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Packing 1-Plane Hamiltonian Cycles in Complete Geometric Graphs

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Abstract. Counting the number of Hamiltonian cycles that are contained in a geometric graph is **#P**-complete even if the graph is known to be planar. A relaxation for problems in plane geometric graphs is to allow the geometric graphs to be 1-plane, that is, each of its edges is crossed at most once. We consider the following question: For any set P of n points in the plane, how many 1-plane Hamiltonian cycles can be packed into a complete geometric graph K_n? We investigate the problem by taking three different situations of P, namely, when P is in convex position and when P is in wheel configurations position. Finally, for points in general position we prove the lower bound of $k-1$ where $n = 2^k + h$ and $0 \leq h < 2^k$. In all of the situations, we investigate the constructions of the graphs obtained.

1. Introduction

Let P be a set of n points in general position in the plane with no three points being collinear. A *geometric graph* is a graph $G = (P, E)$ that consists of a set of vertices P, which are points in the plane, and a set of edges, E, which are straight-line segments whose endpoints belong to P. A *complete* geometric graph K_n is a geometric graph on a set P of n points that has an edge joining every pair of points in P. Two edges are disjoint if they have no point in common. Two subgraphs are *edge-disjoint* if they do not share any edge.

A geometric graph is said to be *plane* (or non-crossing) if its edges do not cross each other. A geometric graph is said to be *1-plane* if every edge is allowed to have at most one crossing. Note that the terms plane graph and 1-plane graph refer to a geometric object, while to be planar or 1-planar are properties of the underlying abstract graph.

By an *edge packing* of a graph G we mean a set of edge-disjoint subgraphs of G. By an *edge partition* of G we mean an edge packing of G with no edge left over, that is the union of all subgraphs in the packing is equal to G. Dor and Tarsi [11] proved that the problem of partitioning a given graph G is NP-complete.

It is often useful to restrict the subgraphs of G to a certain class or property. Among all subgraphs of K_n, plane spanning trees, plane Hamiltonian cycles or paths, and plane perfect matchings, are of interest [1–3, 8] i.e., one may look for the maximum number of these subgraphs that can be packed into K_n.

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A long-standing open question is to determine if the edges of K_n, where n is even, can be partitioned into $\frac{n}{2}$ plane spanning trees? Bernhart and Kanien [5] give an affirmative answer for the problem when the points are in convex position. Bose et al. [8] proved that every complete geometric graph K_n can be partitioned into at most $n - \sqrt{\frac{n}{12}}$ plane trees. Aichholzer et al. [3] showed that $\Omega(\sqrt{n})$ plane spanning trees can be packed into K_n. Recently, this result has been improved to $\lfloor n/3 \rfloor$ plane spanning trees by Biniaz and García [6]. On another hand Biniaz et al. [7] showed that at least $\lceil \log_2 n \rceil - 1$ plane perfect matchings can be packed into K_n.

A *cycle* is a closed path in which the first and last vertices are the same. A *Hamiltonian cycle* is a cycle in a graph that passes through every vertex exactly once, except for the vertex that is both the beginning and end, which is visited twice. Finding a Hamiltonian cycle in a graph is **NP**-complete even if the graph is known to be planar [14]. Moreover, counting the number of Hamiltonian cycles that are contained in a graph is **#P**-complete even if the graph is known to be planar [17].

Many past researchers have attempted the problem of counting the number of plane Hamiltonian cycles on a given graph, which may not necessarily be edge-disjoint [4, 12, 19–21] and many others.

A relaxation refers to problems in plane geometric graphs which allows them to be 1-plane. In other words, each of its edges is crossed at most once. It is not always possible to finding more than one edge-disjoint plane Hamiltonian cycle for a given set of points. Hence, a relaxation is considered in order to solve it as a 1-plane, rather than a plane problem. Thus, the following problem is considered:

A relaxation for problems in plane geometric graphs is to allow the geometric graphs to be 1-plane. Finding more than one edge-disjoint plane Hamiltonian cycle for a given set of points is not always possible to achieve. Hence, we consider a relaxation on the Hamiltonian cycles from being plane to being 1-plane and we study the following problem:

Problem 1.1. *For any set of n points in the plane, how many 1-plane Hamiltonian cycles can be packed into a complete geometric graph K_n?*

Note that, 1-plane relaxation on Hamiltonian cycle or path have received a considerable amount of attention in geometric graph [16]. In particular, many papers considered the problem of 1-plane Hamiltonian alternating cycle (see [9], [15] and [10]).

For simplicity, we will write 1-PHC to refer to a 1-plane Hamiltonian cycle.

1.1. Headlines and Results

We study the problem of packing 1-PHCs into a complete geometric graph K_n for a given set of n points in the plane. Since a complete graph K_n on n vertices has $n(n-1)/2$ edges and a Hamiltonian cycle has n edges, therefore, the number of edge-disjoint Hamiltonian cycles in K_n cannot exceed $(n-1)/2$.

In Section 2, we show that $\lfloor \frac{n}{3} \rfloor$ is a tight bound for the number of 1-PHCs that can be packed into K_n for any given set in convex position.

In Section 3, we show that for a set of points in regular wheel configuration, $\lfloor \frac{n-1}{3} \rfloor$ edge-disjoint 1-PHCs can be packed into K_n, whereby this bound is tight.

In the latter portion of this paper, Section 4, point sets in general position are considered. We know that for $n \geq 3$, and by a minimum weight Hamiltonian cycle in K_n, a trivial lower bound of 1 is obtained since it is a plane cycle. Furthermore, in Section 4, an algorithm (henceforth, Algorithm A) is presented to draw a 1-PHC for any set of points in general position in the plane. The main findings of this paper prove that there are at least $k-1$ 1-PHCs that can be packed into K_n, where $n = 2^k + h$ and $0 \leq h < 2^k$.

2. 1-PHCs for Point Sets in Convex Position

In this section, we study the problem of packing 1-PHCs on a well-known restricted position of a point set which is the convex position. It will shown that for any point set P in convex position, there are at most

$\lfloor \frac{n}{3} \rfloor$ edge-disjoint 1-PHCs that can be packed into K_n and this bound is tight (Theorem 2.4). Throughout this section, for simplicity, we consider all vertices in counter-clockwise order.

Suppose $P = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ is a set of n points in convex position. Let G be a geometric graph on P. Edges of the form v_iv_{i+1}, $i = 0, 1, 2, \ldots, n-1$, are called the *boundary edges* of G. A non-boundary edge in G is called a *diagonal edge*.

Proposition 2.1. *Let P be a set of n points in convex position in the plane where $n \geq 3$. Suppose C is a 1-PHC on P that has a diagonal edge v_iv_j which divides P into two parts, both including v_i and v_j. Then the following statements hold:*

(1) If any part has an odd number of vertices, then C has at least one boundary edge on this part.

(2) If any part has an even number of vertices, then C has at least two boundary edges on this part.

Proof: Let C be a 1-PHC on P. Assume that C contains the diagonal edge v_iv_j that divides P into two parts P_1 and P_2. Assume that $P_1 = \{v_j, v_{j+1}, ..., v_i\}$ and $|P_1|$ is odd. By induction on $|P_1|$, if $|P_1| = 3$, then $P_1 = \{v_j, v_{j+1}, v_i\}$ and either the boundary edge v_jv_{j+1} or v_iv_{j+1} in C; otherwise, there are two crossings, a contradiction.

Assume that $|P_1| \geq 5$ is odd and the proposition is true when $m < |P_1|$, and m is an odd number. We claim that either $v_{i+1}v_{j+1}$ or $v_{i-1}v_{j-1}$ is an edge in C. To prove our claim, suppose that neither $v_{i+1}v_{j+1}$ nor $v_{i-1}v_{j-1}$ are in C.

