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On Extremal Bipartite Graphs with a Given Connectivity

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Abstract. Let $I(G)$ be a topological index of a graph. If $I(G+e) < I(G)$ (or $I(G+e) > I(G)$, respectively) for each edge $e \notin G$, then $I(G)$ is decreasing (or increasing, respectively) with addition of edges. In this paper, we determine the extremal values of some monotonic topological indices which decrease or increase with addition of edges, and characterize the corresponding extremal graphs among bipartite graphs with a given connectivity.

1. Introduction and Preliminaries

Throughout this paper we consider only simple and connected graphs with order greater than 6. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, we denote by $N_G(v)$ the neighborhood of v in G. $d_G(v) = |N_G(v)|$ is called the degree of v in G. In particular, let $\delta(G) = \min\{d_G(v)|v \in V(G)\}$. For vertices $u, v \in V(G)$, the distance $d_G(u,v)$ is defined as the length of a shortest path between u and v in G. Let $G+e$ denotes the graph obtained from G by adding an edge $e \notin E(G)$. For $S \subseteq V(G)$, the induced subgraph of G is denoted by $G[S]$. As usual, P_n and K_n denote a path and a complete graph on n vertices, respectively.

A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. If G contains every edge joining a vertex of X with a vertex of Y, then G is a complete bipartite graph and is denoted by $K_{p,q}$, where $p = |X|$ and $q = |Y|$.

A cut vertex (edge) of a graph is a vertex (an edge) whose removal increases the number of components of the graph. A vertex (An edge) cut of a graph is a set of vertices (edges) whose removal disconnects the graph. The vertex connectivity $\kappa(G)$ (respectively, the edge connectivity $\kappa'(G)$) of a graph G is the minimum number of vertices (respectively, the minimum number of edges) whose deletion yields the resulting graph disconnected or a singleton. It is well known that $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ for any graph G.

Topological indices, also known as molecular descriptors, are graph invariants that map a (molecular) graph to a real number. They are used for modeling physicochemical, toxicologic, biological, and other properties of chemical compounds and more significantly in the nonempirical quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR). Up to now, thousands of

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topological indices have been introduced and extensively studied in chemical graph theory. Some well-known topological indices are distance-based (the Wiener index, the Harary index, the Kirchhoff index, the eccentricity distance sum, etc.), vertex-degree-based (the Zagreb index, the Randić index, the atom-bond connectivity index, etc.) and energy-based (the energy, the Laplacian energy, the matching energy, etc.), respectively.

But from another perspective, many important topological indices have the monotonicity [4], i.e., decrease (or increase, respectively) with addition of edges, such as the Wiener index, the eccentricity distance sum, the Kirchhoff index, the Merrifield-Simmons index are monotonically decreasing with the addition of edges, and the Zagreb index, the Hosoya index, the atom-bond connectivity index, the Estrada index, the matching energy are monotonically increasing with the addition of edges.

The vertex connectivity and edge connectivity are known as two very important graph parameters. In the past several years, a lot of endeavors have been devoted to studying the extremal values of some topological indices in terms of vertex (edge) connectivity. Gutman and Zhang [15] determined the graph with minimum Wiener index among all graphs of order n and vertex (edge) connectivity k. The extremal values of Zagreb and hyper-Wiener indices with a given connectivity obtained by Behtoei et al. [1]. Li and Zhou [19] studied the extremal properties of the first and second Zagreb index when connectivity is at most k. More results for graphs with a given connectivity can be found in [27, 29, 31] and the references cited therein. In 2012, Nath and Paul [22] determined the minimum distance spectral radius among all connected bipartite graphs of order n with a given matching number and a vertex connectivity. After that, these results for bipartite graphs haven been extended to the distance Laplacian spectral radius by Liu et al. [23], to the Wiener index by Li and Song [18] and the Estrada index by Huang et al. [16]. On the basis of the monotonicity, unified approaches for various topological indices of graphs (or bipartite graphs) in terms of some kinds of graph parameters were proposed in [4, 5, 29].

In this paper, we continue to study the mathematical properties of the monotonic topological indices and concentrate on the extremal values of some topological indices in bipartite graphs with a given connectivity. Furthermore, the extremal graphs of these topological indices are determined completely.

Let $I(G)$ be a topological index of a graph. If $I(G+e) < I(G)$ (or $I(G+e) > I(G)$, respectively) for each edge $e \notin G$, then $I(G)$ decreases (or increases, respectively) with addition of edges. The following result shows a common structural characteristic of the extremal graphs for monotonic topological indices over all bipartite graphs with n vertices and a given connectivity.

