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Limit Theorems for Asymptotic Circular mth-Order Markov Chains Indexed by an m-Rooted Homogeneous Tree

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Abstract. In this paper, we give the definition of an asymptotic circular mth-order Markov chain indexed by an m rooted homogeneous tree. By applying the limit property for a sequence of multi-variables functions of a nonhomogeneous Markov chain indexed by such tree, we estabish the strong law of large numbers and the asymptotic equipartition property (AEP) for asymptotic circular mth-order finite Markov chains indexed by this homogeneous tree. As a corollary, we can obtain the strong law of large numbers and AEP about the mth-order finite nonhomogeneous Markov chain indexed by the m rooted homogeneous tree.

1. Introduction

Let $\mathbf{N}$ be the set of all natural numbers, for a positive integer d, two numbers $a, b \in \mathbf{N}$ are said to be congruent modulo d, written:

$$a \equiv b(\text{mod } d) \qquad 0 \le b \le d-1,$$

if their difference $a-b$ is an integer multiple of d (or d divides $a-b$). Equivalently, $a \equiv b(\text{mod } d)$ can also be thought of as asserting that the remainders of the division of both a and b by d are the same. We call the set of natural numbers $x \in \mathbf{N}$, which satisfies the relation $x \equiv a(\text{mod } d)$, to be the residue class of a modulo d in set $\mathbf{N}$, and denote by

$$[a] = \{x | x \in \mathbf{N}, x \equiv a(\text{mod } d)\}.$$

Obviously, the natural numbers set $\mathbf{N}$ can be divided into d subsets by the congruent relation of modulo d as follows:

$$\begin{aligned}
&[0] = \{0, d, 2d, 3d, \cdots, nd, \cdots\} \\
&[1] = \{1, d+1, 2d+1, 3d+1, \cdots, nd+1, \cdots\} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&[d-1] = \{d-1, d+d-1, 2d+d-1, 3d+d-1, \cdots, nd+d-1, \cdots\}
\end{aligned}$$

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For our later purpose, we may let

$$[d] = \{d, 2d, 3d, \cdots, nd, \cdots\},$$

that is, $[d] = [0] \setminus \{0\}$. For more details about the knowledge of number theory, please see ([5]).

Now let's start to introduce the definition of an asymptotic circular mth-order Markov chain indexed by an m-rooted homogeneous tree and give some notations about tree graph and such Markov chain.

A tree T is a connected graph and does't contain any loop. Given any two vertices $s \neq t \in T$, let $\overline{st}$ be the unique path connecting s and t. Define the graph distance $d(s, t)$ to be the number of edges contained in the path $\overline{st}$.

Let $T_{C,N}$ be an infinite Cayley tree with root 0, in which the root 0 has only N neighbours and all other vertices have $N + 1$ neighbors. For each vertex t, there is a unique path from 0 to t, and $|t| = d(0, t)$ for the number of edges on this path. We denote the first predecessor of t by ${}^1t$, the second predecessor of t by ${}^2t$, and denote by ${}^nt$ the n-th predecessor of t. Now we can formulate an m rooted Cayley tree $\tilde{T}_{C,N}$ by the Cayley tree $T_{C,N}$ with the root connecting another ray, where there are $m-1$ edges and m vertices denoted by $0, {}^10, {}^20, \cdots, {}^{m-1}0$ (see Figure 1). When the context permits, this type of tree is simply denoted by T.

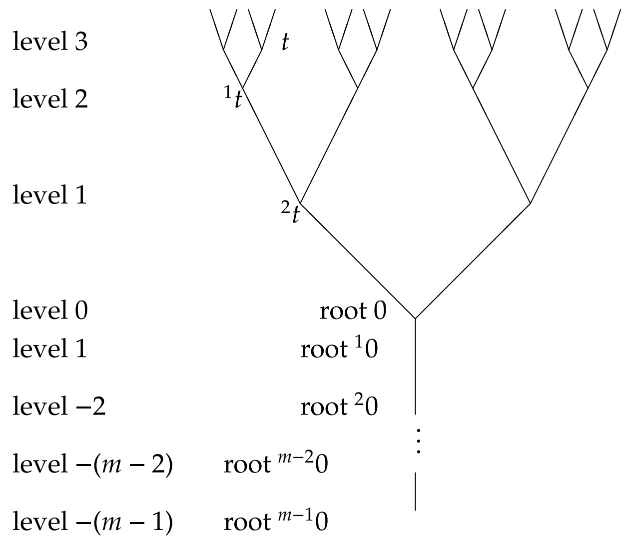

Figure 1 An m rooted Cayley tree $\tilde{T}_{C,2}$

For any two vertices s and t of a tree T, write $s \leq t$ if s is on the unique path from the root 0 to t. For any two vertices s and $t(s, t \notin \{0, {}^10, {}^20, \cdots, {}^{m-1}0\})$, denote by $s \wedge t$ the vertex farthest from 0 satisfying $s \wedge t \leq s$ and $s \wedge t \leq t$. The set of all vertices with distance n from the root 0 is called the n-th generation of T, which is denoted by L_n. That is, L_n is the set of all vertices on level n from the root 0. By analogy, the root -1 is on the level L_{-1}, root -2 is on the level L_{-2}, root $-(m-1)$ is on the level $L_{-(m-1)}$. We denote $X^A := \{X_t, t \in A\}$ and $|A|$ is the number of vertices of set A.

For each nonnegative integer n, we denote by $T^{(n)}$ the subtree of an m rooted Cayley tree T containing the vertices from level 0 to level n. Similarly, for any $i \in \{0, 1, 2, \cdots, d-1\}$, we denote by $T^{(n)}_{[i]}$ the union of all vertices in $[i]$-generations from level 0 to level n.

Definition 1.1(Nonhomogeneous Markov chains indexed by tree T(see [8])) *Let T be an m-rooted infinite homogeneous tree $\tilde{T}_{C,N}$, $\{X_t, t \in T\}$ a stochastic process in a finite state space $\mathcal{X} = \{1, 2, \cdots, b\}$ defined on probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Denote $x_1^m = (x_1, x_2, \cdots, x_m) \in \mathcal{X}^m$. Let*

$$p = \{p(x_1^m), x_1^m \in \mathcal{X}^m\} \tag{1.1}$$

be a distribution on $\mathcal{X}^m$*, and*

$$P_n = (P_n(y|x_1^m)), \ \ y \in \mathcal{X}, x_1^m \in \mathcal{X}^m \tag{1.2}$$

be a transition probability matrix on $\mathcal{X}^{m+1}$*. For every vertex* $t \in L_{n+1}, n \geq 0$*, if*

$$\begin{aligned}&\boldsymbol{P}(X_t = y|X_{1_t} = x_1, X_{2_t} = x_2, X_{3_t} = x_3, \cdots, X_{m_t} = x_m \ \ and \ \ X_s = x_s \ \ for \ \ t \wedge s \leq {}^m t)\\ &= \boldsymbol{P}(X_t = y|X_{1_t} = x_1, X_{2_t} = x_2, X_{3_t} = x_3, \cdots, X_{m_t} = x_m) = P_n(y|x_1^m) \ \ \forall y \in \mathcal{X}, x_1^m \in \mathcal{X}^m,\end{aligned}$$

and

$$\boldsymbol{P}(X_{{m-1}_0} = x_1, X_{{m-2}_0} = x_2, \cdots, X_{1_0} = x_{m-1}, X_0 = x_m) = p(x_1^m) \ \ \forall x_1^m \in \mathcal{X}^m, \tag{1.3}$$

then we call $\{X_t, t \in T\}$ *to be an* $\mathcal{X}$*-valued mth-order* ***nonhomogeneous Markov chain indexed by*** m***-rooted infinite homogeneous tree*** *with the initial distribution* (1.1) *and transition matrices* (1.2)*, or called tree-indexed mth-order nonhomogeneous Markov chains.*

Remark 1.1: In definition 1.1, if for all $n \in \mathbf{N}$,

$$P_n = P = (P(y|x_1^m)), \ \ y \in \mathcal{X}, x_1^m \in \mathcal{X}^m, \tag{1.4}$$

then $\{X_t, t \in T\}$ will be called $\mathcal{X}$-valued mth-order homogeneous Markov chains indexed by tree $\tilde{T}_{C,N}$.