Since C is a 1-PHC, then C contains an edge $v_lv_k \notin \{v_{i+1}v_{j+1}, v_{i-1}v_{j-1}\}$ crosses v_iv_j where $k \in P_1 - \{v_i, v_j\}$ and $l \in P_2 - \{v_i, v_j\}$. When $k \notin \{j+1, i-1\}$, then there is $p \in \{j+2, j+3, ..., k-1\}$ or $p \in \{k+1, k+2, ..., i-2\}$ such that the edge v_pv_q, that incident to v_p and belongs to C, crosses either v_iv_j or v_lv_k, a contradiction.

When $k \in \{j+1, i-1\}$, without loss of generality assume that $k = j+1$ and then $l \neq i+1$ (by assumption). Thus, there is $p \in \{j+2, j+3, ..., i-1\}$ or $p \in \{i+1, i+2, ..., l-1\}$ such that the edge v_pv_q, that belongs to C, crosses either v_iv_j or v_lv_k, a contradiction.

Therefore, either $v_{i+1}v_{j+1} \in E(C)$ or $v_{i-1}v_{j-1} \in E(C)$. Without loss of generality, assume that $v_{i+1}v_{j+1} \in E(C)$.

Let C' be a subgraph of C induced by P_1. Since $v_{i+1}v_{j+1}$ not in C', then the degree of v_j, v_{j+1} will be $d(v_j) = d(v_{j+1}) = 1$ in C'.

Let $C_1 = C' \cup \{v_jv_{j+1}\}$. It is clear that C_1 is a 1-PHC on P_1 since v_jv_{j+1} is a boundary edge. Note that v_iv_j and v_jv_{j+1} are boundary edges in C_1 but are not boundary edges in C.

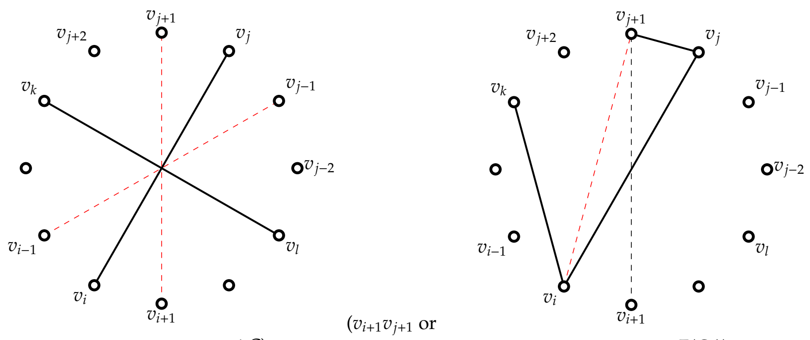
When the boundary edge $v_iv_{i-1} \in E(C_1)$, the claim in the proposition is hold. Hence, assume that the boundary edge $v_iv_{i-1} \notin E(C_1)$. Then there is a diagonal edge $v_iv_k \in E(C_1)$ that divides P_1 into two parts $P_{1,1} = \{v_i, v_{i-1}, ..., v_k\}$ and $P_{1,2} = \{v_i, v_j, v_{j+1}, ..., v_k\}$.

If $k = j+1$, there is a contradiction since C_1 is not a union of disjoint cycles (since $v_iv_jv_{j+1}$ will be cycle). Hence, either $k \in \{j+2, j+4, ..., i-2\}$ or $k \in \{j+3, j+5, ..., i-3\}$. In the case that $k \in \{j+2, j+4, ..., i-2\}$ then $P_{1,1} = \{v_k, v_{k+1}, ..., v_{i-1}, v_i\}$. Then clearly, $|P_{1,1}|$ is odd (since $|P_1|$ is odd and when subtract even number still odd number). By the induction hypothesis, C_1 has at least one boundary edge on $P_{1,1}$. Thus, C has at least one boundary edge on P_1. In thae case that $k \in \{j+3, j+5, ..., i-3\}$, then $P_{1,2} = \{v_i, v_j, v_{j+1}, ..., v_k\}$. Then clearly, $|P_{1,2}|$ is odd.

Now, we assert that $P_{1,2}$ has at least one boundary edge different from v_iv_j and v_jv_{j+1}.

(*) Assume on the contrary that C_1 has the only two boundary edges v_iv_j and v_jv_{j+1} on $P_{1,2}$. Note that $\{v_{j+1}, ..., v_{k-1}\}$ has at least two vertices. However, C_1 matches all the vertices in $\{v_{j+1}, ..., v_{k-1}\}$ with at least two crossings with the edge v_iv_k since C_1 has no boundary edge on $\{v_{j+1}, ..., v_{k-1}\}$; this is a contradiction (since C_1 is a 1-PHC). Thus, C_1 has at least one boundary edge different from v_iv_j and v_jv_{j+1} on $P_{1,2}$. This proves (1).

Assume that $|P_1|$ is even. By induction on $|P_1|$, if $|P_1| = 4$, then $P_1 = \{v_j, v_{j+1}, v_{j+2}, v_i\}$ such that either $v_jv_{j+1}, v_{j+1}v_{j+2} \in E(C)$ or $v_iv_{j+2}, v_{j+1}v_{j+2} \in E(C)$; otherwise, there is a contradiction since C is a 1-PHC. Assume that $|P_1| \geq 6$ is even and the proposition is true when $m < |P_1|$, m is even.



($v_{i+1}v_{j+1}$ or $v_{i-1}v_{j-1} \notin C$) (when $v_iv_{i-1} \notin E(C_1)$)

Figure 1: Illustration cases of Proposition 1

By repeating the same argument in (1), we conclude that either $v_{i+1}v_{j+1} \in E(C)$ or $v_{i-1}v_{j-1} \in E(C)$. Without loss of generality, assume that $v_{i+1}v_{j+1} \in E(C)$.

Let C' be a subgraph of C induced by P_1. It is clear that $d(v_j) = d(v_{j+1}) = 1$ in C'. Let $C_1 = C' \cup \{v_jv_{j+1}\}$. Then C_1 is a 1-PHC on P_1 since v_jv_{j+1} is a boundary edge. Recall that v_iv_j and v_jv_{j+1} are boundary edges in C_1 but are not boundary edges in C.

In the case that C_1 has the boundary edge v_iv_{i-1}, either $v_{i-1}v_{i-2} \in E(C)$ and then the claim in the proposition is true, or $v_{i-1}v_{i-2} \notin E(C)$ and then the diagonal edge $v_{i-1}v_k \in E(C)$ for some $k \in \{j+2, j+3, ..., i-3\}$. Hence, the diagonal $v_{i-1}v_k$ divides P_1 into two parts $P_{1,1} = \{v_k, v_{k+1}, ..., v_{i-2}, v_{i-1}\}$ and $P_{1,2} = \{v_{i-1}, v_i, v_j, v_{j+1}, ..., v_k\}$.

$|P_{1,1}|$, whether odd or even, C has at least one boundary edge on $P_{1,1}$ by part (1) or by induction, respectively.

In the case that C_1 does not have the boundary edge v_iv_{i-1}, then $v_iv_k \in E(C)$ for some $k \in \{j+2, j+3, ..., i-2\}$. Hence, the diagonal edge v_iv_k divides P_1 into two parts $P_{1,1} = \{v_k, v_{k+1}, ..., v_{i-1}, v_i\}$ and $P_{1,2} = \{v_i, v_j, v_{j+1}, ..., v_k\}$.

Now, either both $|P_{1,1}|$ and $|P_{1,2}|$ are odd, then C has at least one boundary edge on $P_{1,1}$ by part (1) and it has at least one boundary edge different from v_iv_j and v_jv_{j+1} on $P_{1,2}$ by argument (*). Or, both $|P_{1,1}|$ and $|P_{1,2}|$ are even. By the induction hypothesis, C_1 has at least two boundary edges on $P_{1,1}$. Thus, C has at least two boundary edges on P_1. This completes the proof. □

As a direct consequence of Proposition 2.1, we have the following corollary.