Proposition 1.1. *Let G be a bipartite graph with the minimal I-value (the maximal I-value) for the topological index I which decreases (increases) with addition of edges among all bipartite graphs with n vertices and vertex connectivity s. Let S be a minimum vertex cut of G. If $G-S$ has a nontrivial component, then $G-S$ has exactly two components G_1 and G_2, and $G[S \cup V(G_1)]$ and $G[S \cup V(G_2)]$ are complete bipartite graphs.*

Proof. Without loss of generality, we assume that I is a topological index which decreases with addition of edges. Let G be a connected bipartite graph with the minimal I-value among all the bipartite graphs with n vertices and vertex connectivity $\kappa(G) = s$. S is a minimum vertex cut of G and $G_1, G_2, \ldots, G_k$ are the components of $G-S$. If $k \geq 3$ or $G[S \cup V(G_i)]$ is not a complete bipartite graph, then we can add some appropriate edges between $G_1, G_2, \ldots, G_{k-1}$ or $G[S \cup V(G_i)]$ such that the result graph G' is still bipartite with n vertices and vertex connectivity s. However, $I(G') < I(G)$, a contradiction. Thus, the proof is completed. □

2. The extremal values of some topological indices in bipartite graphs with a given connectivity

In this section, we will discuss the extremal graphs with respect to some topological indices, including the Merrifield-Simmons index, the first Zagreb index, the Harary index, the hyper-Wiener index, the multiplicative Wiener index, among all bipartite graphs with a given connectivity.

The Merrifield-Simmons index or σ-index of a graph G, denoted by $\sigma(G)$, was introduced by Merrifield and Simmons [21]. It is defined as the total number of independent vertex sets of G, including the empty vertex set, that is

$$\sigma(G) = \sum_{k\geq 0}^{n} i(G;k),$$

where $i(G;k)$ is the number of k-independent vertex sets of G, and $i(G;0) = 1$.

The first Zagreb index is one of the oldest graph invariants, introduced by Gutman and Trinajestić [14]. Denoted by M_1 and defined as the sum of squares of the vertex degrees, it is given by

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)] = \sum_{u\in V(G)} d_G(u)^2.$$

The Harary index of a graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [24] and by Ivanciuc et al. [17] in 1993 for the characterization of molecular graphs. The Harary index is defined as

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)}.$$

The hyper-Wiener index of G, denoted by $WW(G)$, was first introduced by Randić [25] and defined as

$$WW(G) = \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} \left[d_G(u,v) + d_G(u,v)^2\right].$$

The multiplicative Wiener index [12, 13] of graph G was put forward as

$$\pi(G) = \prod_{\{u,v\}\subseteq V(G)} d_G(u,v).$$

The above five indices are widely studied. Some recent results can be found in [2]-[11],[20],[26],[28],[30].

The following lemma is a direct consequence of the definitions above.

Lemma 2.1. *Let G be a connected graph of order n and not isomorphic to the complete graph K_n. Then for each $e \notin E(G)$, we have* (i) $\sigma(G + e) < \sigma(G)$; (ii) $M_1(G + e) > M_1(G)$; (iii) $H(G + e) > H(G)$; (iv) $WW(G + e) < WW(G)$; (v) $\pi(G + e) < \pi(G)$.

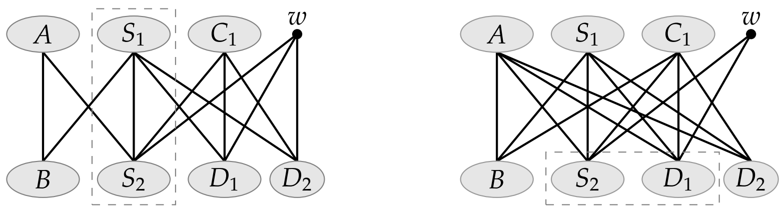

Figure 1: The graphs G and G' in Lemma 2.2.

Lemma 2.2. *Let G be a bipartite graph with n vertices and vertex connectivity s. If S is a minimum vertex cut of G such that $G - S$ has two nontrivial components $G_1 = K_{a,b}$ and $G_2 = K_{c,d}$, then there is a bipartite graph G' with n vertices and vertex connectivity s such that*

(i) $\sigma(G') < \sigma(G)$; (ii) $M_1(G') > M_1(G)$; (iii) $H(G') > H(G)$; (iv) $WW(G') < WW(G)$; (v) $\pi(G') < \pi(G)$.