Definition 1.2 *Let T be an m-rooted infinite homogeneous tree* $\tilde{T}_{C,N}$*,* $\{X_t, t \in T\}$ *be* $\mathcal{X}$*-valued mth-order nonhomogeneous Markov chains indexed by a tree T defined as* ***definition 1.1****. Let* $Q_0, Q_1, Q_2, \cdots, Q_{d-1}$ *be a sequence of stochastic matrices, where* $Q_l = (Q_l(y|x_1^m))_{y\in\mathcal{X}, x_1^m\in\mathcal{X}^m}$ *for* $0 \leq l \leq d-1$*. If*

$$\lim_{r\to\infty} P_{rd+l}(y|x_1^m) = Q_l(y|x_1^m), \quad y \in \mathcal{X}, x_1^m \in \mathcal{X}^m, l = 0, 1, \cdots, d-1, \tag{1.5}$$

then $\{X_t, t \in T\}$ *is called an* ***asymptotic circular*** m***th-order Markov chain indexed by tree*** T*. Especially, if*

$$P_{rd+l} = Q_l, \quad l = 0, 1, 2, \cdots, d-1; r = 0, 1, 2, \cdots, \tag{1.6}$$

then we call $\{X_t, t \in T\}$ *to be a* ***circular*** m***th-order Markov chain indexed by tree*** T*.*

Remark 1.2 It is easy to see that definition 1.2 is a special case of definition 1.1. If $N = 1$, then our model is reduced to the asymptotic circular mth-order Markov chains on the line.

The subject of tree-indexed processes has been deeply studied and made abundant achievements. Benjamini and Peres ([1]) introduced the notion of the tree-indexed Markov chains and studied the recurrence and ray-recurrence for them. Berger and Ye ([9]) have studied the existence of entropy rate for some stationary random fields on a homogeneous tree. Ye and Berger (see [9, 10]), by using Pemantle's result([7]) and a combinatorial approach, have studied a Shannon-McMillan theorem with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu ([11]) and Yang([13]) proved a strong law of large numbers for Markov chains fields on a homogeneous tree (a particular case of tree-indexed Markov chains and PPG-invariant random fields). Yang and Ye([14]) have established a Shannon-McMillan theorem with convergence almost surely for nonhomogeneous Markov chains on a homogeneous tree. Huang and Yang (see [3]) has studied the Shannon-McMillan theorem in the sense of almost surely for finite homogeneous Markov chains indexed by a uniformly bounded infinite tree. As we have known that Zhong, Yang and Liang ([15]) have studied the asymptotic equipartition property with convergence almost surely for asymptotic circular Markov chains, which is under condition of convergence in Cesàro sense.

Arbitrary stochastic process can be approximated by a high-order Markov process, therefore it is greatly significant to study the subject of the high-order Markov process. Of course, to study the high-order Markov process indexed by trees is also of great significance of its own. Yang and his coauthors have already tried to do it and made some good results. Yang and Liu ([12])have studied the asymptotic equipartition property for mth-order nonhomogeneous Markov information source. Shi and Yang ([8]) also studied the strong law of large numbers and a Shannon-McMillan theorem with a.e convergence for mth-order nonhomogeneous Markov chains indexed by an m-rooted homogeneous tree.

In this article, we introduce the model of an asymptotic circular mth-order Markov chain indexed by an m-rooted homogeneous tree defined as definition 1.2 and mainly establish the strong law of large numbers and Shannon-McMillan theorem with convergence almost surely for this model. This paper will generalize the results of Shi and Yang ([8]) .

Let $\delta_j(\cdot)$ be the Kronecker delta function in $\mathcal{X}$, that is

$$\delta_j(i) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j, \end{cases} \quad i, j \in \mathcal{X}.$$

For $k \in \mathbf{N}$, let

$$I^{[l]}(k) = \begin{cases} 1 & \text{if } k \in [l]; \\ 0 & \text{if } k \bar{\in} [l], \end{cases} \quad l = 0, 1, \cdots, d-1, d.$$

Denote

$$\begin{aligned}
&0_m = \{^{m-1}0, ^{m-2}0, \cdots, ^{1}0, 0\} \\
&\tilde{0}_m = \{^{m-1}0, ^{m-2}0, \cdots, ^{1}0\} \\
&X_n^1(t) = \{X_{^n t}, X_{^{n-1} t}, \cdots, X_{^2 t}, X_{^1 t}\} \\
&X_n^0(t) = \{X_{^n t}, X_{^{n-1} t}, \cdots, X_{^2 t}, X_{^1 t}, X_t\}
\end{aligned}$$

Let $x_n^1(t)$ and $x_n^0(t)$ be the realizations of $X_n^1(t)$ and $X_n^0(t)$ respectively.

Recall that $T^{(n)}$ is the subtree of an m rooted homogeneous tree T containing the vertices from level 0 to level n and $T_{[i]}^{(n)}$ is the union of all vertices in $[i]$-generations from level 0 to level n for any $i \in \{0, 1, 2, \cdots, d-1\}$. Let $S_n(i_1^m)$ and $S_n(i_1^{m+1})$ be the number of $i_1^m = (i_1, i_2, \cdots, i_m)$ in the ordered items of the sequence $\{X_{m-1}^0(t), t \in T^{(n)}\}$ and the number of $i_1^{m+1} = (i_1, i_2, \cdots, i_m, i_{m+1})$ in the ordered items of the sequence of $\{X_m^0(t), t \in T^{(n)} \setminus \{0\}\}$ respectively. For each $l = 0, 1, 2, \cdots, d-1, d$, we also let $S_n^{[l]}(i_1^m)$ and $S_n^{[l]}(i_1^{m+1})$ be the number of $i_1^m = (i_1, i_2, \cdots, i_m)$ in the ordered items of the sequence $\{X_{m-1}^0(t), t \in T_{[l]}^{(n)}\}$ and the number of $i_1^{m+1} = (i_1, i_2, \cdots, i_m, i_{m+1})$ in the ordered items of the sequence of $\{X_m^0(t), t \in T_{[l]}^{(n)} \setminus \{0\}\}$ respectively. For simplicity of the notations, we denote

$$\delta_{i_1^m}(X_{m-1}^0(t)) = \delta_{i_1}(X_{^{m-1}t})\delta_{i_2}(X_{^{m-2}t}) \cdots \delta_{i_m}(X_t) \tag{1.7}$$

$$\delta_{i_1^{m+1}}(X_m^0(t)) = \delta_{i_1}(X_{^m t})\delta_{i_2}(X_{^{m-1}t}) \cdots \delta_{i_{m+1}}(X_t) \tag{1.8}$$

Thus we have

$$S_n(i_1^m) = \sum_{k=0}^{n} \sum_{t \in L_k} \delta_{i_1^m}(X_{m-1}^0(t)), \tag{1.9}$$

$$S_n^{[l]}(i_1^m) = \sum_{k=0}^{n} \sum_{t \in L_k} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t)), \tag{1.10}$$

$$S_n(i_1^{m+1}) = \sum_{k=1}^{n} \sum_{t \in L_k} \delta_{i_1^{m+1}}(X_m^0(t)), \tag{1.11}$$

$$S_n^{[l]}(i_1^{m+1}) = \sum_{k=1}^{n} \sum_{t \in L_k} I^{[l]}(k-1)\delta_{i_1^{m+1}}(X_m^0(t)). \tag{1.12}$$

Obviously, we have

$$S_n^{[d]}(i_1^m) = S_n^{[0]}(i_1^m) - \delta_{i_1^m}(X_{0_m}), \tag{1.13}$$

$$S_n^{[d]}(i_1^{m+1}) = S_n^{[0]}(i_1^{m+1}). \tag{1.14}$$

2. Some lemmas

Let A be any mth-order transition matrix. Now we define another stochastic matrix as follows:

$$\bar{A} = (\bar{A}(j_1^m|i_1^m)) \quad i_1^m, j_1^m \in \mathcal{X}^m, \tag{2.1}$$

where

$$\bar{A}(j_1^m|i_1^m) = \begin{cases} A(j_m|i_1^m), & \text{as } j_v = i_{v+1}, v = 1, 2, \cdots, m-1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

$\bar{A}$ is called the m-dimensional stochastic matrix determined by the mth-order transition matrix A.

Now we give three lemmas which are very useful to prove our main results.