Corollary 2.2. *Let P be a set of n points in convex position in the plane where $n \geq 3$. Suppose C is a 1-PHC on P. Then the following statements hold:*
(1) If n is even, C has at least two boundary edges.
(2) If n is odd, C has at least three boundary edges.

Proof: Let C be a 1-PHC on a set P of n points. If all edges of C are boundary edges, then the claim in the lemma holds. Thus, assume that C contains a diagonal edge v_iv_j.

If n is even, v_iv_j divides P into two parts, each part having an odd (even) number of vertices. Then by Proposition 2.1, C has at least one boundary edge (two boundary edges) on each part.

If n is odd, v_iv_j divides P into two parts, and one part has an odd number of vertices. By Proposition 2.1, C has at least one boundary edge on this part, while C has at least two boundary edges on the second part, which has an even number of vertices. □

Suppose G is a geometric graph on a set in convex position P that has a diagonal edge v_iv_j. A boundary edge v_kv_{k+1} is called *on the right side* of v_iv_j if $i \leq k < j$ and *on the left side* of v_iv_j if $j \leq k < i$. A diagonal edge is said to *have a boundary edge on each side* if there are two boundary edges, on left and right sides of the diagonal edge. A boundary edge v_kv_{k+1} is called a *single boundary edge in G* if the two boundary edges $v_{k-1}v_k$ and $v_{k+1}v_{k+2}$ are not in G.

Proposition 2.3. *Let P be a set of n points in convex position in the plane where $n \geq 4$. Suppose C is a 1-PHC on P. Then the following statements hold:*

(1) If C has only two boundary edges $\{v_kv_{k+1}\}$ for $k \in \{r, s\}$, then C has the edges v_kv_{k+2} and $v_{k+1}v_{k-1}$.

(2) If C has only three boundary edges, then C has at least one single boundary edge v_rv_{r+1} with the edges v_rv_{r+2} and $v_{r+1}v_{r-1}$.

Proof: Let C be a 1-PHC on P containing only two boundary edges v_kv_{k+1} for $k \in \{r, s\}$. Assume on the contrary that at least one of the two edges $\{v_kv_{k+2}, v_{k+1}v_{k-1}\}$ is not in C for some $k \in \{r, s\}$.

Without loss of generality, assume that $v_rv_{r+2} \notin E(C)$. Then v_rv_i and $v_{r+1}v_j \in E(C)$ for some $r+3 \leq i \leq r-2$ and $r+3 \leq j \leq r-1$.

If v_rv_{r+1} and v_sv_{s+1} are in consecutive order, then all the remaining edges of C are diagonal edges where v_rv_{r+1} and v_sv_{s+1} are on the same side of each one. Hence, the other side of any diagonal edge does not contain any boundary edge, which contradicts Proposition 2.1. Thus v_kv_{k+1} is a single boundary edge for each $k \in \{r, s\}$ and then $i \neq r-1$ and $j \neq r+2$.

If v_rv_i and $v_{r+1}v_j$ are crossing, then there is a vertex v_t where $r+2 \leq t < i$ such that at least one of the two edges incident to v_t in C crosses v_rv_i, which is a contradiction since C is a 1-PHC.

If v_rv_i and $v_{r+1}v_j$ are not crossing, then by Proposition 2.1, C has at least one boundary edge on the left side of v_rv_i and at least another boundary edge on the right side of $v_{r+1}v_j$ (both different from v_rv_{r+1}), which is a contradiction since C has only two boundary edges. This proves (1).

Let C contain three boundary edges v_kv_{k+1} for $k \in \{r, s, t\}$. Suppose that no single boundary edge is in C. That is, the boundary edges in C are in consecutive order. But all the remaining edges of C are diagonal edges where v_kv_{k+1} for each $k \in \{r, s, t\}$ are on the same side of each one. Hence, the other side of any diagonal edge does not contain any boundary edge, which contradicts Proposition 2.1. Thus C has at least one single boundary edge.

Assume on the contrary that, if v_kv_{k+1} is a single boundary edge in C for some $k \in \{r, s, t\}$, then either one of three cases follow: v_kv_{k+2} is not in C, $v_{k+1}v_{k-1}$ is not in C, or both are not in C.

Without loss of generality, assume that v_rv_{r+1} is a single boundary edge in C and $v_rv_{r+2} \notin E(C)$. Then v_rv_i and $v_{r+1}v_j$ are in C, where $r+3 \leq i < r-1$ and $r+2 < j \leq r-2$.

If v_rv_i and $v_{r+1}v_j$ are crossing, then there is a vertex v_t where $r+2 \leq t < j$ such that at least one of the two edges incident to v_t in C crosses v_rv_i, which is a contradiction since C is a 1-PHC.

If v_rv_i and $v_{r+1}v_j$ are not crossing, then by Proposition 2.1, C has at least one boundary edge on the set $\{v_{i+1}, v_{i+2}, \ldots, v_r\}$ and at least one boundary edge on the set $\{v_{r+1}, v_{r+2}, \ldots, v_j\}$; otherwise, there is a contradiction, since C has only three boundary edges.

This implies that there is a single boundary edge v_sv_{s+1} where $i \leq s < r-1$ such that either v_sv_{s+2} or $v_{s+1}v_{s-1}$ is not in C (by assumption). Without loss of generality, assume that $v_sv_{s+2} \notin E(C)$. Let v_sv_p and $v_{s+1}v_q$ is in C.

In the case that $p \notin \{j, j+1, ..., s-1\}$, v_sv_p crosses both v_rv_i and $v_{r+1}v_j$, which is a contradiction since C is a 1-PHC. Then by Proposition 2.1, C has at least one boundary edge on a vertex set $\{v_j, v_{j+1}, \ldots, v_{s-1}\}$, which is a contradiction since C has only three boundary edges. This completes the proof. □

Theorem 2.4. *Let P be a set of n points in convex position on the plane where $n \geq 3$. Then there exist k edge-disjoint 1-PHCs $C_1, C_2, \ldots, C_k$ on P that can be packed into K_n where $k \leq \lfloor \frac{n}{3} \rfloor$.*

Proof: (1) Let $n = 2m$, which is even. Suppose $P = \{v_0, v_1, ..., v_{n-1}\}$ is a set of n points in convex position on the plane. By Lemma 2.2, every 1-PHC on P contains at least two boundary edges. On the other hand, P has n boundary edges; that is, the number of 1-PHCs does not exceed $n/2$.

We claim that if C_i and C_j are two edge-disjoint 1-PHCs each having only two boundary edges, then any boundary edge of C_i can not be in consecutive order with a boundary edge of C_j. To prove the claim, assume on the contrary that $v_r v_{r+1} \in E(C_i)$ and $v_{r+1} v_{r+2} \in E(C_j)$. By Proposition 2.3, the edge $v_r v_{r+2} \in E(C_i) \cap E(C_j)$, which is a contradiction since C_i and C_j are edge-disjoint 1-PHCs ($E(C_i) \cap E(C_j) = \phi$, where $i \neq j$).

Note that a single boundary edge in C_i can be adjacent to any two consecutive boundary edges in C_j where $i \neq j$; that is, $C_i \cup C_j$ can have three boundary edges in consecutive order. Therefore, the number of 1-PHCs that can be packed into K_n is at most $\lfloor \frac{n}{3} \rfloor$, making the bound tight.

Now, it will be shown how to pack $\lfloor \frac{n}{3} \rfloor$ 1-PHCs into K_n. To ensure that the boundary edges of all cycles $\{C_1, C_2, \dots, C_k\}$ are in consecutive order, let $\{C_1, C_2, \dots, C_k\}$ be divided into two sets A and B where each 1-PHC in A has only two boundary edges, and each 1-PHC in B has only four boundary edges (which are two couples of boundary edges and each couple has two boundary edges in consecutive order). The boundary edges are arranged with the property that a single boundary edge in $C \in A$ is in consecutive order with a couple of boundary edges in $C' \in B$. This property is depicted in Figure 2(a).

