}(B)} = \mathcal{K}$. Obviously, $0 \in \sigma_p(M_X)$ or $\overline{\mathcal{R}(M_X)} = \mathcal{H} \oplus \mathcal{K}$ for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Hence, $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X)$.

Case 2: A is left invertible and $n(B) > d(A)$. From the proof of *Case* 2 of Theorem 2.12, we obtain that M_X is noninjective for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Therefore, $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X)$.

Case 3: A is left invertible, $\mathcal{R}(B)$ is not closed, $\overline{\mathcal{R}(B)} = \mathcal{K}$ and $n(B) = d(A) < \infty$. Then M_X has the matrix form as in (6) for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Also, the relation (8) holds true. If X_4 in (8) is injective, then $\overline{\mathcal{R}(M_X)} = \mathcal{H} \oplus \mathcal{K}$; If X_4 in (8) is noninjective, then M_X is noninjective. Hence, $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X)$.

Case 4: A is left invertible, $\mathcal{R}(B)$ is closed, and $n(B) < \infty$ or $d(A) < \infty$. Then, M_X admits the following block representation

$$M_X = \begin{pmatrix} A_1 & X_1 & X_2 \\ 0 & X_3 & X_4 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{N}(B) \\ \mathcal{N}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{pmatrix}$$

for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Clearly, $A_1 : \mathcal{H} \to \mathcal{R}(A)$ and $B_1 : \mathcal{K} \to \mathcal{R}(B)$ are invertible. Thus there exists the invertible operators

$$U = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & -X_3B_1^{-1} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{R}(B) \\ \mathcal{R}(B)^\perp \end{pmatrix}$$

and V as in (7) such that

$$UM_XV = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & X_4 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In view of $n(B) < \infty$ or $d(A) < \infty$, we see that X_4 is a finite rank operator. It follows from $\mathcal{R}(M_X)$ is closed for every $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Therefore, $0 \notin \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X)$.

Corollary 2.16. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{r,2}(M_X) \subseteq \sigma_{r,2}(A) \cup \sigma_{r,2}(B) \quad \textit{for every } X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$$

if and only if $(\sigma_c(A) \cap \sigma_{p,2}(B)) \cup (\sigma_c(A) \cap \sigma_{r,1}(B)) \cup (\sigma_{r,1}(A) \cap \sigma_c(B)) \cup \Delta_3 \cup \Delta_4 = \emptyset$, *where* Δ_3 *and* Δ_4 *as in Theorem 2.15.*

Proof. In the similar way as the proof of Corollary 2.5, using Theorem 2.15, we obtain the desired result.

Corollary 2.17. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then,*

$$\sigma_{r,2}(M_X) = \sigma_{r,2}(A) \cup \sigma_{r,2}(B) \quad \text{for every } X \in \mathcal{B}(\mathcal{K},\mathcal{H})$$

if and only if $(\sigma_c(A) \cap \sigma_{p,2}(B)) \cup (\sigma_c(A) \cap \sigma_{r,1}(B)) \cup (\sigma_{r,1}(A) \cap \sigma_c(B)) \cup \Delta_3 \cup \Delta_4 = \emptyset$, *and the following statements are fulfilled:*

(i) $\lambda \in \sigma_{r,2}(A)$ *implies* $\lambda \in \sigma_r(B) \cup \rho(B)$;

(ii) $\lambda \in \sigma_{r,2}(B)$ *implies* $\lambda \in \sigma_c(A) \cup \sigma_{r,2}(A) \cup \rho(A) \cup \{\lambda \in \sigma_{r,1}(A) : d(A-\lambda) < \infty\}$, *where* Δ_3 *and* Δ_4 *as in Theorem 2.15.*

Proof. Sufficiency. By Corollary 2.16, we get $\sigma_{r,2}(M_X) \subseteq \sigma_{r,2}(A) \cup \sigma_{r,2}(B)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Now, we prove the opposite inclusion. Assume that $\lambda = 0$. If $0 \in (\sigma_{r,2}(A) \cap \rho(B)) \cup (\sigma_{r,2}(B) \cap \rho(A))$, then $0 \in \sigma_{r,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. If $0 \in \sigma_{r,2}(A) \cap \sigma_r(B)$, then $0 \in \sigma_r(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. This, together with $0 \in \sigma_{r,2}(A) \subseteq \sigma_l(A) \subseteq \sigma_l(M_X)$ implies that $0 \in \sigma_{r,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Similarly, if $0 \in \sigma_c(A) \cap \sigma_{r,2}(B)$, then $0 \in \sigma_{r,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Now let $0 \in \sigma_{r,2}(B) \cap \sigma_{r,1}(A)$ and $d(A) < \infty$. Then we get $0 \in \sigma_m(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ from the proof of *Case 2* of Theorem 2.12. This implies that $0 \in \sigma_{r,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Therefore, $\sigma_{r,2}(A) \cup \sigma_{r,2}(B) \subseteq \sigma_{r,2}(M_X)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$.

Necessity. Assume to the contrary that there exists $\lambda_0 \in \mathbb{C}$, such that one of the assertions (i) and (ii) fails to hold. There are three possible cases.

Case 1: $\lambda_0 \in (\sigma_{r,2}(A) \cap \sigma_p(B)) \cup (\sigma_{r,2}(B) \cap \sigma_p(A))$. Take $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$ by $X_0 = 0$. Then $\lambda_0 \in \sigma_p(M_{X_0})$, and hence $\lambda_0 \notin \sigma_{r,2}(M_{X_0})$. This contradicts the assumption $\sigma_{r,2}(M_X) = \sigma_{r,2}(A) \cup \sigma_{r,2}(B)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, since $\lambda_0 \in \sigma_{r,2}(A) \cup \sigma_{r,2}(B)$.