Lemma 2.1 *Set $R_0 = \bar{Q}_0\bar{Q}_1\bar{Q}_2\cdots\bar{Q}_{d-1}$, for $0 \le l \le d-1$, where $\bar{Q}_l$ is the $m-$dimensional transition matrix which is determined by the mth-order transition matrix Q_l. If all the elements of $\{Q_l, 0 \le l \le d-1\}$ are positive, that is*

$$Q_l = (Q_l(j|i_1^m)),\ Q_l(j|i_1^m) > 0,\ \forall j \in \mathcal{X}, i_1^m \in \mathcal{X}^m, 0 \le l \le d-1, \tag{2.3}$$

then R_0 is ergodic.

Proof. We say a matrix $A > 0$ if every element of A is positive. Then we have $Q_l > 0$ for $0 \le l \le d-1$, since all the elements of $\{Q_l, 0 \le l \le d-1\}$ are positive. Let $\{\xi_n, n \ge 0\}$ be an mth-order circular Markov chain with mth-order transition matrices $P_n = (P_n(j|i_1^m)), i_1^m \in \mathcal{X}^m, j \in \mathcal{X}$ which satisfies that

$$P_{td+l} = Q_l, \quad t = 0, 1, 2, ..., 0 \le l \le d-1. \tag{2.4}$$

Obviously, $P_{td+l} > 0$ for all $t = 0, 1, 2, ..., 0 \le l \le d-1$. We also let $\{\bar{P}_n\}$ be the m-dimensional stochastic matrix determined by the mth-order transition matrix $\{P_n\}$. Now we let $\eta_n = \xi_n^{n+m-1} = (\xi_n, \cdots, \xi_{n+m-1})$, if $n = td + l, t = 0, 1, 2, ..., 0 \le l \le d-1$, we have

$$\begin{aligned} \bar{P}_n(\eta_{n+1} = j_1^m|\eta_n = i_1^m) &= \bar{P}_{td+l}(j_1^m|i_1^m) = \bar{Q}_l(j_1^m|i_1^m) \\ &= \begin{cases} Q_l(j_m|i_1^m), & \text{as } j_v = i_{v+1}, v = 1, 2, \cdots, m-1; \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

So that $\{\eta_n, n \ge 0\}$ is an 1th-order circular Markov chain with its circular transition matrices $\{\bar{Q}_l, 0 \le l \le d-1\}$. For $\forall i_1^m, j_1^m \in \mathcal{X}^m$,

$$\begin{aligned} &\bar{P}_0\bar{P}_1\bar{P}_2\cdots\bar{P}_{m-1}(j_1^m|i_1^m) = P(\eta_m = j_1^m|\eta_0 = i_1^m) \\ = \; &P(\xi_m^{2m-1} = j_1^m|\xi_0^{m-1} = i_1^m) \\ = \; &P(\xi_m = j_1, \xi_{m+1} = j_2, \cdots, \xi_{2m-1} = j_m|\xi_0 = i_1, \xi_1 = i_2, \cdots, \xi_{m-1} = i_m) \\ = \; &P_0(j_1|i_1^m)P_1(j_2|i_2^m, j_1)\cdots P_{m-1}(j_m|i_m, j_1^{m-1}) > 0. \end{aligned} \tag{2.6}$$

Now denote

$$S = \bar{P}_0\bar{P}_1\bar{P}_2\cdots\bar{P}_{d-1}\bar{P}_d\cdots\bar{P}_{m-1},$$

naturally, we have $S > 0$ by (2.6).

If $m = nd$, we have

$$R_0^n = (\bar{Q}_0\bar{Q}_1\bar{Q}_2\cdots\bar{Q}_{d-1})^n = S > 0.$$

which implies that R_0 is ergodic.

On the other hand, if $m < nd$, then

$$R_0^n = S\bar{P}_m\bar{P}_{m+1}\cdots\bar{P}_{nd-1}.$$

By the definitions of $\bar{P}_m, \bar{P}_{m+1}, \cdots, \bar{P}_{nd-1}$, we assert that, for each $m \le r \le nd-1$, every column of $\bar{P}_r$ has an positive element. Then it follows that $R_0^n > 0$, so that R_0 is ergodic.

The proof of Lemma 2.1 is completed.

Lemma 2.2 *Let $\bar{Q}_0, \bar{Q}_1, \bar{Q}_2, \cdots, \bar{Q}_{d-1}$ be a sequence of $m-$dimensional transition matrices which is determined by the sequence of mth-order transition matrix $\{Q_l, 0 \le l \le d-1\}$. Set $R_0 = \bar{Q}_0\bar{Q}_1\bar{Q}_2\cdots\bar{Q}_{d-1}, R_1 = \bar{Q}_1\bar{Q}_2\cdots\bar{Q}_{d-1}\bar{Q}_0, \cdots, R_{d-1} = \bar{Q}_{d-1}\bar{Q}_0\bar{Q}_1\cdots\bar{Q}_{d-2}$. Suppose that R_0 is **ergodic**, then $R_1, R_2, \cdots, R_{d-1}$ are also **ergodic.** Let $R_l^n(j_1^m|i_1^m)$ be the $n-$step transition probability determined by a stochastic matrix R_l, then*

$$\lim_{n\to\infty} R_l^n(j_1^m|i_1^m) = \pi^l(j_1^m), l = 0, 1, 2, \cdots, d-1, \tag{2.7}$$

where $\pi^l = (\pi^l(i_1^m))_{i_1^m\in\mathcal{X}^m}$ is the unique stationary distribution determined by the m-dimensional transition matrix R_l.

Proof. A constant stochastic matrix is a stochastic matrix that has identical rows. Since the stochastic matrix R_0 is **ergodic**, then there is a constant stochastic matrix S_0 such that

$$\lim_{n\to\infty} R_0^n = S_0. \tag{2.8}$$

for which each row of S_0 is the stationary distribution determined by stochastic matrix R_0.

Let $S_l = S_0\prod_{i=0}^{l-1}\bar{Q}_i, l = 1, 2, \cdots, d-1$. Obviously, S_l is also a constant stochastic matrix. By induction, for each positive integer $n \ge 1$, it is easy to see,

$$R_l^n = (\prod_{i=l}^{d-1}\bar{Q}_i)R_0^{n-1}(\prod_{i=0}^{l-1}\bar{Q}_i),$$

then we have

$$\begin{aligned} R_l^n - S_l &= (\prod_{i=l}^{d-1}\bar{Q}_i)R_0^{n-1}(\prod_{i=0}^{l-1}\bar{Q}_i) - (\prod_{i=l}^{d-1}\bar{Q}_i)S_0(\prod_{i=0}^{l-1}\bar{Q}_i) \\ &= (\prod_{i=l}^{d-1}\bar{Q}_i)(R_0^{n-1} - S_0)(\prod_{i=0}^{l-1}\bar{Q}_i) \end{aligned} \tag{2.9}$$

By (2.8), it follows that

$$\lim_{n\to\infty}(R_l^n - S_l) = (\prod_{i=l}^{d-1}\bar{Q}_i)\lim_{n\to\infty}(R_0^{n-1} - S_0)(\prod_{i=0}^{l-1}\bar{Q}_i) = 0. \tag{2.10}$$

so that $R_1, R_2, \cdots, R_{d-1}$ are ergodic, and (2.7) holds accordingly. The proof of this lemma is completed.

Let $\{X_t, t \in T\}$ be an asymptotic mth-order circular Markov chain indexed by an m-rooted homogeneous tree T defined as **definition 1.2** and $\{g_n(x_1^{m+1}), n \ge 1\}$ be a sequence of functions defined on $\mathcal{X}^{m+1}$. Denote

$$G_n(\omega) = \sum_{k=1}^{n}\sum_{t\in L_k} E[g_k(X_m^0(t))|X_m^1(t)], \tag{2.11}$$

$$H_n(\omega) = \sum_{k=1}^{n}\sum_{t\in L_k} g_k(X_m^0(t)). \tag{2.12}$$

Lemma 2.3(See [8]) *Let $\{X_t, t \in T\}$ be a nonhomogeneous mth-order Markov chain indexed by an m rooted Cayley tree T defined as* ***definition 1.1*** *and $\{g_n(x_1^{m+1}), t \in T\}$ be a sequence of uniformly bounded functions defined on $\mathcal{X}^{m+1}$. That is, there is a constant $k > 0$ such that $|g_n(x_1^{m+1})| \le k$ for all $n \ge 0$. Let $G_n(\omega)$ and $H_n(\omega)$ be defined as above two equations (2.11) and (2.12) respectively, $\{a_n, n \ge 1\}$ be a sequence of nonnegative random variables. Set*

$$\Omega_0 = \{\lim_{n\to\infty} a_n = \infty, \limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \sum_{t\in L_k} E[|g_k(X_m^0(t))||X_m^1(t)] = M(\omega) < \infty\}, \tag{2.13}$$

Then

$$\lim_{n\to\infty} \frac{H_n(\omega) - G_n(\omega)}{a_n} = 0 \qquad a.e. \ on \ \Omega_0. \tag{2.14}$$

3. Strong law of large numbers

In this section, we mainly focus on studying the strong law of large numbers for an asymptotic mth-order circular Markov chain indexed by an m rooted Cayley tree T.