Proof. Let (X, Y) be the partition of G, $S_1 = S \cap X$, $S_2 = S \cap Y$, $A = X \cap V(G_1)$, $B = Y \cap V(G_1)$, $C = X \cap V(G_2)$ and $D = Y \cap V(G_2)$. Then $|S| = s$, $|A| = a$, $|B| = b$, $|C| = c$ and $|D| = d$. Since S is a minimum vertex cut of G, we have $|S_1| = s_1 \leq \min\{b, d\}$ and $|S_2| = s_2 \leq \min\{a, c\}$. Let $D_1 \subseteq D$ and $D_2 = D - D_1$ such that $|D_1| = s_1$, $w \in C$, $C_1 = C - \{w\}$ and

$$G' = G - \{wv : v \in D_2\} + \{yz : y \in B, z \in C_1\} + \{pq : p \in A, q \in D\}$$

(see Figure 1). Then G' is also a bipartite graph with n vertices and vertex connectivity s, where $S_2 \cup D_1$ is a minimum vertex cut of G'.

(i) For the Merrifield-Simmons index, we have to separately consider the following two cases.

Case 1. $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ (i.e., $s_1 \geq 1$ and $s_2 \geq 1$). Without loss of generality, we may assume that $a = \max\{a, b, c, d\}$. By calculating immediately, we get

$$\sigma(G) = \binom{n}{0} + \binom{n}{1} + \sum_{i=2}^{a+s_1+c} \binom{a+s_1+c}{i} + \sum_{i=2}^{b+s_2+d} \binom{b+s_2+d}{i} + \left(\sum_{i=1}^{a} \binom{a}{i}\right)\left(\sum_{j=1}^{d} \binom{d}{j}\right) + \left(\sum_{i=1}^{b} \binom{b}{i}\right)\left(\sum_{j=1}^{c} \binom{c}{j}\right)$$

and

$$\sigma(G') = \binom{n}{0} + \binom{n}{1} + \sum_{i=2}^{a+s_1+c} \binom{a+s_1+c}{i} + \sum_{i=2}^{b+s_2+d} \binom{b+s_2+d}{i} + \sum_{i=1}^{b+d-s_1} \binom{b+d-s_1}{i}.$$

Then

$$\begin{aligned}
\sigma(G') - \sigma(G) = & \sum_{i=1}^{b+d-s_1} \binom{b+d-s_1}{i} - \left(\sum_{i=1}^{a} \binom{a}{i}\right)\left(\sum_{j=1}^{d} \binom{d}{j}\right) - \left(\sum_{i=1}^{b} \binom{b}{i}\right)\left(\sum_{j=1}^{c} \binom{c}{j}\right) \\
= & 2^{b+d-s_1} - 1 - (2^a - 1)(2^d - 1) - (2^b - 1)(2^c - 1) \\
\leq & 2^{b+d-s_1} - 1 - (2^b - 1)(2^d - 1) - (2^b - 1)(2^c - 1) \qquad (\text{by } a \geq b) \\
\leq & 2^{b+d-s_1} - 1 - 2^{b-1}(2^d + 2^c - 2) \qquad (\text{by } 2^b - 1 \geq 2^{b-1}) \\
= & (2^{b+d-s_1} - 2^{b+d-1}) - 1 - 2^{b-1}(2^c - 2) < 0.
\end{aligned}$$

Case 2. $S_1 = \emptyset$ or $S_2 = \emptyset$, i.e., $s_1 = 0$ or $s_2 = 0$. Without loss of generality, we assume that $s_1 = 0$ and $s_2 = s$. Equivalently, we can denote G by $\overline{K_s}\nabla(K_{a,b} \cup K_{c,d})$ which obtained by joining each vertex of $\overline{K_s}$ to a-part in $K_{a,b}$ and each vertex of c-part in $K_{c,d}$, respectively.

If $c < d$ or $a < b$, then we will prove in the following that $\sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c+1,d-1})) < \sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c,d}))$ or $\sigma(\overline{K_s}\nabla(K_{a+1,b-1} \cup K_{c,d})) < \sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c,d}))$.

Note that $\sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c,d})) = \sigma(K_{a,b})\sigma(K_{c,d}) + \sigma(\overline{K_b})\sigma(\overline{K_d})(\sigma(\overline{K_s}) - 1) = (2^a + 2^b - 1)(2^c + 2^d - 1) + 2^{b+d}(2^s - 1)$, for $c < d$, we have

$$\begin{aligned}
& \sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c+1,d-1})) - \sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c,d})) \\
= & (2^a + 2^b - 1)(2^{c+1} + 2^{d-1} - 1) + 2^{b+d-1}(2^s - 1) - (2^a + 2^b - 1)(2^c + 2^d - 1) - 2^{b+d}(2^s - 1) \\
= & (2^a + 2^b - 1)(2^c - 2^{d-1}) - 2^{b+d-1}(2^s - 1) < 0.
\end{aligned}$$

By the similar way, we can prove that $\sigma(\overline{K_s}\nabla(K_{a+1,b-1} \cup K_{c,d})) < \sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c,d}))$ for $a < b$.

