



Minimal Translation Surfaces in the Three-Dimensional Heisenberg Group

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Abstract. We define and classify minimal translation surfaces in the 3-dimensional Heisenberg group $H(1, 1)$.

1. Preliminaries

Historically, translation surfaces were first studied in the Euclidean 3-dimensional space and they are represented as graphs $z = f(x) + g(y)$, where f and g are smooth functions. Scherk proved in 1835 that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|,$$

where a is a non-zero constant. See [S].

Since then, minimal translation surfaces were generalized in several directions. For example, the Euclidean space \mathbb{E}^3 was replaced with other spaces of dimension 3 – usually being 3-dimensional Lie groups and the notion of “translation” was often replaced by using the group operation. See for example [ILM], [Lop], [LM], [Seo], [Y-13], or [YLK]. For more results on translation surfaces see e.g. [DGW], [Liu], [MN]. Another generalizations of Scherk surfaces are: affine translation surfaces in Euclidean 3-space [LY], affine translation surfaces in affine 3-dimensional spaces [YF] and translation surfaces in Galilean 3-space [Y-17]. On the other hand, Scherk surfaces were generalized to minimal translation surfaces in Euclidean spaces of arbitrary dimensions. See e.g. [DVZ], [MM], [MPRH], [VWY]. Moreover, in [DVVW], translation surfaces in the Euclidean 3-space are thought as sum of two curves. Similar approach appears also in [MM].

In the present paper we define the notion of translation surfaces in the Heisenberg group $H(1, 1)$ as the formal analogue to those in the Euclidean 3-space. Note that, because of the absence of an affine structure on $H(1, 1)$, we are not allowed to say that this concept is intrinsic for $H(1, 1)$, as the group operation is not involved. See e.g. [Lop, ILM].

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The 3-dimensional Heisenberg group $H(1, 1)$ is an example of a connected, two-step nilpotent, real Lie group with one-dimensional center. It can be represented as

$$H(1, 1) = \left(\left\{ a^* = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}, \cdot \right), \tag{1.1}$$

where dot denotes the usual matrix multiplication. A global coordinate system (x, y, z) may be defined on $H(1, 1)$ by setting $x(a^*) = a$, $y(a^*) = b$, $z(a^*) = c$. Hence, we can consider the left invariant metric on $H(1, 1)$ as follows

$$\tilde{g} = dx^2 + dy^2 + (dz - xdy)^2. \tag{1.2}$$

An orthonormal basis of left invariant vector fields on $H(1, 1)$ is obtained as

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad \xi = \frac{\partial}{\partial z}. \tag{1.3}$$

Then, the Levi-Civita connection $\tilde{\nabla}$ of $H(1, 1)$ is given by

$$\tilde{\nabla}_\xi X = \tilde{\nabla}_X \xi = -\frac{1}{2}Y, \quad \tilde{\nabla}_\xi Y = \tilde{\nabla}_Y \xi = \frac{1}{2}X, \quad \tilde{\nabla}_X Y = -\tilde{\nabla}_Y X = \frac{1}{2}\xi, \tag{1.4}$$

the other derivatives being zero. See for example [GC, K].

A regular surface in $H(1, 1)$ is a 2-dimensional submanifold, which can be expressed, locally, as graphs of smooth functions of two variables, that is one can write $z = f_3(x, y)$, $y = f_2(x, z)$ or $x = f_1(y, z)$. Of course, interesting classes of surfaces appear when the functions involved have specific forms. In this paper, we study minimal translation surfaces in $H(1, 1)$.

Definition 1.1. A surface M in Heisenberg group $H(1, 1)$ is a translation surface if it is given by an isometric immersion $\mathcal{F} : U \subset \mathbb{R}^2 \rightarrow H(1, 1)$ of the form

$$\mathcal{F}(u, v) = (u, v, f(u) + g(v)), \quad (\text{Type I}) \tag{1.5}$$

or

$$\mathcal{F}(u, v) = (u, f(u) + g(v), v), \quad (\text{Type II}) \tag{1.6}$$

or

$$\mathcal{F}(u, v) = (f(u) + g(v), u, v), \quad (\text{Type III}) \tag{1.7}$$

where f and g are smooth functions on opens of \mathbb{R} .

Remark that in the Euclidean space, the three coordinates x , y and z are interchangeable, while in $H(1, 1)$ they are not. So, this is the reason for which we have to consider three types of translation surfaces in $H(1, 1)$.