For each $i = 0, 1, \dots, \lfloor \frac{m}{3} \rfloor - 1$ and $j = 0, 1, \dots, \lfloor \frac{m}{3} \rfloor - 1$ where $m \geq 2$. Let $C_i = v_{3i} v_{3i+1} v_{3i-1} v_{3i+3} v_{3i-3} \dots v_{3i-m} v_{2i+(m+1)}, v_{3i-m-1} v_{2i+(m+3)} \dots v_{3i-(2m-2)} v_{3i}$, and let $C_j = v_{3j+2} v_{3j+1} v_{3j+4} v_{3j-1} \dots v_{3j+m+1} v_{3j-m+2}, v_{3j+m+3} v_{3j-m}, \dots v_{3j+(2m)} v_{3j-(2m-3)} v_{3j+2}$.

Here the operations on the subscripts are reduced modulo $n-1$.

Note that $C_3 \in A$ is obtained from $C_1 \in A$ by rotating the edges of C_1 where v_i take the place of v_{i+3}. Hence, $C_1, C_3 \in A$ are edge-disjoint 1-PHCs. Similarly, $C_4 \in B$ is obtained from $C_2 \in B$ by rotating the edges of $C_2 \in B$ where v_i take the place of v_{i+3}. Hence, $C_2, C_4 \in A$ are edge-disjoint 1-PHCs. Furthermore, each cycle in A or B matches all vertices in P, that is each cycle is a Hamiltonian cycle.

(2) Now, let $n = 2m + 1$, which is odd. By Lemma 2.2, every 1-PHC in P contains at least three boundary edges. On the other hand, P has n boundary edges. Therefore, the number of 1-PHCs that can be packed into K_n is at most $\lfloor \frac{n}{3} \rfloor$, making the bound tight.

Now, will be shown how to pack $\lfloor \frac{n}{3} \rfloor$ 1-PHCs into K_n. To ensure that the boundary edges of all cycles $\{C_1, C_2, \dots, C_k\}$ are in consecutive order, let each 1-PHC have two boundary edges in consecutive ordered and one single boundary edge. The boundary edges are arranged with the property that a single boundary edge in C is in consecutive order with two consecutive boundary edges in C' and vice versa. This property is depicted in Figure 2(b).

For each $i = 0, 1, \dots, \lfloor \frac{n}{3} \rfloor - 1$. Let $C_i = v_{(m+2)i} v_{(m+2)i+1} v_{(m+2)i-1} v_{(m+2)i+3} v_{(m+2)i-3} \dots v_{(m+2)i-(2m-1)} v_{(m+2)i}$.

Here the operations on the subscripts are reduced modulo $n-1$.

Note that C_3 is obtained from C_1 by rotating the edges of C_1 where v_i take the place of v_{i+3}. Hence, C_1, C_3 are edge-disjoint 1-PHCs. Similarly, C_4 is obtained from C_2 by rotating the edges of C_2 where v_i take the place of v_{i+3}. Hence, C_2, C_4 are edge-disjoint 1-PHCs. Furthermore, each cycle matches all vertices in P, that is each cycle is a Hamiltonian cycle. □

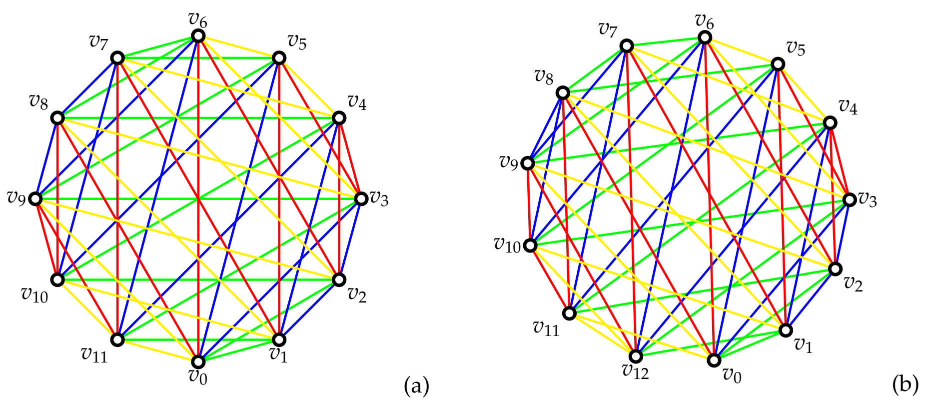


Figure 2: 1-PHCs on point sets in convex position: (a) $n = 12$ and (b) $n = 13$.

A vertex is pendent if it is of one degree. In the next section, the following additional result will be required.

Lemma 2.5. *Let P be a set of n points in convex position in the plane where $n \geq 3$. Suppose T is a 1-plane Hamiltonian path (1-PHP) on P with two pendent vertices v_i and v_j. Then the following statements hold:*

(1) T has at least one boundary edge when $|i-j| = 1$, and each diagonal edge in T has at least one boundary edge on one side.

(2) T has at least two boundary edges when $|i-j| > 1$, and each diagonal edge in T has at least one boundary edge on each side.

Proof: Let T be a 1-PHP on P with two pendent vertices v_i and v_j. Assume that $|i-j| = 1$, and let $C = T \cup \{v_iv_j\}$. Then C is a 1-PHC since v_iv_j is a boundary edge. By Lemma 2.2, C has at least two boundary edges when n is even and three boundary edges when n is odd. Note that v_iv_j is a boundary edge in C since $|i-j| = 1$. Observe that by Proposition 2.1, each diagonal edge in C has at least one boundary edge on each side. This proves (1).

Assume that $|i-j| > 1$. By induction on n, if $n = 4$, the statement is trivially true. Assume that $n \geq 5$ and the lemma is true when $m < n$.

In the case that all the edges in T are boundary edges, the statement holds. Hence, assume that there is a diagonal edge $v_rv_s \in E(T)$.

Let P_1 and P_2 be two sets of points of P on each side of v_rv_s that both include v_r and v_s. Let T_1 and T_2 be the edges of T on P_1 and P_2, respectively. It is clear that T_i, $i = 1, 2$ is a 1-PHP on P_i and P_i is in convex position.

By the induction hypothesis, T_i has at least two boundary edges. Note that v_rv_s is a boundary edge in T_i, but it is not boundary edge in T. This proves (2). □

3. Points in Wheel Configuration

In this section, we turn to another special configuration. We say a set P of n points, is in *regular wheel configuration* if $n-1$ of its points are regularly spaced on a circle $C(P)$ with one point x in the center of $C(P)$. The x is termed the *center* of P. For simplicity, we consider all vertices of $C(P)$ in counter-clockwise order.

Note that when n is even, that is $|C(P)|$ (the order of $C(P)$) is odd, and since $C(P)$ is regularly spaced on a circle, a line passing through any two points in $C(P)$ does not contain x. On the other hand, when n is odd, that is $|C(P)|$ is even, and by regularity of $C(P)$, x lies on a line that passes through any two points v_i and v_j in $C(P)$ such that $|i-j| = \frac{n-1}{2}$ which contradicts the general position assumption. Hence, the case is considered when n is even.

An edge of the form xv is called a *radial edge*, and every 1-PHC on P contains exactly two radial edges.

Lemma 3.1. *Let P be a set of n points in regular wheel configuration in the plane where $n \geq 4$, is even. Suppose C is a 1-PHC on P. Then C has at least two boundary edges and each diagonal edge in C has at least one boundary edge on each side.*

Proof: The lemma is trivially true when n=4. Hence assume $n \geq 6$. Suppose x is the center of P and $v_0v_1 \cdots v_{n-2}v_0$ is the circle $C(P)$. Assume that C is a 1-PHC where xv_i and xv_j are two radial edges of C. It is clear that $C - \{v_ix, v_jx\}$ is a 1-PHP on $C(P)$. By Lemma 2.5, $C - \{v_ix, v_jx\}$ has at least two boundary edges except the case when $|i-j| = 0$, which possibly has only one boundary edge. Furthermore, each diagonal edge in $C - \{v_ix, v_jx\}$ has at least one boundary edge on each side.