Therefore, for $a < b$ or $c < d$, one can always construct a new bipartite graph $\overline{K_s}\nabla(K_{a,b} \cup K_{c+1,d-1})$ (or $\overline{K_s}\nabla(K_{a+1,b-1} \cup K_{c,d})$ with n vertices and vertex connectivity s such that $\sigma(\overline{K_s}\nabla(K_{a,b} \cup K_{c+1,d-1})) < \sigma(G)$ (or $\sigma(\overline{K_s}\nabla(K_{a+1,b-1} \cup K_{c,d})) < \sigma(G)$). So, in what follows, we assume that $a \geq b \geq 1$ and $c \geq d \geq 1$.

Subcase 2.1. $a \geq b \geq 2$ and $c \geq d \geq 2$.

$$\begin{aligned}\sigma(G') - \sigma(G) =& \sum_{i=1}^{b+d} \binom{b+d}{i} - \left(\sum_{i=1}^{a} \binom{a}{i}\right)\left(\sum_{j=1}^{d} \binom{d}{j}\right) - \left(\sum_{i=1}^{b} \binom{b}{i}\right)\left(\sum_{j=1}^{c} \binom{c}{j}\right)\\ =& 2^{b+d} - 1 - (2^a - 1)(2^d - 1) - (2^b - 1)(2^c - 1)\\ \leq& 2^{b+d} - 1 - (2^b - 1)(2^d - 1) - (2^b - 1)(2^d - 1)\\ =& 2^{b+1} + 2^{d+1} - 2^{b+d} - 3 < 0.\end{aligned}$$

Subcase 2.2. $c > d = 1$ and $a \geq b \geq 1$ (or $a > b = 1$ and $c \geq d \geq 1$).

$$\begin{aligned}\sigma(G') - \sigma(G) =& 2^{b+d} - 1 - (2^a - 1)(2^d - 1) - (2^b - 1)(2^c - 1)\\ =& 2^{b+1} - 2^a - (2^b - 1)(2^c - 1)\\ \leq& 2^{b+1} - 2^{b-1}(2^c - 1) - 2^a\\ =& 2^{b+1} - 2^{b+c-1} + 2^{b-1} - 2^a < 0.\end{aligned}$$

Analogously, one can show that $\sigma(G') - \sigma(G) < 0$ for $a > b = 1$ and $c \geq d \geq 1$.

Subcase 2.3. $a = b = c = d = 1$. Then $s = 1$ and $G = P_5$. G is a spanning subgraph of G', where G' is obtained from $K_{2,2}$ by attaching a pendant edge. So, $\sigma(G') < \sigma(G)$ from Lemma 2.1.

(ii) For the first Zagreb index, we have

$$\begin{aligned}M_1(G') - M_1(G) =& \left(d_{G'}(w)^2 - d_G(w)^2\right) + \sum_{x \in A} \left(d_{G'}(x)^2 - d_G(x)^2\right) + \sum_{x \in B} \left(d_{G'}(x)^2 - d_G(x)^2\right)\\ &+ \sum_{x \in C_1} \left(d_{G'}(x)^2 - d_G(x)^2\right) + \sum_{x \in D_1} \left(d_{G'}(x)^2 - d_G(x)^2\right) + \sum_{x \in D_2} \left(d_{G'}(x)^2 - d_G(x)^2\right)\\ >& \left(d_{G'}(w)^2 - d_G(w)^2\right) + \sum_{x \in A} \left(d_{G'}(x)^2 - d_G(x)^2\right)\\ =& [(s_1 + s_2)^2 - (d + s_2)^2] + a[(b + s_2 + d)^2 - (b + s_2)^2]\\ =& (s_1^2 + 2s_1s_2 - d^2 - 2s_2d) + a(d^2 + 2bd + 2s_2d)\\ \geq& (s_1^2 + 2s_1s_2 - d^2 - 2s_2d) + (d^2 + 2bd + 2s_2d)\\ =& s_1^2 + 2s_1s_2 + 2bd > 0.\end{aligned}$$

(iii) Note that $d_{G'}(x, y) + 2 = d_G(x, y) = 3$ for $x \in A$ and $y \in D$; $d_{G'}(w, y) - 2 = d_G(w, y) = 1$ for $y \in D_2$; $d_{G'}(x, y) + 2 = d_G(x, y) = 3$ for $x \in B$ and $y \in C_1$, and the distances of other pairs of vertices are unchanged. So, we have

$$\begin{aligned}H(G') - H(G) =& \sum_{x \in A, y \in D} \left(\frac{1}{d_{G'}(x, y)} - \frac{1}{d_G(x, y)}\right) + \sum_{x \in B, y \in C_1} \left(\frac{1}{d_{G'}(x, y)} - \frac{1}{d_G(x, y)}\right) + \sum_{y \in D_2} \left(\frac{1}{d_{G'}(w, y)} - \frac{1}{d_G(w, y)}\right)\\ =& \frac{2}{3}ad + \frac{2}{3}b(c - 1) - \frac{2}{3}(d - s_1) = \frac{2}{3}(ad + bc - b - d + s_1) > 0.\end{aligned}$$