Case 2: $\lambda_0 \in \sigma_{r,2}(A) \cap \sigma_c(B)$. This implies that $\overline{\lambda_0} \in \sigma_{p,1}(A^*) \cap \sigma_c(B^*)$. From the proof of *Case 3* of Corollary 2.6, we obtain $M^*_{X_0} - \overline{\lambda_0}$ is ingective for some $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$, and hence $\overline{\mathcal{R}(M_{X_0})} = \mathcal{H} \oplus \mathcal{K}$. Therefore $\lambda_0 \notin \sigma_{r,2}(M_{X_0})$.

Case 3: $\lambda_0 \in \sigma_{r,2}(B) \cap \sigma_{r,1}(A)$ and $d(A - \lambda_0) = \infty$. By Theorem 2.12, we obtain $\lambda_0 \in \sigma_{r,1}(M_{X_0})$ for some $X_0 \in \mathcal{B}(\mathcal{K},\mathcal{H})$, and hence $\lambda_0 \notin \sigma_{r,2}(M_{X_0})$. Therefore $\lambda_0 \notin \sigma_{r,2}(M_{X_0})$.

Remark 2.18. Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$. From [6, Lemma1], we get that $\sigma(M_X) \subseteq \sigma(A) \cup \sigma(B)$ for every $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$. But the inclusion is not true for 1,2-point spectrum and 2-residual spectrum.

Remark 2.19. A description of the set $\bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_X)$ was given in [13] (see (1)). From Theorem 2.12 and Theorem 2.15, we obtain that

$$\bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_r(M_X) = \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,1}(M_X) \cup \bigcup_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{r,2}(M_X).$$

Corollary 2.20. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then $\lambda \in \sigma_{r,1}(M_{X_1})$ and $\lambda \in \sigma_{r,2}(M_{X_2})$ for certain $X_1, X_2 \in \mathcal{B}(\mathcal{K},\mathcal{H})$, if and only if one of the statements (a)–(e) is fulfilled:*

(*a*) $\lambda \in \sigma_{r,1}(A) \cap \rho_m(B)$ *and* $n(B-\lambda) = d(A-\lambda) = \infty$;

(*b*) $\lambda \in \sigma_{r,1}(A) \cap \sigma_m(B)$ *and* $d(A-\lambda) = \infty$.

Proof. The result is immediately from Theorem 2.12 and Theorem 2.15.

We conclude this section with two illustrating examples of the previous results.

Example 2.21. *Let $\mathcal{H} = \mathcal{K} = \ell^2$. Consider the operators $A \in \mathcal{B}(\ell^2)$ and $B \in \mathcal{B}(\ell^2)$ defined by*

$$Ax = (0, x_1, \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{3}}, \cdots), \quad Bx = (x_3, x_4, x_5, \cdots)$$

for $(x_1, x_2, x_3, \cdots) \in \ell^2$. Then, we claim there exist $X_1 \in \mathcal{B}(\ell^2)$ and $X_2 \in \mathcal{B}(\ell^2)$ such that $0 \in \sigma_{p,1}(M_{X_1})$ and $0 \in \sigma_{p,2}(M_{X_2})$.

Indeed, it is clear that $0 \in \sigma_{p,1}(B) \cap \sigma_r(A)$ and $2 = n(B) > d(A) = 1$. By Corollary 2.11, we obtain that there exist $X_1 \in \mathcal{B}(\ell^2)$ and $X_2 \in \mathcal{B}(\ell^2)$ such that $0 \in \sigma_{p,1}(M_{X_1}) \cap \sigma_{p,2}(M_{X_2})$. In fact, if taking $X_2 = 0$ and $X_1 \in \mathcal{B}(\ell^2)$ by

$$X_1 x = (x_1, 0, 0, 0, \cdots)$$

for $(x_1, x_2, x_3, \cdots) \in \ell^2$, we immediately see $0 \in \sigma_{p,1}(M_{X_1})$ and $0 \in \sigma_{p,2}(M_{X_2})$.

Example 2.22. *Let $\mathcal{H} = \mathcal{K} = \ell^2$. Consider the operators $A \in \mathcal{B}(\ell^2)$ and $B \in \mathcal{B}(\ell^2)$ defined by*

$$Ax = (x_1, 0, x_2, 0, x_3, 0, \cdots), \quad Bx = (x_1, x_3, x_5, \cdots)$$

for $(x_1, x_2, x_3, \cdots) \in \ell^2$. Then, we claim there exist $X_1 \in \mathcal{B}(\ell^2)$ and $X_2 \in \mathcal{B}(\ell^2)$ such that $0 \in \sigma_{r,1}(M_{X_1})$ and $0 \in \sigma_{r,2}(M_{X_2})$.

Direct calculations show that $0 \in \sigma_{r,1}(A) \cap \rho_m(B)$ and $n(B) = d(A) = \infty$. By Corollary 2.20, there exist $X_1 \in \mathcal{B}(\ell^2)$ and $X_2 \in \mathcal{B}(\ell^2)$ such that $0 \in \sigma_{r,1}(M_{X_1}) \cap \sigma_{r,2}(M_{X_2})$. In fact, define $X_1 \in \mathcal{B}(\ell^2)$ and $X_2 \in \mathcal{B}(\ell^2)$ by

$$X_1 x = (0, 0, 0, x_2, 0, x_4, 0, x_6, \cdots),$$
$$X_2 x = (0, 0, 0, \tfrac{1}{2}x_2, 0, \tfrac{1}{4}x_4, 0, \tfrac{1}{6}x_6, \cdots)$$

for $(x_1, x_2, x_3, \cdots) \in \ell^2$. Then we can check that $0 \in \sigma_{r,1}(M_{X_1})$ and $0 \in \sigma_{r,2}(M_{X_2})$.

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