Assume that $j = i+1$. Suppose that $C - \{v_ix, v_jx\}$ has only one boundary edge e. Let $C' = C - \{v_ix, v_jx\} \cup \{v_iv_j\}$. Then C' is a 1-PHC on $C(P)$ and has only two boundary edges v_iv_j and e. By Proposition 2.3, C' has the crossing edges v_iv_{i+2} and $v_{i+1}v_{i-1}$; that is, v_iv_{i+2} and $v_{i+1}v_{i-1}$ are edges in C. Then the radial edges xv_i cross v_jv_{i-1} in C and xv_j crosses v_iv_{j+1} in C (since $n \geq 6$), which is a contradiction since C is a 1-PHC. Thus C has at least two boundary edges. □

Proposition 3.2. *Let P be a set of n points in regular wheel configuration in the plane where $n \geq 8$, is even. Suppose C is a 1-PHC on P. If C has only two boundary edges $v_k v_{k+1}$ for $k \in \{r, s\}$. Then C has the edges $v_k v_{k+2}$ and $v_{k+1} v_{k-1}$.*

Proof: It is not difficult to verify that the proposition is not true when $n = 6$. Hence, assume $n \geq 8$. Suppose x is the center of P and $v_0 v_1 \cdots v_{n-2} v_0$ is in the circle $C(P)$. Let C be a 1-PHC on P that contains only two boundary edges $v_k v_{k+1}$ for $k \in \{r, s\}$.

It is clear that $v_k v_{k+1}$ for $k \in \{r, s\}$ are single boundary edges otherwise, by Lemma 3.1 any diagonal edge in C where $v_r v_{r+1}$ and $v_s v_{s+1}$ on one side has a third boundary edge on the other sides, a contradiction. Before proceeding, the following observation shall be noted.

(O1) If $v_p v_q$ any edge in C, then $r + 1 \leq p \leq s$ and $s + 1 \leq q \leq r$, otherwise, by Lemma 3.1 C has at least three boundary edge, a contradiction.

Assume on the contrary that at least one of the two edges $\{v_k v_{k+2}, v_{k+1} v_{k-1}\}$ is not in C for some $k \in \{r, s\}$. Without loss of generality, assume that $v_r v_{r+2} \notin E(C)$. Suppose that $v_r v_i$ and $v_{r+1} v_j \in E(C)$, where $i \neq j$, we consider the following two cases.

Case (1): Suppose that $x \notin \{v_i, v_j\}$. By (O1) $r + 3 \leq i \leq s$ and $s + 1 \leq j \leq r - 1$. Hence, $v_r v_i$ and $v_{r+1} v_j$ are crossing. Thus, there is a vertex v_t where $r + 2 \leq t < i$ such that at least one of the two edges matches v_t in C crosses $v_r v_i$, a contradiction since C is a 1-PHC.

Case (2): Suppose that either $v_i = x$ or $v_j = x$. Without loss of generality, assume that $v_i = x$. Let $v_{r-1} v_l$ and $v_{r+1} v_j$ be two edges in C. By (O1) $s + 1 \leq j \leq r - 1$ and $r + 2 \leq l \leq s$ where $v_s v_{s+1}$ is the second boundary edge. By regularity of $C(P)$, $v_r x$ and $v_{r+1} v_j$ are crossing, then $j = r - 1$; otherwise, there is a vertex v_t where $j + 1 \leq t < r$ such that at least one of the two edges matches v_t into C crosses $v_{r+1} v_j$, in both cases there is a contradiction. Thus $v_{r+1} v_{r-1}$ in C.

Now, if $l > r + 3$. Then the edges in C that incident on v_{r+2} and v_{r+3} crosses $v_{r-1} v_l$, a contradiction since C is a 1-PHC. By regularity of $C(P)$ and $n \geq 8$ if $l = r + 3$, then $v_{r-1} v_l$ crosses $v_r x$ which is a contradiction since $v_{r+1} v_{r-1} \in E(C)$ and crosses $v_r x$. This completes the proof. □

We now present the main result of this section.

Theorem 3.3. *Let P be a set of n points in regular wheel configuration in the plane where $n \geq 10$, is even. Then there exist k edges-disjoint 1-PHCs $C_1, C_2, \ldots, C_k$ on P that can be packed into K_n where $k \leq \lfloor \frac{n-1}{3} \rfloor$.*

Proof: Suppose x is the center of P and $v_0 v_1 \cdots v_{n-2} v_0$ is the circle $C(P)$. By Lemma 3.1, every 1-PHC in P contains at least two boundary edges. On the other hand, $C(P)$ has $n - 1$ boundary edges; that is, the number of 1-PHCs does not exceed $n - 1/2$.

By Proposition 3.2, if C_i and C_j are two edge-disjoint 1-PHCs each having only two boundary edges, then any boundary edge of C_i can not be in consecutive order with a boundary edge of C_j.

Note that a single boundary edge in C_i can be adjacent to two consecutive boundary edges in C_j where $i \neq j$; that is, $C_i \cup C_j$ can have three boundary edges in consecutive order. Therefore, the number of 1-PHCs that can be packed into K_n is at most $\lfloor \frac{n-1}{3} \rfloor$ and this bound is tight.

Now, it will be shown how to pack $\lfloor \frac{n-1}{3} \rfloor$ 1-PHCs into K_n. To ensure that the boundary edges of all cycles $\{C_1, C_2, \ldots, C_k\}$ are in consecutive order. Then each 1-PHC should have three boundary edges (where two of them are in consecutive order) with the property that a single boundary edge in C_i is adjacent to the two consecutive boundary edges in C_j and vice versa. This property is depicted in Figure 3. For each $i = 0, 1, \ldots, \lfloor \frac{n-1}{3} \rfloor - 1$ and $r = \lfloor \frac{m+1}{2} \rfloor$. Let $C_i = v_{(m+1)i} v_{(m+1)i+1} v_{(m+1)i-1} v_{(m+1)i+3} v_{(m+1)i-3} \cdots v_{(m+1)i+r}\, x\, v_{(m+1)i+5} v_{(m+1)i-5} \cdots v_{(m+1)i-(2m-3)} v_{(m+1)i}$. Here the operations on the subscripts are reduced modulo $2n - 1$. □

4. 1-PHCs on Point Sets in General Position

In this section, a set P of n points are considered in general position in the plane i.e., no three points are collinear. For $n = 2^k + h$ where $0 \leq h < 2^k$, it will be shown that there are at least $k - 1$ edge-disjoint 1-PHCs

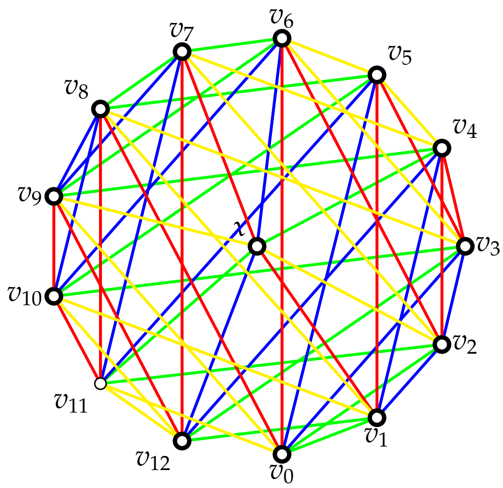


Figure 3: 1-PHCs on a set of points in regular wheel configuration, $n = 14$.

on P (Theorem 4.4). For this purpose, some ingredients are presented that will be used to prove the main result in this section.