(iv) Similarly, one can obtain that

$$WW(G') - WW(G) = 5(d - s_1 - ad - bc + b) < 0.$$

(v) For the multiplicative Wiener index, we have

$$\frac{\pi(G')}{\pi(G)} = \prod_{x \in A, y \in D} \frac{d_{G'}(x, y)}{d_G(x, y)} \prod_{x \in B, y \in C_1} \frac{d_{G'}(x, y)}{d_G(x, y)} \prod_{y \in D_2} \frac{d_{G'}(w, y)}{d_G(w, y)} = \left(\frac{1}{3}\right)^{ad} \left(\frac{1}{3}\right)^{bc-b} \left(\frac{1}{3}\right)^{-d+s_1} = \left(\frac{1}{3}\right)^{ad+bc-b-d+s_1} < 1.$$

And $\pi(G') < \pi(G)$. $\square$

Let $\widetilde{K}(x, n-s-x-1)$ be the graph depicted in Figure 2, where n, x and s are non-negative integers and $0 \le x \le n-2s-1$. It is easy to see that $\widetilde{K}(0, n-s-1)$ is just the complete bipartite graph $K_{s,n-s}$.

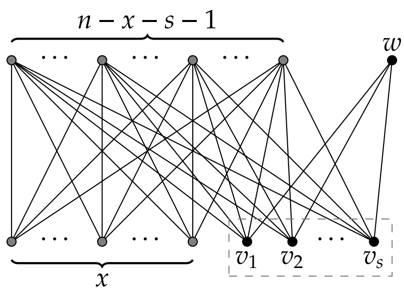

Figure 2: The bipartite graph $\widetilde{K}(x, n-x-s-1)$.

Theorem 2.3. *Let G be a connected bipartite graph on n vertices with vertex connectivity s.*

(i) *If $1 \le s \le \frac{n-1}{2}$ and n is odd, then $\sigma(G) \ge 3 \cdot 2^{\frac{n-1}{2}} + 2^{\frac{n-2s-1}{2}} - 2$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$;*

(ii) *If $1 \le s \le \frac{n}{2}$ and n is even, then $\sigma(G) \ge 2^{\frac{n+2}{2}} + 2^{\frac{n-2s}{2}} - 2$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2s}{2}, \frac{n-2}{2})$.*

Proof. Let G be a bipartite graph with the minimal Merrifield-Simmons index among all bipartite graphs with n vertices and vertex connectivity s. Then there is a non-negative integer x ($0 \le x \le n-2s-1$) such that $G \cong \widetilde{K}(x, n-s-x-1)$ from Proposition 1.1 and Lemmas 2.1 and 2.2.

By the definition of the Merrifield-Simmons index, we have

$$\begin{aligned}\sigma(\widetilde{K}(x, n-s-x-1)) =& \binom{n}{0} + \binom{n}{1} + \sum_{i=2}^{n-s-x} \binom{n-s-x}{i} + \sum_{i=2}^{s+x} \binom{s+x}{i} + \sum_{i=1}^{x} \binom{x}{i} \\ =& 2^{n-s-x} + 2^{s+x} + 2^x - 2 = 2^{n-s} \cdot 2^{-x} + (2^s+1) \cdot 2^x - 2.\end{aligned}$$

Let $f(x) = 2^{n-s} \cdot 2^{-x} + (2^s+1) \cdot 2^x - 2$, then it is easy to get that

$$\min f(x) = \begin{cases} f(\frac{n-2s-1}{2}) = 3 \cdot 2^{\frac{n-1}{2}} + 2^{\frac{n-2s-1}{2}} - 2, & \text{if } n \text{ is odd;} \\ f(\frac{n-2s}{2}) = 2^{\frac{n+2}{2}} + 2^{\frac{n-2s}{2}} - 2, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we have (i) $\sigma(G) \ge f(\frac{n-2s-1}{2}) = 3 \cdot 2^{\frac{n-1}{2}} + 2^{\frac{n-2s-1}{2}} - 2$ for odd n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$; and (ii) $\sigma(G) \ge f(\frac{n-2s}{2}) = 2^{\frac{n+2}{2}} + 2^{\frac{n-2s}{2}} - 2$ for even n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s}{2}, \frac{n-2}{2})$. □

Corollary 2.4. *Let G be a connected bipartite graph on n vertices with edge connectivity r (or minimum degree r).*