Theorem 3.1 *Let $\{X_t, t \in T\}$ be an asymptotic mth-order circular Markov chain indexed by an m rooted Cayley tree T defined as* ***definition 1.2***, *for $l = 0, 1, 2, \cdots, d-1$ and $\forall n \in \mathbf{N}$, let $S_n^{[l]}(j_1^m)$ be defined as* (1.11). *Then for all $i_1^m \in \mathcal{X}^m$, we have*

$$\lim_{n\to\infty} \Big\{ \frac{1}{|T_{[0]}^{(n)}|} S_n^{[0]}(j_1^m) - \frac{1}{|T_{[d-1]}^{(n-1)}|} \sum_{i_1^m \in \mathcal{X}^m} S_{n-1}^{[d-1]}(i_1^m) \bar{Q}_{d-1}(j_1^m | i_1^m) \Big\} = 0 \quad a.e. \tag{3.1}$$

$$\lim_{n\to\infty} \Big\{ \frac{1}{|T_{[l]}^{(n)}|} S_n^{[l]}(j_1^m) - \frac{1}{|T_{[l-1]}^{(n-1)}|} \sum_{i_1^m \in \mathcal{X}^m} S_{n-1}^{[l-1]}(i_1^m) \bar{Q}_{d-1}(j_1^m | i_1^m) \Big\} = 0, \ 1 \le l \le d-1 \ a.e.. \tag{3.2}$$

where $\bar{Q}_l$ is the $m-$dimensional transition matrix which is determined by the mth-order transition matrix Q_l for $0 \le l \le d-1$.

Proof . For $t \in L_k$ and $k \in \mathbf{N}$ and $l = 1, 2, \cdots, d$, in **Lemma 2.3** letting $g_k(X_m^0(t)) = I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))$ and $a_n = |T_{[l]}^{(n)}|$, obviously $|g_k(X_{m-1}^0(t))| \le 1$. It is easy to see

$$\limsup_{n\to\infty} \frac{1}{|T_{[l]}^{(n)}|} \sum_{k=1}^{n} \sum_{t\in L_k} E[|g_k(X_m^0(t))||X_m^1(t)] \le \limsup_{n\to\infty} \frac{1}{|T_{[l]}^{(n)}|} \sum_{k=1}^{n} \sum_{t\in L_k} I^{[l]}(k) \le 1. \ a.e. \tag{3.3}$$

which implies that $\Omega_0 = \Omega$. Then we have

$$\begin{aligned} G_n(\omega) &= \sum_{k=1}^{n} \sum_{t\in L_k} E[g_k(X_m^0(t))|X_m^1(t)] \\ &= \sum_{k=1}^{n} \sum_{t\in L_k} E[I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))|X_m^1(t)] \\ &= \sum_{k=1}^{n} \sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^{m-1}}(X_{m-1}^1(t)) P_{k-1}(i_m|X_m^1(t)) \\ &= \sum_{k=1}^{n} \sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^{m-1}}(X_{m-1}^1(t)) P_{k-1}(i_m|X_{^mt}, i_1^{m-1}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i_0\in\mathcal{X}} I^{[l]}(k)\delta_{i_0}(X_{m^t})\delta_{i_1^{m-1}}(X_{m-1}^1(t))P_{k-1}(i_m|i_0,i_1^{m-1}),\\
&= \sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i_0\in\mathcal{X}} I^{[l]}(k)\delta_{i_0^{m-1}}(X_m^1(t))P_{k-1}(i_m|i_0^{m-1})\\
&= N\sum_{k=0}^{n-1}\sum_{t\in L_k}\sum_{i_0\in\mathcal{X}} I^{[l-1]}(k)\delta_{i_0^{m-1}}(X_{m-1}^0(t))P_k(i_m|i_0^{m-1})\\
&= N\sum_{r=0}^{\lfloor\frac{n-l}{d}\rfloor}\sum_{t\in L_{rd+l-1}}\sum_{i_0\in\mathcal{X}} I^{[l-1]}(k)\delta_{i_0^{m-1}}(X_{m-1}^0(t))P_{rd+l-1}(i_m|i_0^{m-1})
\end{aligned}
$$

here and thereafter $\lfloor a\rfloor$ is the largest integer less than a.

$$
\begin{aligned}
H_n(\omega) &= \sum_{k=1}^{n}\sum_{t\in L_k} g_k(X_m^0(t))\\
&= \sum_{k=1}^{n}\sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))\\
&= S_n^{[l]}(i_1^m) - \delta_{i_1^m}(X_{0_m})I^{[l]}(0).
\end{aligned}
$$

Combining above two equations and (2.14), we get

$$
\lim_{n\to\infty}\Big\{\frac{1}{|T_{[l]}^{(n)}|}S_n^{[l]}(i_1^m) - \frac{1}{|T_{[l-1]}^{(n-1)}|}\cdot
\sum_{r=0}^{\lfloor\frac{n-l}{d}\rfloor}\sum_{t\in L_{rd+l-1}}\sum_{i_0\in\mathcal{X}}\delta_{i_0^{m-1}}(X_{m-1}^0(t))P_{rd+l-1}(i_m|i_0^{m-1})\Big\} = 0. \quad a.e. \tag{3.4}
$$

where we have used the following fact

$$
|T_{[l]}^{(n)}| = N^l + N^{d+l} + \cdots + N^{\lfloor\frac{n-l}{d}\rfloor d+l}, \qquad \lim_{n\to\infty}\frac{|T_{[l]}^{(n)}|}{|T_{[l-1]}^{(n-1)}|} = N. \tag{3.5}
$$

By (1.11), we have

$$
\begin{aligned}
&\sum_{i_0\in\mathcal{X}} S_{n-1}^{[l-1]}(i_0^{m-1})Q_{l-1}(i_m|i_0^{m-1})\\
&= \sum_{r=0}^{\lfloor\frac{n-l}{d}\rfloor}\sum_{t\in L_{rd+l-1}}\sum_{i_0\in\mathcal{X}}\delta_{i_0^{m-1}}(X_{m-1}^0(t))Q_{l-1}(i_m|i_0^{m-1})
\end{aligned} \tag{3.6}
$$

it follows from (3.4) – (3.6) that

$$
\begin{aligned}
&\lim_{n\to\infty}\{\frac{1}{|T_{[l]}^{(n)}|}S_n^{[l]}(i_1^m) - \frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{i_0\in\mathcal{X}} S_{n-1}^{[l-1]}(i_0^{m-1})Q_{l-1}(i_m|i_0^{m-1})\}\\
&= \lim_{n\to\infty}\{\frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{i_0\in\mathcal{X}}\sum_{r=0}^{\lfloor\frac{n-l}{d}\rfloor}\sum_{t\in L_{rd+l-1}}\delta_{i_0^{m-1}}(X_{m-1}^0(t))(P_{rd+l-1}(i_m|i_0^{m-1}) - Q_{l-1}(i_m|i_0^{m-1}))\}\\
&\le \lim_{n\to\infty}\sup_{0\le r\le\lfloor\frac{n-l}{d}\rfloor}\sup_{i_0^{m-1}\in\mathcal{X}^m, i_m\in\mathcal{X}}|P_{rd+l-1}(i_m|i_0^{m-1}) - Q_{l-1}(i_m|i_0^{m-1})|\cdot\lim_{n\to\infty}\frac{|T_{[l-1]}^{(n-1)}|}{|T_{[l-1]}^{(n-1)}|}\\
&= 0. \quad a.e.
\end{aligned}
$$

which implies that

$$\lim_{n\to\infty}\Big\{\frac{1}{|T^{(n)}_{[l]}|}S^{[l]}_n(i_1^m)-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{i_0\in\mathcal{X}}S^{[l-1]}_{n-1}(i_0^{m-1})Q_{l-1}(i_m|i_0^{m-1})\Big\}=0,\ l=1,2,\cdots,d\ a.e.. \tag{3.7}$$

By (2.2) and (3.7), we have

$$\lim_{n\to\infty}\Big\{\frac{1}{|T^{(n)}_{[l]}|}S^{[l]}_n(i_1^m)-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)\bar{Q}_{l-1}(i_1^m|j_1^m)\Big\}=0,\ l=1,2,\cdots,d\ a.e.. \tag{3.8}$$

Combining with (1.14) and (3.8), it follows that (3.1) and (3.2) are really true.