4.1. Bisect Lines for a Set of Points

Let P be a set of m points in general position in the plane. A line l is said to *bisect* a set P if both open half spaces defined by l contain precisely $\frac{m}{2}$ points. It is no loss of generality to assume m is odd since otherwise, any point v may be removed and any line that bisects $P - \{w\}$ also bisects P.

Let P_1 and P_2 be two point sets in the plane. If H_1 and H_2 are two convex polygons containing P_1 and P_2 respectively, then it is said that H_1 and H_2 are disjoint if there is a line that separates them. Moreover, if P is a disjoint union of two point sets P_1 and P_2, the ham-sandwich cut theorem guarantees the existence of a line that simultaneously bisects P_1 and P_2 (see for example [13],[18]).

Lemma 4.1. *Let P be a set of n points in the general position where $n \geq 2$. Suppose there is a line separating a given set $\{v_1, v_2, \ldots, v_k\}$ from P where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Then there is a line that bisects P into P_1 and P_2 with $\{v_1, v_2, \ldots, v_k\} \subset P_1$.*

Proof: If $k = \lfloor \frac{n}{2} \rfloor$, then the lemma is trivially true. Hence, assume that $k < \lfloor \frac{n}{2} \rfloor$.

Let $u_1 \in P - \{v_1, v_2, \ldots, v_k\}$ be such that all points in $P - \{v_1, v_2, \ldots, v_k\}$ are on one side of the line ℓ_1 passing through u_1 and w_1 for some $w_1 \in \{v_1, v_2, \ldots, v_k\}$ and let $P_1 = \{u_1, v_1, v_2, \ldots, v_k\}$. Furthermore, let L_1 be a line parallel to ℓ_1 such that all points in $P - \{u_1, v_1, v_2, \ldots, v_k\}$ are on one side of L_1.

If $|P_1| = \lfloor \frac{n}{2} \rfloor$, then the proof is complete. Otherwise, repeat the argument with $u_2 \in P - \{u_1, v_1, v_2, \ldots, v_k\}$ and $w_2 \in \{u_1, v_1, v_2, \ldots, v_k\}$ so that all points in $P - \{u_1, v_1, v_2, \ldots, v_k\}$ are on one side of the line ℓ_2 passing through u_2w_2, and let $P_1 = \{u_1, u_2, v_1, v_2, \ldots, v_k\}$ with the line L_2 similarly defined. By repeating the argument where necessary, the conclusion of the lemma is reached. □

Lemma 4.2. *Let L be a line that bisects a set P of m points in the general position into P_1 and P_2 where $m \geq 6$, and let $l_1^{\perp}$ be a line perpendicular to L and all points in P are on one side of $l^{\perp}$. Suppose $\{u, v\} \subset P_1$ is a given set such that less than half points of P_1 are between $l_1^{\perp}$ and a line $l_2^{\perp}$, which is perpendicular to L, passing through any point in $\{v, w\}$. Then there is a line that bisects P_1 and P_2 into $P_{i,j}$, for each $i = 1, 2$ with $j = 1, 2$ and $\{v, w\} \subseteq P_{1,k}$ for some $k \in \{1, 2\}$.*

Proof: The lemma is trivially true if $m = 5$ with $P_1 = \{v, w\}$. Hence, assume that $m \geq 6$. By hum-sandwich cut theorem, there is a line l_h that bisects P_1 and P_2 in the plane into sets $\{P_{1,1}, P_{1,2}, P_{2,1}, P_{2,2}\}$. It is no loss of generality to assume $P_{1,2}$ is between l_h and $l^{\perp}$. Assume on the contrary that $\{v, w\} \nsubseteq P_{1,j}$ for each $j = 1, 2$. Then all the points of $P_{1,2}$ are between $l^{\perp}$ and the line, perpendicular to L, a contradiction. Thus $\{v, w\} \subseteq P_{1,2}$. □

4.2. Drawing a 1-PHC on a Set of Points

We shall give a description of an algorithm for drawing a 1-PHC on a set of points in general position in the plane. In what follows, let $l(v_1, v_2)$ be a line passing through the two points v_1 and v_2.

Algorithm (A)

1. Find a line l that bisects P into P_1 and P_2 where either $|P_1| = |P_2|$ or $|P_1| = |P_2| + 1$.
2. Find a line $l^{\perp}$ such that $l^{\perp}$ is perpendicular to l and all points in P are on one side of $l^{\perp}$.
3. Find $CH(P_i)$, the convex hull of P_i, for each $i = 1, 2$ and select $v_i \in CH(P_i)$ such that all the points in $P_1 \cup P_2 - \{v_1, v_2\}$ are between $l^{\perp}$ and the line $l(v_1, v_2)$. Let v_1v_2 be an edge in C and let $v_i^* = v_i$, for each $i = 1, 2$.
4. If $P_i = \{v_i^*\}$ for $i = 1, 2$, let $v_1^*v_2^*$ be an edge in C and Stop. If $P_2 = \{v_2^*\}$ and $P_1 = \{v_1^*, w\}$ let v_1^*w and v_2^*w be edges in C and Stop. Otherwise, let $P_i = P_i - \{v_i^*\}$ for each $i = 1, 2$.
5. Find $CH(P_i)$, for $i = 1, 2$ and select $v_i \in CH(P_i)$, $i = 1, 2$, be such that all the points in $P_1 \cup P_2 - \{v_1^*, v_2^*\}$ are between $l^{\perp}$ and the line $l(v_1, v_2)$.
6. If no point of $\{v_1^*, v_2^*\}$, is between $l^{\perp}$ and the line $l(v_1, v_2)$. Let $v_1^*v_2$ and $v_2^*v_1$ be edges in C. Repeat Step (4) with v_i taking the place of v_i^* for each $i = 1, 2$.
7. For some $i \in \{1, 2\}$, if v_{3-i}^* is not between $l^{\perp}$ and the line $l(v_i, v_i^*)$ and all points in $P_1 \cup P_2 - \{v_{3-i}^*\}$ are between $l^{\perp}$ and the line $l(v_i, v_i^*)$, let $v_iv_{3-i}^*$ and v_iv_{3-i} be two edges in C.
8. If $\{v_iv_{3-i}^*, v_iv_{3-i}\} \subset E(C)$, let $P_i = P_i - \{v_i\}$ and repeat Step (4) with v_{3-i} taking the place of v_{3-i}^*.

The edge v_1^*w in Step (4) is termed a "stone" and shall be denoted by $st(v, w)$.

Proof: From Step (1), suppose *(i)* $|P_1| = |P_2|$. By Step (3), $v_1v_2 \in E(C)$, $v_1^* = v_1$ and $v_2^* = v_2$. By Step (4), $P_i = P_i - \{v_i^*\}$ for each $i = 1, 2$. By Step (5), $v_i \in CH(P_i)$, $i = 1, 2$.

By Step (6), we have

Case (1): No point of $\{v_1^*, v_2^*\}$, is between $l^{\perp}$ and the line $l(v_1, v_2)$. Then $v_1^*v_2, v_2^*v_1 \in E(C)$. It is clear that $v_1^*v_2, v_2^*v_1$ are cross edges (from above hypothesis). Moreover, by repeating the same Steps (4) into (6) the resulting graph is a 1-PHC.

Case (2): There is a point of $\{v_1^*, v_2^*\}$, is between $l^{\perp}$ and the line $l(v_1, v_2)$. Without loss of generality assume that v_1^*, is between $l^{\perp}$ and the line $l(v_1, v_2)$. Clearly, $v_2^*v_1$ and v_1v_2 are non crossing edges. Moreover, $v_2^*v_1$ and v_1v_2 do not cross any edge in C. Therefore, by repeating the same Steps (4) into (7) the resulting graph is a 1-PHC.

(ii) $|P_1| = |P_2| + 1$. Steps from (1) into (8) guarantee the existence of 1-PHC on $P - \{w\}$.