(i) *If $1 \le r \le \frac{n-1}{2}$ and n is odd, then $\sigma(G) \ge 3 \cdot 2^{\frac{n-1}{2}} + 2^{\frac{n-2r-1}{2}} - 2$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2})$;*

(ii) *If $1 \le r \le \frac{n}{2}$ and n is even, then $\sigma(G) \ge 2^{\frac{n+2}{2}} + 2^{\frac{n-2r}{2}} - 2$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2r}{2}, \frac{n-2}{2})$.*

Proof. Note that the edge connectivity (or the minimum degree) of $\widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2})$ and $\widetilde{K}(\frac{n-2r}{2}, \frac{n-2}{2})$ is r. Let G be a bipartite graph with the minimal Merrifield-Simmons index among all bipartite graphs with n vertices and edge connectivity r (or minimum degree r). Recall that $\kappa(G) \le \kappa'(G) \le \delta(G)$, then the vertex connectivity of G is at most r. By Theorem 2.3, we have (i) $\sigma(G) \ge \sigma(\widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2}))$ for odd n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2})$; (ii) $\sigma(G) \ge \sigma(\widetilde{K}(\frac{n-2r}{2}, \frac{n-2}{2}))$ for even n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2r}{2}, \frac{n-2}{2})$. □

Theorem 2.5. *Let G be a connected bipartite graph on n vertices with vertex connectivity s.*

(i) *If $\frac{n-2}{2} \le s \le \frac{n}{2}$, then $M_1(G) \le sn(n-s)$ with equality if and only if $K_{s,n-s}$;*

(ii) *If $1 \le s \le \frac{n-3}{2}$ and n is odd, then $M_1(G) \le \frac{n^3-3n^2+3n-1}{4} + sn + s^2$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$;*

(iii) *If $1 \le s \le \frac{n-4}{2}$ and n is even, then $M_1(G) \le \frac{n^3-3n^2+2n}{4} + sn + s^2 + s$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$.*

Proof. Let G be a bipartite graph with the maximum first Zagreb index among all bipartite graphs of order n with vertex connectivity s. By the same way as in the Theorem 2.3, we can confirm that $G \cong \widetilde{K}(x, n-s-x-1)$, and by the definition of the first Zagreb index, we have

$$M_1(\widetilde{K}(x, n-s-x-1)) = g(x) = (1-n)x^2 + (n^2 - 2sn - 2n + 1)x + sn^2 - s^2n.$$

If $\frac{n-2}{2} \le s \le \frac{n}{2}$, and note that x $(0 \le x \le n/2)$ is a non-negative integer, then we have

$$M_1(G) \le g(0) = s(n-s)^2$$

with equality if and only if $G \cong \widetilde{K}(0, n-s-1)$, i.e., $G \cong K_{s,n-s}$.

If $1 \le s < \frac{n-2}{2}$, then it is not difficultly to verify that

$$\max g(x) = \begin{cases} g(\frac{n-2s-1}{2}) = \frac{n^3-3n^2+3n-1}{4} + sn + s^2, & \text{if } n \text{ is odd;} \\ g(\frac{n-2s-2}{2}) = \frac{n^3-3n^2+2n}{4} + sn + s^2 + s, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we have $M_1(G) \le g(\frac{n-2s-1}{2}) = \frac{n^3-3n^2+3n-1}{4} + sn + s^2$ for odd n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$; $M_1(G) \le g(\frac{n-2s-2}{2}) = \frac{n^3-3n^2+2n}{4} + sn + s^2 + s$ for even n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$. □

By the similar argument as in the proof of Corollary 2.4, we can show the following results.

Corollary 2.6. *Let G be a connected bipartite graph on n vertices with edge connectivity r (or minimum degree r).*

(i) *If $\frac{n-2}{2} \le r \le \frac{n}{2}$, then $M_1(G) \le rn(n-r)$ with equality if and only if $K_{r,n-r}$;*

(ii) *If $1 \le r \le \frac{n-3}{2}$ and n is odd, then $M_1(G) \le \frac{n^3-3n^2+3n-1}{4} + rn + r^2$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2})$;*

(iii) *If $1 \le r \le \frac{n-4}{2n}$ and n is even, then $M_1(G) \le \frac{n^3-3n^2+2n}{4} + rn + r^2 + r$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2r-2}{2}, \frac{n}{2})$.*

Theorem 2.7. *Let G be a connected bipartite graph on n vertices with vertex connectivity s.*

(i) *If $1 \le s \le \frac{n-1}{2}$ and n is odd, then $H(G) \le \frac{3n^2}{8} - \frac{7n}{12} + \frac{2s}{3} + \frac{5}{24}$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$;*