Theorem 3.2 *Let $\{X_t, t\in T\}$ be an mth-order asymptotic circular Markov chain indexed by an m rooted homogeneous tree T defined as* ***definition 1.2****, for $l=0,1,2,\cdots,d-1$ and $\forall n\in\mathbf{N}$, let $S^{[l]}_n(j_1^m)$ be defined as* (1.11)*, R_l defined as Lemma 2.2 and R_0 be ergodic. Then for all $i_1^m\in\mathcal{X}^m$, we have*

$$\lim_{n\to\infty}\frac{S^{[l]}_n(j_1^m)}{|T^{(n)}_{[l]}|}=\pi^l(j_1^m),\qquad a.e.. \tag{3.9}$$

where $\pi^l=(\pi^l(j_1^m))_{j_1^m\in\mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_l for each $l=0,1,2,\cdots,d-1$.

Proof: For $l=1,2,\cdots,d$, multiplying (3.8) by $\bar{Q}_l(k_1^m|i_1^m)$ and adding them together for all $i_1^m\in\mathcal{X}^m$, by using (3.8) again, we have

$$\begin{aligned}
&\lim_{n\to\infty}\Big\{\frac{1}{|T^{(n)}_{[l]}|}\sum_{i_1^m\in\mathcal{X}^m}S^{[l]}_n(i_1^m)\bar{Q}_l(k_1^m|i_1^m)\\
&-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}\sum_{i_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)\bar{Q}_{l-1}(i_1^m|j_1^m)\bar{Q}_l(k_1^m|i_1^m)\Big\}\\
=\ &\lim_{n\to\infty}\Big\{\Big[\frac{1}{|T^{(n)}_{[l]}|}\sum_{i_1^m\in\mathcal{X}^m}S^{[l]}_n(i_1^m)\bar{Q}_l(k_1^m|i_1^m)-\frac{1}{|T^{(n+1)}_{[l+1]}|}S^{[l+1]}_{n+1}(k_1^m)\Big]\\
&+\Big[\frac{1}{|T^{(n+1)}_{[l+1]}|}S^{[l+1]}_{n+1}(k_1^m)-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}\sum_{i_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)\bar{Q}_{l-1}(i_1^m|j_1^m)\bar{Q}_l(k_1^m|i_1^m)\Big]\Big\}\\
=\ &\lim_{n\to\infty}\Big[\frac{1}{|T^{(n+1)}_{[l+1]}|}S^{[l+1]}_{n+1}(k_1^m)-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)(\bar{Q}_{l-1}\bar{Q}_l)(k_1^m|j_1^m)\Big]=0\quad a.e.
\end{aligned} \tag{3.10}$$

Noting the basic fact that $S^{[l+d-1]}_{n+d-1}(k_1^m)=S^{[l-1]}_{n+d-1}(k_1^m)-\delta_{k_1^m}(X_{0_m})\delta_1(l)$ and $|T^{(n+d-1)}_{[l+d-1]}|=|T^{(n+d-1)}_{[l-1]}|-\delta_1(l)$ for $l=1,2,\cdots,d$, by induction we have

$$\begin{aligned}
&\lim_{n\to\infty}\Big\{\frac{S^{[l+d-1]}_{n+d-1}(k_1^m)}{|T^{(n+d-1)}_{[l+d-1]}|}-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)(\bar{Q}_{l-1}\bar{Q}_l\cdots\bar{Q}_{d-1}\bar{Q}_0\bar{Q}_1\cdots\bar{Q}_{l-2})(k_1^m|j_1^m)\Big\}\\
=\ &\lim_{n\to\infty}\Big\{\frac{S^{[l-1]}_{n+d-1}(k_1^m)-\delta_{k_1^m}(X_{0_m})\delta_1(l)}{|T^{(n+d-1)}_{[l-1]}|-\delta_1(l)}-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)R_{l-1}(k_1^m|j_1^m)\Big\}=0\quad a.e.,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\Big\{\frac{S^{[l-1]}_{n+d-1}(k_1^m)}{|T^{(n+d-1)}_{[l-1]}|}-\frac{1}{|T^{(n-1)}_{[l-1]}|}\sum_{j_1^m\in\mathcal{X}^m}S^{[l-1]}_{n-1}(j_1^m)R_{l-1}(k_1^m|j_1^m)\Big\}=0\quad a.e. \tag{3.11}$$

Multiplying (3.11) by $R_{l-1}(i_1^m|k_1^m)$ and adding them together for all $k_1^m \in \mathcal{X}^m$, by using (3.11) again, we arrive at

$$\begin{aligned}
&\lim_{n\to\infty}\Big\{\frac{\sum_{k_1^m\in\mathcal{X}^m} S_{n+d-1}^{[l-1]}(k_1^m)R_{l-1}(i_1^m|k_1^m)}{|T_{[l-1]}^{(n+d-1)}|}\\
&-\frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{k_1^m\in\mathcal{X}^m}\sum_{j_1^m\in\mathcal{X}^m} S_{n-1}^{[l-1]}(j_1^m)R_{l-1}(k_1^m|j_1^m)R_{l-1}(i_1^m|k_1^m)\Big\}\\
=\;&\lim_{n\to\infty}\Big\{[\frac{1}{|T_{[l-1]}^{(n+d-1)}|}\sum_{k_1^m\in\mathcal{X}^m} S_{n+d-1}^{[l-1]}(k_1^m)R_{l-1}(i_1^m|k_1^m)-\frac{S_{n+2d-1}^{[l-1]}(i_1^m)}{|T_{[l-1]}^{(n+2d-1)}|}]\\
&+[\frac{S_{n+2d-1}^{[l-1]}(i_1^m)}{|T_{[l-1]}^{(n+2d-1)}|}-\frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{j_1^m\in\mathcal{X}^m} S_{n-1}^{[l-1]}(j_1^m)R_{l-1}^2(i_1^m|j_1^m)]\Big\}\\
=\;&\lim_{n\to\infty}[\frac{S_{n+2d-1}^{[l-1]}(i_1^m)}{|T_{[l-1]}^{(n+2d-1)}|}-\frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{j_1^m\in\mathcal{X}^m} S_{n-1}^{[l-1]}(j_1^m)R_{l-1}^2(i_1^m|j_1^m)]=0 \quad a.e.
\end{aligned}\tag{3.12}$$

By induction again we get

$$\lim_{n\to\infty}[\frac{S_{n+Md-1}^{[l-1]}(i_1^m)}{|T_{[l-1]}^{(n+Md-1)}|}-\frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{j_1^m\in\mathcal{X}^m} S_{n-1}^{[l-1]}(j_1^m)R_{l-1}^M(i_1^m|j_1^m)]=0 \quad a.e. \tag{3.13}$$

Since

$$\lim_{M\to\infty} R_{l-1}^M(i_1^m|j_1^m)=\pi^{l-1}(i_1^m);\qquad \sum_{j_1^m\in\mathcal{X}^m} S_{n-1}^{[l-1]}(j_1^m)=|T_{[l-1]}^{(n-1)}|, \tag{3.14}$$

combining (3.13) and (3.14), it follows that

$$\lim_{n\to\infty}\frac{S_n^{[l-1]}(i_1^m)}{|T_{[l-1]}^{(n)}|}=\pi^{l-1}(i_1^m),\quad \text{for } l=1,2,\cdots,d, i_1^m\in\mathcal{X}^m \quad a.e.. \tag{3.15}$$

it follows that (3.9) is true. The proof of theorem 3.2 is completed.