From Step (4) the last two points $v_i^* \in CH(P_i)$, $i = 1, 2$ satisfying $w \in P_1$ are between $l^{\perp}$ and the line joining v_1^* and v_2^*. Hence the edge v_1^*w and v_2^*w do not crosse any edge in path on $P - \{w\}$. □

A 1-PHC obtained by Algorithm (A) which contains a stone, is depicted in Figure 4.

4.3. A Joining Between Two 1-PHCs

In this section, it will be shown how to extract a 1-PHC by joining two edge-disjoint 1-PHCs. Let $P(1)$ and $P(2)$ be two disjoint point sets in general position in the plane. Suppose $C(1)$ and $C(2)$ are two edge-disjoint 1-PHCs on $P(1)$ and $P(2)$, respectively.

The edges $u_1u_2 \in E(C(1))$ and $v_1v_2 \in E(C(2))$ are called *joining edges* of $C(1)$ and $C(2)$ if the graph resulting from removing them and adding the edges u_1v_1 and u_2v_2 (or u_1v_2 and u_2v_1) still an edge-disjoint 1-PHC on $P_1 \cup P_2$. The edges u_1v_1 and u_2v_2 (or u_1v_2 and u_2v_1) are termed a *connection edges* of $C(1)$ and $C(2)$.

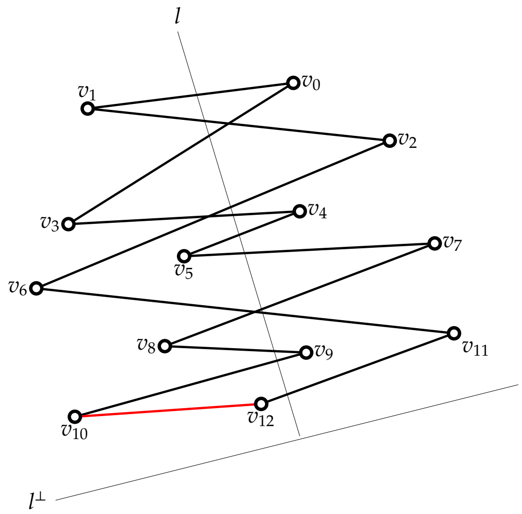


Figure 4: An example of Algorithm A**.**

Suppose that there are two crossing edges u_1u_2 and u_3u_4 in $C(1)$ such that the graph resulting from removing them and adding two non-crossing edges u_1u_3 and u_2u_4 still a 1-PHC. Then the edges u_1u_3 and u_2u_4 are termed *created* edges in $C(1)$. Two joining edges of $C(1)$ and $C(2)$ are called *created joining* edges if at least one of them is a created edge.

Lemma 4.3. *Let P be a set of n points in general position in the plane where $n \geq 8$, and let C be a 1-PHC on P, and let P be bisect into two disjoint sets P_1 and P_2. Suppose $C(1)$ and $C(2)$ are two 1-PHCs on P_1 and P_2, respectively, where $C(1)$ and $C(2)$ have no joining edges. Then $C(1)$ and $C(2)$ can be joined by two joining created edges.*

Proof: Let C be a 1-PHC on P. In so doing, the set P has been split into P_1 and P_2. Assume that $C(1)$ and $C(2)$ are two 1-PHCs on P_1 and P_2, respectively since $n \geq 8$, and let $C(1)$ and $C(2)$ have no joining edges.

Case (1): When $C(i)$ has at least one crossing for each $i = 1, 2$ (since $n \geq 8$ and $|C(1)| \geq 4$ and $|C(2)| \geq 4$). Let $\{u_1u_2, u_3u_4\}$ and $\{v_1v_2, v_3v_4\}$ be the two crossing edges in $C(1)$ and $C(2)$, respectively. By removing the crossing edges in $C(1)$ and $C(2)$ and adding the non-crossing edges $\{u_1u_4, u_2u_3\}$ and $\{v_1v_4, v_2v_3\}$ in $C(1)$ and $C(2)$, respectively we obtain two created edges in each of $C(1)$ and $C(2)$. Chose $e_1 \in \{u_1u_4, u_2u_3\}$ and $e_2 \in \{v_1v_4, v_2v_3\}$ such that no points between them. Without loss of generality assume that no points between the two created edges u_1u_4 and v_1v_4.

Note that at least one edge in a set $A = \{u_1v_1, u_1v_4, u_4v_1, u_4v_4\}$ is not in C since A is 4-cycle graph and $|V(C)| \geq 8$ is not union of two cycles. This means

(1) If C has only three edges of A. Then there is an edge $u'v' \in A \cap E(C)$ such that $d_C(u') = d_C(v') = 2$. Thus there are two joining edges one in original $C(1)$ incident on u' and another in original $C(2)$ incident on v', a contradiction (since $C(1)$ and $C(2)$ have no joining edges).

(2) If C has only two edges $\{uv, u'v'\} \subset A$. Then (i) $\{uv, u'v'\}$ share no vertex, and hence u_1u_4 and v_1v_4 are joining created edges since $A - \{uv, u'v'\}$ are connection edges. (ii) $\{uv, u'v'\}$ share on a vertex. It is no loss of generality to assume that $u = u' = u_1$. That is, $\{u_1v_1, u_1v_4\} \subset A$, then the created edge u_1v_1 and the edge incident on u_1 such as u_1u^* for some $u^* \notin A$ are created joining edges since $A - \{u_1v_1, u_1v_4\}$ is not in C and at most one of the two edges of u^*v_1, u^*v_4 is not in C since $A - \{u_1v_1, u_1v_4\} \cup \{u^*v_1, u^*v_4\}$ is 4-cyles.

(3) If C has only one edge $u'v' \in A$. Then u_1u_4 and v_1v_4 are joining created edges since there are two connections edge in $A - \{u'v'\}$.

Case (2): When $C(i)$ has at most one crossing for some $i \in \{1, 2\}$. By removing the crossing edges and adding two created edges we obtain two plane cycles $C(1)$ and $C(2)$. As in case(1), assume that no points between the two edges $u_1u_2 \in E(C(1))$ and another $v_1v_2 \in E(C(2))$. By repeating the similar argument in case (1) it can be seen that $C(1)$ and $C(2)$ have two joining created edges. □

4.4. Packing 1-PHCs into a Point Set

We conclude this paper with the following main result.

Theorem 4.4. *Let P be a set of n points in general position in the plane where $n = 2^k + h$, with $0 \leq h < 2^k$. Then there exist at least $k-1$ edge-disjoint 1-PHCs $C_1, C_2, \ldots, C_{k-1}$ on P that can be packed into K_n.*

Proof: First we apply Algorithm (A) to obtain the first 1-PHC C_1. In so doing, the set P has been bisected into P_1 and P_2 by l_1. Let P_1 be on the left of l_1 and P_2 on the right of l_1.

If P_1 has no stone, then by hum-sandwich cut theorem there is a line l_2 that simultaneously bisects P_1 and P_2 into $P_{i,j}$, $i = 1,2$ with $j = 1,2$ which are label in the anticlockwise order and either $|P_{1,1}| = |P_{1,2}|$ or $|P_{1,1}| = |P_{1,2}| + 1$.

If P_1 has a stone $st(v, w)$, then there follows two possible cases:

Case (1): By Lemma 4.2, there is a line l_2 that simultaneously bisects P_1 and P_2 into $P_{i,j}$, $i = 1,2$ with $j = 1,2$. and $\{v, w\} \subsetneq P_{1,2}$ and either $|P_{1,1}| = |P_{1,2}|$ or $|P_{1,1}| = |P_{1,2}| + 1$.

Case (2): By Lemma 4.1, there is a line l_2 that bisects P_1 into $P_{1,1}$ and $P_{1,2}$ and with $\{v, w\} \subsetneq P_{1,2}$ and either $|P_{1,1}| = |P_{1,2}|$ or $|P_{1,1}| = |P_{1,2}| + 1$. Furthermore, there is a line l'_2 that bisects P_2 into $P_{2,1}$ and $P_{2,2}$ and either $|P_{2,1}| = |P_{2,2}|$ or $|P_{2,1}| = |P_{2,2}| + 1$.