(ii) *If $1 \le s \le \frac{n-2}{2}$ and n is even, then $H(G) \le \frac{3n^2}{8} - \frac{7n}{12} + \frac{2s}{3} + \frac{1}{6}$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$.*

Proof. Let G be a bipartite graph with the minimum Harary index among all bipartite graphs of order n with vertex connectivity s. By the same way as in the previous Theorem, we can confirm that $G \cong \widetilde{K}(x, n-s-x-1)$, and by the definition of the Harary index, we have

$$H(\widetilde{K}(x, n-s-x-1)) = \psi(x) = \frac{-6x^2 + (6n - 12s - 8)x + 3n^2 - 3n - 6s^2 + 6sn}{12}.$$

Then, its is easy to see that

$$\max \psi(x) = \begin{cases} \psi(\frac{n-2s-1}{2}) = \frac{3n^2}{8} - \frac{7n}{12} + \frac{2s}{3} + \frac{5}{24}, & \text{if } n \text{ is odd;} \\ \psi(\frac{n-2s-2}{2}) = \frac{3n^2}{8} - \frac{7n}{12} + \frac{2s}{3} + \frac{1}{6}, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we have $H(G) \le \psi(\frac{n-2s-1}{2}) = \frac{3n^2}{8} - \frac{7n}{12} + \frac{2s}{3} + \frac{5}{24}$ for odd n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$. $H(G) \le \psi(\frac{n-2s-2}{2}) = \frac{3n^2}{8} - \frac{7n}{12} + \frac{2s}{3} + \frac{1}{6}$ for even n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$. □

By the similar argument as in the proof of Corollary 2.4, we can show the following results.

Corollary 2.8. *Let G be a connected bipartite graph on n vertices with edge connectivity r (or minimum degree r).*

(i) *If* $1 \le r \le \frac{n-1}{2}$ *and n is odd, then* $H(G) \le \frac{3n^2}{8} - \frac{7n}{12} + \frac{2r}{3} + \frac{5}{24}$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2})$;

(ii) *If* $1 \le r \le \frac{n-2}{2}$ *and n is even, then* $H(G) \le \frac{3n^2}{8} - \frac{7n}{12} + \frac{2r}{3} + \frac{1}{6}$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2r-2}{2}, \frac{n}{2})$.

Theorem 2.9. *Let G be a connected bipartite graph on n vertices with vertex connectivity s.*

(i) *If* $\frac{n-3}{2} \le s \le \frac{n}{2}$, *then* $WW(G) \ge 2s^2 - 2sn + \frac{3}{2}n^2 - \frac{3}{2}n$ *with equality if and only if* $G \cong K_{s,n-s}$;

(ii) *If* $1 \le s \le \frac{n-5}{2}$ *and n is odd, then* $WW(G) \ge n^2 + n - 5s - 3$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2s-3}{2}, \frac{n+1}{2})$;

(iii) *If* $1 \le s \le \frac{n-4}{2}$ *and n is even, then* $WW(G) \ge n^2 + n - 5s - 3$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$.

Proof. Let G be a bipartite graph with the minimum hyper-Wiener index among all bipartite graphs of order n with vertex connectivity s. By the same way as in the previous Theorem, we can confirm that $G \cong \widetilde{K}(x, n-s-x-1)$, and by the definition of the hyper-Wiener index, we have

$$WW(\widetilde{K}(x, n-s-x-1)) = \varphi(x) = 2x^2 + (4s - 2n + 5)x + 2s^2 - 2sn + \frac{3}{2}n^2 - \frac{3}{2}n.$$

For $\frac{n-3}{2} \le s \le \frac{n}{2}$, it is clear to see that $\min \varphi(x) = \varphi(0) = 2s^2 - 2sn + \frac{3}{2}n^2 - \frac{3}{2}n$, and the extremal graph is attained by $K_{s,n-s}$. On the other side, for $1 \le s \le \frac{n-4}{2}$,

$$\min \varphi(x) = \begin{cases} \varphi(\frac{n-2s-3}{2}) = n^2 + n - 5s - 3, & \text{if } n \text{ is odd;} \\ \varphi(\frac{n-2s-2}{2}) = n^2 + n - 5s - 3, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we have $WW(G) \ge n^2 + n - 5s - 3$ for odd n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-3}{2}, \frac{n+1}{2})$; $WW(G) \ge n^2 + n - 5s - 3$ for even n, with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$. □

By the similar argument as in the proof of Corollary 2.4, we can show the following results.