For any nonnegative integer n, there exist two nonnegative integer numbers r and l such that

$$n = rd + l. \quad \text{for } l = 0,1,\cdots,d-1 \tag{3.16}$$

For such $l = 0,1,\cdots,d-1$, we have

$$\begin{aligned}
|T_{[i]}^{(rd+l)}| &= \begin{cases} N^i+N^{d+i}+N^{2d+i}+\cdots+N^{rd+i} & \text{for } 0\le i\le l;\\ N^i+N^{d+i}+N^{2d+i}+\cdots+N^{(r-1)d+i} & \text{for } l+1\le i\le d-1,\end{cases}\\
&= \begin{cases} \frac{N^i(1-N^{rd+d})}{1-N^d} & \text{for } 0\le i\le l;\\ \frac{N^i(1-N^{rd})}{1-N^d} & \text{for } l+1\le i\le d-1.\end{cases}
\end{aligned}\tag{3.17}$$

Noting that

$$|T^{(rd+l)}| = 1+N+N^2+\cdots+N^{rd+l}=\frac{1-N^{rd+l+1}}{1-N}, \tag{3.18}$$

it is easy to see

$$
\begin{aligned}
\lim_{r\to\infty}\frac{|T^{(rd+l)}_{[i]}|}{|T^{(rd+l)}|} &= \begin{cases} \frac{N^i(1-N)}{1-N^d}\times \lim_{r\to\infty}\frac{1-N^{rd+d}}{1-N^{rd+l+1}} & \text{for } 0\le i\le l;\\ \frac{N^i(1-N)}{1-N^d}\times \lim_{r\to\infty}\frac{1-N^{rd}}{1-N^{rd+l+1}} & \text{for } l+1\le i\le d-1,\end{cases}\\
&= \begin{cases} \frac{N^{d-l-1+i}(1-N)}{1-N^d} & \text{for } 0\le i\le l;\\ \frac{N^{i-l-1}(1-N)}{1-N^d} & \text{for } l+1\le i\le d-1,\end{cases}
\end{aligned}
\tag{3.19}
$$

Combining (3.9) and (3.19), we can arrive at the following corollary easily.

Corollary 3.3 *Under the same conditions of Theorem 3.2, for each $l = 0,1,\cdots,d-1$, we have*

$$
\lim_{r\to\infty}\frac{S^{[i]}_{rd+l}(j_1^m)}{|T^{(rd+l)}|} = \begin{cases} \frac{N^{d-l-1+i}(1-N)}{1-N^d}\times\pi^i(j_1^m) & \text{for } 0\le i\le l;\\ \frac{N^{i-l-1}(1-N)}{1-N^d}\times\pi^i(j_1^m) & \text{for } l+1\le i\le d-1, \quad a.e.\end{cases}
\tag{3.20}
$$

where $\pi^l = (\pi^l(j_1^m))_{j_1^m\in\mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_l for each $l = 0,1,2,\cdots,d-1$.

For each $l = 0,1,\cdots,d-1$, noting that

$$
S_{rd+l}(j_1^m) = \sum_{i=0}^{d-1} S^{[i]}_{rd+l}(j_1^m).
\tag{3.21}
$$

We can also derive another corollary from above one as follows:

Corollary 3.4 *Under the same conditions of Theorem 3.2, we have*

$$
\lim_{r\to\infty}\frac{S_{rd+l}(j_1^m)}{|T^{(rd+l)}|} = \sum_{i=0}^{l}\frac{N^{d-l-1+i}(1-N)}{1-N^d}\times\pi^i(j_1^m) + \sum_{i=l+1}^{d-1}\frac{N^{i-l-1}(1-N)}{1-N^d}\times\pi^i(j_1^m) \quad a.e.
\tag{3.22}
$$

where $\pi^i = (\pi^i(j_1^m))_{j_1^m\in\mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_i for each $i = 0,1,2,\cdots,d-1$.

Theorem 3.5 *Let $\{X_t, t\in T\}$ be an asymptotic mth-order circular Markov chain indexed by an m rooted Cayley tree T defined as* ***definition 1.2****, for $l = 0,1,2,\cdots,d-1$ and $\forall n\in\mathbf{N}$, let $S_n^{[l]}(i_1^{m+1})$ be defined as* (1.11). *Then for all $i_1^{m+1}\in\mathcal{X}^{m+1}$, we have*

$$
\lim_{n\to\infty}\frac{S_n^{[l]}(i_1^{m+1})}{|T^{(n)}_{[l]}|} = \pi^l(i_1^m)Q_l(i_{m+1}|i_1^m) \qquad a.e.
\tag{3.23}
$$

where $\pi^l = (\pi^l(i_1^m))_{i_1^m\in\mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_l for each $l = 0,1,2,\cdots,d-1$.

Proof : For $t\in L_k$ and $k\in\mathbf{N}$, letting $g_k(X_m^0(t)) = I^{[l]}(k-1)\delta_{i_1^{m+1}}(X_m^0(t))$, and $a_n = |T^{(n)}_{[l]}|$ in **Lemma 2.2**, apparently

$|g_k(X_m^0(t))| \le 1$, it is easy to see $\Omega_0 = \Omega$. Then we have

$$\begin{aligned} G_n(\omega) &= \sum_{t\in T^{(n)}\setminus\{0\}} E[g_k(X_m^0(t))|X_m^1(t)] \\ &= \sum_{k=1}^{n}\sum_{t\in L_k} E[I^{[l]}(k-1)\delta_{i_1^{m+1}}(X_m^0(t))|X_m^1(t)] \\ &= \sum_{k=1}^{n}\sum_{t\in L_k}\sum_{x_t\in\mathcal{X}} I^{[l]}(k-1)\delta_{i_1^m}(X_m^1(t))\delta_{i_{m+1}}(x_t)P_{k-1}(x_t|X_m^1(t)), \\ &= \sum_{k=1}^{n}\sum_{t\in L_k} I^{[l]}(k-1)\delta_{i_1^m}(X_m^1(t))P_{k-1}(i_{m+1}|i_1^m), \\ &= N\sum_{k=0}^{n-1}\sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))P_k(i_{m+1}|i_1^m), \end{aligned}$$

$$H_n(\omega) = \sum_{t\in T^{(n)}\setminus\{0\}} g_k(X_m^0(t)) = \sum_{k=1}^{n}\sum_{t\in L_k} I^{[l]}(k-1)\delta_{i_1^{m+1}}(X_m^0(t)) = S_n^{[l]}(i_1^{m+1}).$$

Combining above two equations and (2.14), we get

$$\lim_{n\to\infty}\frac{1}{|T_{[l]}^{(n)}|}\Big\{S_n^{[l]}(i_1^{m+1}) - N\sum_{k=0}^{n-1}\sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))P_k(i_{m+1}|i_1^{m+1})\Big\} = 0 \quad a.e. \tag{3.24}$$

Now we assert that

$$\lim_{n\to\infty}\frac{1}{|T_{[l]}^{(n)}|}\Big\{S_n^{[l]}(i_1^{m+1}) - NS_{n-1}^{[l]}(i_1^m)Q_l(i_{m+1}|i_1^m)\Big\} = 0 \quad a.e. \tag{3.25}$$

In fact, by (3.24),(1.6) and (1.11), we obtain

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{|T_{[l]}^{(n)}|}\Big\{S_n^{[l]}(i_1^{m+1}) - NS_{n-1}^{[l]}(i_1^m)Q_l(i_{m+1}|i_1^m)\Big\} \\ =\ &\lim_{n\to\infty}\frac{1}{|T_{[l]}^{(n)}|}\Big\{N\sum_{k=0}^{n-1}\sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))P_k(i_{m+1}|i_1^m) - NS_{n-1}^{[l]}(i_1^m)Q_l(i_{m+1}|i_1^m)\Big\} \\ =\ &\lim_{n\to\infty}\frac{1}{|T_{[l-1]}^{(n-1)}|}\Big\{\sum_{k=0}^{n-1}\sum_{t\in L_k} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))(P_k(i_{m+1}|i_1^m) - Q_l(i_{m+1}|i_1^m))\Big\} \\ \le\ &\lim_{n\to\infty}\frac{1}{|T_{[l-1]}^{(n-1)}|}\sum_{r=0}^{\lfloor\frac{n-l-1}{d}\rfloor}\sum_{t\in L_{rd+l}} \delta_{i_1^m}(X_{m-1}^0(t))|P_{rd+l}(i_{m+1}|i_1^m) - Q_l(i_{m+1}|i_1^m)| \\ \le\ &\lim_{n\to\infty}\sup_{0\le r\le\lfloor\frac{n-l-1}{d}\rfloor}\sup_{i_1^{m+1}\in\mathcal{X}^{m+1}} |P_{rd+l}(i_{m+1}|i_1^m) - Q_l(i_{m+1}|i_1^m)|\cdot\lim_{n\to\infty}\frac{|T_{[l]}^{(n-1)}|}{|T_{[l-1]}^{(n-1)}|} = 0\ a.e., \end{aligned} \tag{3.26}$$

which implies (3.25), here the second equation holds because of (3.5). Then our conclusion (3.23) can be derived from (3.9) and (3.25).