In all cases, label the parts $P_{i,j}$ in the anticlockwise order. In case (1) and case (2), $C(1)$ and $C(2)$ are two edge-disjoint cycles can be joined using either the joining edges or created joining edges (by Lemma 4.3).

To obtain C_3, rename $P_{i,j}$ to be four parts P_1, P_2, P_3, and P_4 arranged in anticlockwise ordered. Then repeat the above operations with P_i taking place P for each $i = 1,2,3,4$ to obtain 1-PHCs $C(i)$. Join $C(i)$ with $C(i+1)$ for $i = 1,2,3$ either by joining edges or by created joining edges.

In general, to obtain C_r where $1 \leq r \leq k-1$, repeat the above operations on parts $P_1, P_2, \ldots, P_{2^{r-1}}$, arranged in anticlockwise ordered, with P_i taking place P for each $i = 1, 2, \ldots, 2^{r-1}$ to obtain 1-PHCs $C(i)$. Join $C(i)$ with $C(i+1)$ for $i = 1, 2, \ldots, 2^{r-1} - 1$ either by joining edges or by created joining edges. □

An example of such geometric graph contains a number of 1-PHCs is shown in Figure 5.

5. Open Problems

In this paper, the problem of packing 1-plane Hamiltonian cycles into complete geometric graphs, on a given set of n points in the plane was investigated. The problem of partitioning the complete geometric graphs into 1-plane Hamiltonian cycles is also of interest. Note that the complete graph K_n has $\frac{n(n-1)}{2}$ edges and every Hamiltonian cycle has n edges. In the case that n is an odd number, then K_n can be partitioned into $\frac{n-1}{2}$ edge-disjoint Hamiltonian cycles. Thus, following open problem is presented:

Problem: Does every complete geometric graph K_n, where n is odd number, have a partition of its edge set into 1-plane Hamiltonian cycles?

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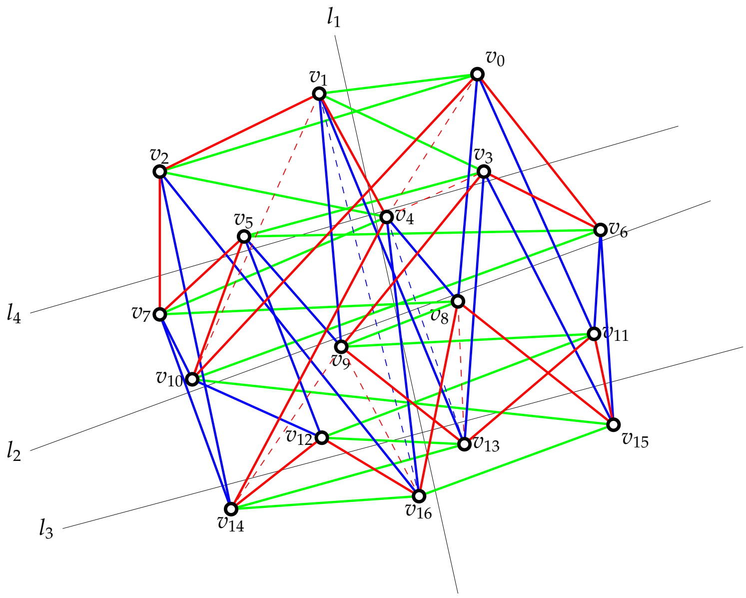

Figure 5: An example of Theorem 4.4

References

[1] O. Aichholzer, A. Asinowski, T. Miltzow, Disjoint compatibility graph of non-crossing matchings of points in convex position, Electronic Journal of Combinatorics 22 (1) (2015) P1.65.

[2] O. Aichholzer, S. Bereg, A. Dumitrescu, A. G. Olaverri, C. Huemer, F. Hurtado, M. Kano, A. Marquez, D. Rappaport, S. Smorodinsky, D. L. Souvaine, J. Urrutia, D. R. Wood, Compatible geometric matchings, Computational Geometry 42(6-7)(2009) 617 - 626.

[3] Aichholzer, T. Hackl, M. Korman, M. v. Kreveld, M. Löffler, A. Pilz, B. Speckmann E. Welzld, Packing plane spanning trees and paths in complete geometric graphs, Information Processing Letters 124 (2017) 35 – 41.

[4] S. Akl, A lower bound on the maximum number of crossing-free Hamilton cycles in a rectilinear drawing of K_n, Ars Combinatoria 7 (1979) 7 - 18.

[5] F. Bernhart, P. C. Kainen, The book thickness of a graph, Journal of Combinatorial Theory, Series B 27(3) (1979) 320 – 331.

[6] A. Biniaz, A. García, Packing Plane Spanning Trees into a Point Set, in: Proceedings of 30th Canadian Conference on Computational Geometry CCCG 2018 (2018) 49 – 53.

[7] A. Biniaz, P. Bose, A. Maheshwari, M. Smid, Packing plane perfect matchings into a point set, Discrete Mathematics & Theorectical Computer Science 17 (2015) 119–142.

[8] P. Bose, F. Hurtado, E. Rivera-Campo, D. R. Wood, Partitions of complete geometric graphs into plane trees, Computational Geometry 34(2) (2006) 116 – 125.

[9] M. Claverol, D. Garijo, F. Hurtado, D. Lara, C. Seara, The alternating path problem revisited, in: Proceedings of XV Spanish Meeting on Computational Geometry (2013) 115 – 118.

[10] M. Claverol, A. García, D. Garijo, C. Seara, J. Tejel, On Hamiltonian alternating cycles and paths, Computational Geometry, In Press (2017).

[11] D. Dor, M. Tarsi, Graph decomposition is NP-complete: A complete proof of Holyers conjecture, SIAM Journal on Computing 26(4) (1997) 1166 - 1187.

[12] A. Dumitrescu, A. Schulz, A. Sheffer, Cs.D. Tóth, Bounds on the maximum multiplicity of some common geometric graphs, Proceedings of 28th Symp. Theorectical Aspects Computer Science (2011) 637 - 648.

[13] H. Edelsbrunner, Algorithms In Combinatorial Geometry, Springer-Verlag, Berlin, (1987).

[14] M.R. Garey, D.S. Johnson, R.E. Tarjan, The planar Hamiltonian circuit problem is NP-complete, SIAM Journal on Computing 5 (1976) 704 - 714.

[15] A. Kaneko, M. Kano, Y. Yoshimoto, Alternating Hamiltonian cycles with minimum number of crossings in the plane, International Journal of Computational Geometry & Applications 10 (2000) 73 - 78.

[16] S. G. Kobourov, G. Liotta, F. Montecchiani, An annotated bibliography on 1-planarity, Computer Science Review 25 (2017) 49 – 67.

[17] M. Liśkiewicz, M. Ogihara, S. Toda, The complexity of counting self-avoiding walks in subgraphs of two-dimensional grids and hypercubes, Theoretical Computer Science 304 (2003), 129 - 156.

[18] C. Lo, J. Matousek, W. Steiger, Algorithms for ham-sandwich cuts, Discrete & Computational Geometry 11(4) (1994) 433 - 452.

[19] T. Motzkin, Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, Bulletin of the American Mathematical Society 54(4) (1948) 352 - 360.

[20] M. Sharir , A. Sheffe, E. Welzl, Counting Plane Graphs: Perfect Matchings, Spanning Cycles, and Kasteleyns Technique, Journal of Combinatorial Theory, Series A 120(4) (2013) 777 – 794.

[21] M. Sharir, E. Welzl, On the number of crossing-free matchings, cycles, and partitions, SIAM Journal on Computing 36(3) (2006) 695 – 720.