Corollary 2.10. *Let G be a connected bipartite graph on n vertices with edge connectivity r (or minimum degree r).*

(i) *If* $\frac{n-3}{2} \le r \le \frac{n}{2}$, *then* $WW(G) \ge 2r^2 - 2rn + \frac{3}{2}n^2 - \frac{3}{2}n$ *with equality if and only if* $G \cong K_{r,n-r}$;

(ii) *If* $1 \le r \le \frac{n-5}{2}$ *and n is odd, then* $WW(G) \ge n^2 + n - 5r - 3$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2r-3}{2}, \frac{n+1}{2})$;

(iii) *If* $1 \le r \le \frac{n-4}{2}$ *and n is even, then* $WW(G) \ge n^2 + n - 5r - 3$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2r-2}{2}, \frac{n}{2})$.

Theorem 2.11. *Let G be a connected bipartite graph on n vertices with vertex connectivity s.*

(i) *If* $1 \le s \le \frac{n-1}{2}$ *and n is odd, then* $\pi(G) \ge 3^{\frac{n-2s-1}{2}} 2^{\frac{n^2-2n+1}{4}}$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$;

(ii) *If* $1 \le s \le \frac{n-2}{2}$ *and n is even, then* $\pi(G) \ge 3^{\frac{n-2s-2}{2}} 2^{\frac{n^2-2n+4}{4}}$ *with equality if and only if* $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$.

Proof. Let G be a bipartite graph with the minimum multiplicative Wiener index among all bipartite graphs of order n with vertex connectivity s. By the same way as in the previous Theorem, we can confirm that $G \cong \widetilde{K}(x, n-s-x-1)$, and by the definition of the multiplicative Wiener index, we have

$$\pi(\widetilde{K}(x, n-s-x-1)) = \exp\left\{\left[x^2 + \left(2s - n + \frac{\ln 3}{\ln 2}\right)x + s^2 - sn + \frac{1}{2}n^2 - \frac{1}{2}n\right] \cdot \ln 2\right\} = \exp\left\{\phi(x) \cdot \ln 2\right\},$$

where $\phi(x) = x^2 + \left(2s - n + \frac{\ln 3}{\ln 2}\right)x + s^2 - sn + \frac{1}{2}n^2 - \frac{1}{2}n$.

It is easy to check that

$$\min \phi(x) = \begin{cases} \phi(\frac{n-2s-1}{2}) = \frac{(n-2s-1)\cdot \ln 3}{2\ln 2} + \frac{n^2-2n+1}{4}, & \text{if } n \text{ is odd;} \\ \phi(\frac{n-2s-2}{2}) = \frac{(n-2s-2)\cdot \ln 3}{2\ln 2} + \frac{n^2-2n+4}{4}, & \text{if } n \text{ is even.} \end{cases}$$

This gives

$$\pi(G) \geq \begin{cases} \exp\left\{\phi(\frac{n-2s-1}{2}) \cdot \ln 2\right\} = 3^{\frac{n-2s-1}{2}} 2^{\frac{n^2-2n+1}{4}}, & \text{if } n \text{ is odd;} \\ \exp\left\{\phi(\frac{n-2s-2}{2}) \cdot \ln 2\right\} = 3^{\frac{n-2s-2}{2}} 2^{\frac{n^2-2n+4}{4}}, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we can conclude that $\pi(G) \geq 3^{\frac{n-2s-1}{2}} 2^{\frac{n^2-2n+1}{4}}$ for odd n with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-1}{2}, \frac{n-1}{2})$ and $\pi(G) \geq 3^{\frac{n-2s-2}{2}} 2^{\frac{n^2-2n+4}{4}}$ for even n with equality if and only if $G \cong \widetilde{K}(\frac{n-2s-2}{2}, \frac{n}{2})$. $\square$

By the similar argument as in the proof of corollary 2.4, we can show the following results.

Corollary 2.12. *Let G be a connected bipartite graph on n vertices with edge connectivity r (or minimum degree r).*

(i) *If $1 \leq r \leq \frac{n-1}{2}$ and n is odd, then $\pi(G) \geq 3^{\frac{n-2r-1}{2}} 2^{\frac{n^2-2n+1}{4}}$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2r-1}{2}, \frac{n-1}{2})$;*

(ii) *If $1 \leq r \leq \frac{n-2}{2}$ and n is even, then $\pi(G) \geq 3^{\frac{n-2r-2}{2}} 2^{\frac{n^2-2n+4}{4}}$ with equality if and only if $G \cong \widetilde{K}(\frac{n-2r-2}{2}, \frac{n}{2})$.*

Remark. Note that the obtained expressions for the minimum (maximum) values of all the topological indices discussed in this paper can be viewed as monotone decreasing (increasing) functions on s (or r), so the results in Theorems 2.3-2.11 and Corollaries 2.4-2.12 also hold for a connected bipartite graph G on n vertices with vertex connectivity at most s (or edge connectivity at most r, minimum degree at most r).

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