By using (3.19) again, **Theorem 3.5** will imply the following corollary apparently.
Corollary 3.6 *Under the same conditions of Theorem 3.5, for each $l = 0, 1, 2, \cdots, d-1$, we have*

$$\lim_{r\to\infty} \frac{S^{[k]}_{rd+l}(i_1^{m+1})}{|T^{(rd+l)}|} = \begin{cases} \frac{N^{d-l-1+k}(1-N)}{1-N^d} \times \pi^k(i_1^m)Q_k(i_{m+1}|i_1^m) & \text{for } 0 \le k \le l; \\ \frac{N^{k-l-1}(1-N)}{1-N^d} \times \pi^k(i_1^m)Q_k(i_{m+1}|i_1^m) & \text{for } l+1 \le k \le d-1, \end{cases} \tag{3.27}$$

where $\pi^l = (\pi^l(i_1^m))_{i_1^m \in \mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_l for each $l = 0, 1, 2, \cdots, d-1$.

Obviously, for each $l = 0, 1, 2, \cdots, d-1$, noting that

$$S_{rd+l}(i_1^{m+1}) = \sum_{i=0}^{d-1} S^{[i]}_{rd+l}(i_1^{m+1}). \tag{3.28}$$

Thus, the following corollary can be derived from the above one easily.
Corollary 3.7 *Under the same conditions of Theorem 3.5, for each $l = 0, 1, 2, \cdots, d-1$, we have*

$$\begin{aligned}\lim_{r\to\infty} \frac{S_{rd+l}(i_1^{m+1})}{|T^{(rd+l)}|} &= \sum_{k=0}^{l} \frac{N^{d-l-1+k}(1-N)}{1-N^d} \times \pi^k(i_1^m)Q_k(i_{m+1}|i_1^m)) \\ &+ \sum_{k=l+1}^{d-1} \frac{N^{k-l-1}(1-N)}{1-N^d} \times \pi^k(i_1^m)Q_k(i_{m+1}|i_1^m) \quad a.e.\end{aligned} \tag{3.29}$$

where $\pi^l = (\pi^l(i_1^m))_{i_1^m \in \mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_l for each $l = 0, 1, 2, \cdots, d-1$.

4. Shannon-McMillan theorem

In this section, we mainly prove the asymptotic equipartition (AEP) for asymptotic mth-order circular Markov chain indexed by an m rooted homogeneous tree $\tilde{T}_{C,N}$.

Let T be the homogeneous tree $\tilde{T}_{C,N}$, $(X_t)_{t\in T}$ be a stochastic process indexed by tree T with state space $\mathcal{X} = \{1, 2, \cdots, b\}$. Denote

$$\tilde{T}^{(n)} = T^{(n)} \bigcup \{\tilde{0}_m\}, \tag{4.1}$$

and

$$P(x^{\tilde{T}^{(n)}}) = P(X^{\tilde{T}^{(n)}} = x^{\tilde{T}^{(n)}}). \tag{4.2}$$

Let

$$f_n(\omega) = -\frac{1}{|\tilde{T}^{(n)}|} \log P(X^{\tilde{T}^{(n)}}), \tag{4.3}$$

$f_n(\omega)$ will be called the entropy density of $X^{\tilde{T}^{(n)}}$. If $(X_t)_{t\in T}$ is an asymptotic mth-order circular Markov chain indexed by the m rooted homogeneous tree T with finite state space $\mathcal{X}$ defined as Definition 1.2, obviously, we have

$$P(x^{\tilde{T}^{(n)}}) = P(X^{\tilde{T}^{(n)}} = x^{\tilde{T}^{(n)}}) = P[X_{0_m} = x_{0_m}] \prod_{k=1}^{n} \prod_{t\in L_k} P_{k-1}(x_t|x_m^1(t)). \tag{4.4}$$

where $x_m^1(t)$ is the realization of $X_m^1(t) = (X_{m_t}, X_{m-1_t}, \cdots, X_{1_t})$. Thus we have

$$f_n(\omega) = -\frac{1}{|\tilde{T}^{(n)}|}[\log P(X_{0_m}) + \sum_{k=1}^{n} \sum_{t\in L_k} \log P_{k-1}(X_t|X_m^1(t))]. \tag{4.5}$$

Obviously, we have

$$\lim_{n\to\infty}\frac{|T^{(n)}|}{|\tilde{T}^{(n)}|}=\lim_{n\to\infty}\frac{|T^{(n)}|}{|T^{(n)}|+m-1}=1. \tag{4.6}$$

so that

$$\lim_{n\to\infty} f_n(\omega)=-\lim_{n\to\infty}\frac{1}{|T^{(n)}|}[\log P(X_{0_m})+\sum_{k=1}^{n}\sum_{t\in L_k}\log P_{k-1}(X_t|X_m^1(t))]. \tag{4.7}$$

The convergence of $f_n(\omega)$ to a constant in a sense (L_1 convergence , convergence in probability, a.e. convergence) is called the Shannon-McMillan theorem or the entropy theorem or the AEP in information theory.

Lemma 4.1 (see Theorem 3 in [8]) *Let $(X_t)_{t\in T}$ be a finite mth-order nonhomogeous Markov chain indexed by an m rooted homogeneous tree T defined as **Definition 1.1**. Let $f_n(\omega)$ be defined by (4.7). Then we have*

$$\lim_{n\to\infty}\{f_n(\omega)-\frac{1}{|T^{(n)}|}\sum_{k=1}^{n}\sum_{t\in L_k}H[P_{k-1}(1|X_m^1(t)),P_{k-1}(2|X_m^1(t)),\cdots,P_{k-1}(b|X_m^1(t))]\}=0,\quad a.e. \tag{4.8}$$

where $H[p_1,p_2,\cdots,p_b]$ is the entropy of the distribution $(p_1,p_2,\cdots,p_b)$, that is

$$H[p_1,p_2,\cdots,p_b]=-\sum_{i=1}^{b}p_i\log p_i.$$

Theorem 4.2 *Let $\{X_t, t\in T\}$ be an asymptotic circular mth-order Markov chain indexed by an m−rooted homogeneous tree $\tilde{T}_{C,N}$ defined as **definition 1.2**. Let $f_n(\omega)$ be defined by (4.7). Then for two nonnegative integer number r and l which satisfy equation (3.16), for each $l=0,1,2,\cdots,d-1$ we have*

$$\begin{aligned}\lim_{r\to\infty} f_{rd+l}(\omega) &= -\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\sum_{k=0}^{l-1}\frac{N^{d-l+k}(1-N)}{1-N^d}\times\pi^k(i_1^m)Q_k(i_{m+1}|i_1^m)\log Q_k(i_{m+1}|i_1^m)\\ &\quad-\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\sum_{k=l}^{d-1}\frac{N^{k-l}(1-N)}{1-N^d}\times\pi^k(i_1^m)Q_k(i_{m+1}|i_1^m)\log Q_k(i_{m+1}|i_1^m)\ \ a.e.\end{aligned} \tag{4.9}$$

where $\pi^l=(\pi^l(i_1^m))_{i_1^m\in\mathcal{X}^m}$ is the unique stationary distribution determined by the transition matrix R_l for each $l=0,1,2,\cdots,d-1$.

Proof of Theorem 4.2 : Letting $\varphi(x)=x\log x, x\geq 0$(suppose $\varphi(0)=0$), obviously $\varphi(x)=x\log x$ is a continuous function at any $x\geq 0$, then by (1.5), for all $i_1^{m+1}\in\mathcal{X}^{m+1}$ and $l=0,1,2,\cdots,d-1$, we have

$$\lim_{r\to\infty}|P_{rd+l}(i_{m+1}|i_1^m)\log P_{rd+l}(i_{m+1}|i_1^m)-Q_l(i_{m+1}|i_1^m)\log Q_l(i_{m+1}|i_1^m)|=0. \tag{4.10}$$

Apparently, Lemma 4.1 can imply that

$$\lim_{n\to\infty}\{f_n(\omega)+\frac{1}{|T^{(n)}|}\sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i_{m+1}\in\mathcal{X}}P_{k-1}(i_{m+1}|X_m^1(t))\log P_{k-1}(i_{m+1}|X_m^1(t))\}=0,\quad a.e. \tag{4.11}$$

Moreover, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i_{m+1}\in\mathcal{X}} P_{k-1}(i_{m+1}|X_m^1(t))\log P_{k-1}(i_{m+1}|X_m^1(t))\\
=&\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\sum_{k=1}^{n}\sum_{t\in L_k}\sum_{l=0}^{d-1} I^{[l]}(k-1)\delta_{i_1^m}(X_m^1(t))P_{k-1}(i_{m+1}|i_1^m)\log P_{k-1}(i_{m+1}|i_1^m)\\
=&N\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\sum_{k=0}^{n-1}\sum_{t\in L_k}\sum_{l=0}^{d-1} I^{[l]}(k)\delta_{i_1^m}(X_{m-1}^0(t))P_k(i_{m+1}|i_1^m)\log P_k(i_{m+1}|i_1^m)\\
=&N\sum_{l=0}^{d-1}\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\sum_{r=0}^{\lfloor\frac{n-l-1}{d}\rfloor}\sum_{t\in L_{rd+l}}\delta_{i_1^m}(X_{m-1}^0(t))P_{rd+l}(i_{m+1}|i_1^m)\log P_{rd+l}(i_{m+1}|i_1^m)
\end{aligned} \tag{4.12}$$

Now we claim that

$$\begin{aligned}
&\lim_{n\to\infty}\{\frac{1}{|T^{(n)}|}\sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i_{m+1}\in\mathcal{X}} P_{k-1}(i_{m+1}|X_m^1(t))\log P_{k-1}(i_{m+1}|X_m^1(t))\\
&-\sum_{l=0}^{d-1}\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\frac{S_{n-1}^{[l]}(i_1^m)}{|T^{(n-1)}|}Q_l(i_{m+1}|i_1^m)\log Q_l(i_{m+1}|i_1^m)\} = 0. \ a.e.
\end{aligned} \tag{4.13}$$

In fact, we have

$$\begin{aligned}
&\lim_{n\to\infty}|\frac{1}{|T^{(n)}|}\sum_{k=1}^{n}\sum_{t\in L_k}\sum_{i_{m+1}\in\mathcal{X}} P_{k-1}(i_{m+1}|X_m^1(t))\log P_{k-1}(i_{m+1}|X_m^1(t))\\
&-\sum_{l=0}^{d-1}\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\frac{S_{n-1}^{[l]}(i_1^m)}{|T^{(n-1)}|}Q_l(i_{m+1}|i_1^m)\log Q_l(i_{m+1}|i_1^m)|\\
\leq&\lim_{n\to\infty}\frac{1}{|T^{(n-1)}|}\sum_{l=0}^{d-1}\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\sum_{r=0}^{\lfloor\frac{n-l-1}{d}\rfloor}\sum_{t\in L_{rd+l}}\delta_{i_1^m}(X_{m-1}^0(t))\cdot\\
&|P_{rd+l}(i_{m+1}|i_1^m)\log P_{rd+l}(i_{m+1}|i_1^m)-Q_l(i_{m+1}|i_1^m)\log Q_l(i_{m+1}|i_1^m)|\\
\leq&\lim_{n\to\infty}\sup_{0\leq r\leq\lfloor\frac{n-l-1}{d}\rfloor}\sup_{i_1^{m+1}\in\mathcal{X}^{m+1}}|P_{rd+l}(i_{m+1}|i_1^m)\log P_{rd+l}(i_{m+1}|i_1^m)-Q_l(i_{m+1}|i_1^m)\log Q_l(i_{m+1}|i_1^m)|\sum_{i_{m+1}\in\mathcal{X}}\frac{\sum_{i_1^m\in\mathcal{X}^m}\sum_{l=0}^{d-1}S_{n-1}^{[l]}(i_1^m)}{|T^{(n-1)}|}\\
=&0 \quad a.e.
\end{aligned} \tag{4.14}$$

The last equation holds because $\mathcal{X}$ is a finite state space and (4.10) holds. Thus the assertion (4.13) is true. So that, for such two nonnegative integer number r and l which satisfy equation (3.16), we have

$$\begin{aligned}
\lim_{r\to\infty} f_{rd+l}(\omega) &= -\sum_{k=0}^{d-1}\sum_{i_1^{m+1}\in\mathcal{X}^{m+1}}\lim_{r\to\infty}\frac{S_{rd+l-1}^{[k]}(i_1^m)}{|T^{(rd+l-1)}|}Q_k(i_{m+1}|i_1^m)\log Q_k(i_{m+1}|i_1^m)\\
&= -\sum_{k=0}^{l-1}\frac{N^{d-l+k}(1-N)}{1-N^d}\times\pi^k(i_1^m)Q_k(i_{m+1}|i_1^m)\ln Q_k(i_{m+1}|i_1^m)\\
&\quad-\sum_{k=l}^{d-1}\frac{N^{k-l}(1-N)}{1-N^d}\times\pi^k(i_1^m)Q_k(i_{m+1}|i_1^m)\ln Q_k(i_{m+1}|i_1^m)\ a.e.
\end{aligned} \tag{4.15}$$

where the second equation can be implied by (3.20). The proof of this theorem is completed.

Let $Q_0 = Q_1 = Q_2 = \cdots = Q_{d-1} = P$, then $\bar{Q}_0 = \bar{Q}_1 = \bar{Q}_2 = \cdots = \bar{Q}_{d-1} = \bar{P}$ so that $R_0 = R_1 = R_2 = \cdots = R_{d-1} = R = \bar{P}^d$. Since the stochastic matrix $\bar{P}$ is ergodic, then $R = \bar{P}^d$ is also ergodic. Suppose that π is the unique stationary distribution determined by R, then we have $\pi^l(i_1^m) = \pi(i_1^m)$, for all $i_1^m \in \mathcal{X}^m$ and $l = 0, 1, 2, \cdots, d-1$, so that it is easy to derive a corollary as follows by doing some simple computations to the results of **Corollary 3.4, Corollary 3.7 and Theorem 4.2** respectively.

Corollary 4.3(See [8]) *Let $(X_t)_{t\in T}$ be a finite nonhomogeneous Markov chain indexed by an $m-$rooted homogeneous tree with finite initial distribution* (1.1) *and finite transition matrices* (1.2). *Let $P = (P(i_{m+1}|i_1^m))_{i_1^{m+1}\in\mathcal{X}^{m+1}}$ be another finite mth-order transition matrix and the m-dimensional stochastic matrix $\bar{P}$ determined by P be ergodic. Let $S_n(i_1^m)$ and $S_n(i_1^{m+1})$ be defined as* (1.10) *and* (1.12) *respectively, and $f_n(\omega)$ be defined by (4.7). If*

$$\lim_{n\to\infty} P_n(i_{m+1}|i_1^m) = P(i_{m+1}|i_1^m), \quad \forall i_1^{m+1} \in \mathcal{X}^{m+1}. \tag{4.16}$$

Then we have

$$\lim_{n\to\infty} \frac{S_n(i_1^m)}{|T^{(n)}|} = \pi(i_1^m) \quad a.e., \tag{4.17}$$

$$\lim_{n\to\infty} \frac{S_n(i_1^m)}{|T^{(n)}|} = \pi(i_1^m)P(i_{m+1}|i_1^m) \quad a.e., \tag{4.18}$$

$$\lim_{n\to\infty} f_n(\omega) = -\sum_{i_1^{m+1}\in\mathcal{X}} \pi(i_1^m)P(i_{m+1}|i_1^m)\log P(i_{m+1}|i_1^m) \qquad a.e.. \tag{4.19}$$

where $\pi = (\pi(i_1^m))$ is the unique stationary distribution determined by the transition matrix $\bar{P}$.

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