The mean curvature of the immersion is defined as $H = \frac{1}{2}(k_1 + k_2)$ where k_1 and k_2 are principal curvatures on an arbitrary point of M . Let $\{\mathcal{F}_u, \mathcal{F}_v\}$ be the basis of T_pM and denote by E, F and G the coefficients of first fundamental form on M , that is

$$E = \tilde{g}(\mathcal{F}_u, \mathcal{F}_u), \quad F = \tilde{g}(\mathcal{F}_u, \mathcal{F}_v), \quad G = \tilde{g}(\mathcal{F}_v, \mathcal{F}_v). \tag{1.8}$$

Then the function H is obtained as

$$H = \frac{G\tilde{g}(N, \tilde{\nabla}_{E_1} E_1) - 2F\tilde{g}(N, \tilde{\nabla}_{E_1} E_2) + E\tilde{g}(N, \tilde{\nabla}_{E_2} E_2)}{2(EG - F^2)}, \tag{1.9}$$

where $\{E_1, E_2\}$ is an arbitrary basis on the surface M [ILM]. Thus the minimality condition for the translation surface M is given as follows:

$$G\tilde{g}(N, \tilde{\nabla}_{\mathcal{F}_u}\mathcal{F}_u) - 2F\tilde{g}(N, \tilde{\nabla}_{\mathcal{F}_u}\mathcal{F}_v) + E\tilde{g}(N, \tilde{\nabla}_{\mathcal{F}_v}\mathcal{F}_v) = 0. \tag{1.10}$$

In the following, we investigate the minimality of translation surfaces of each of the three cases. Some of the first examples of minimal surfaces in the Heisenberg group (with a slightly different metric) can be found in [B].

2. Minimal translation surfaces of type I

Let us consider a translation surface M of type I in $H(1, 1)$ parametrized by $\mathcal{F}(u, v) = (u, v, f(u) + g(v))$. The tangent plane of M is spanned by

$$\mathcal{F}_u = X + f'(u)\xi \quad \text{and} \quad \mathcal{F}_v = Y + (g'(v) - u)\xi, \tag{2.1}$$

while the unit normal N (up to orientation) is given by

$$N = \frac{1}{\alpha} [-f'(u)X - (g'(v) - u)Y + \xi], \tag{2.2}$$

where $\alpha^2 = 1 + f'(u)^2 + (g'(v) - u)^2$.

We obtain the coefficients of the first fundamental form of \mathcal{F} as

$$E = 1 + f'(u)^2, \quad F = f'(u)(g'(v) - u), \quad G = 1 + (g'(v) - u)^2. \tag{2.3}$$

Then, the Levi-Civita connection on the surface is given by

$$\begin{cases} \tilde{\nabla}_{\mathcal{F}_u}\mathcal{F}_u = -f'(u)Y + f''(u)\xi, \\ \tilde{\nabla}_{\mathcal{F}_v}\mathcal{F}_v = (g'(v) - u)X + g''(v)\xi, \\ \tilde{\nabla}_{\mathcal{F}_u}\mathcal{F}_v = \frac{1}{2} \{f'(u)X - (g'(v) - u)Y - \xi\}. \end{cases} \tag{2.4}$$

Consequently, the minimality condition (1.10) may be expressed as follows:

$$f''(u) + g''(v) + f''(u)[g'(v) - u]^2 + f'(u)^2g''(v) + f'(u)[g'(v) - u] = 0. \tag{2.5}$$

Our aim now is to find the solutions of the partial differential equation (PDE) (2.5). Let us assume first that f', f'', g' and g'' are different from zero at every point. Taking successive derivatives with respect to u and v , we obtain

$$2\left(\frac{f'''}{f'f''} + \frac{g'''}{g'g''}\right) = \frac{1}{g'}\left(\frac{1}{f'} + \frac{2uf'''}{f'f''}\right). \tag{2.6}$$

Doing the same for the equation (2.6), that is taking the derivatives with respect to u and v successively, we get

$$\left(\frac{1}{g'}\right)' \left(\frac{1}{f'} + \frac{2uf'''}{f'f''}\right)' = 0. \tag{2.7}$$

Since the first factor cannot vanish, we should have

$$\frac{1}{f'} + \frac{2uf'''}{f'f''} = c_1, \tag{2.8}$$

where c_1 is a constant.

Combining of (2.6) with (2.8) we find

$$\frac{f'''}{f'f''} = \frac{c_1}{2g'} - \frac{g'''}{g'g''}. \tag{2.9}$$

Remark that the left-hand side of equation (2.9) is a function of u , while the right-hand side is a function of v . Therefore, there exist other two constants c_2 and c_3 such that

$$f'' = \frac{c_2}{2}f'^2 + c_3, \tag{2.10}$$

$$\frac{g'''}{g''} = \frac{c_1}{2} - c_2g'. \tag{2.11}$$

Now, plugging (2.10) into (2.8) we obtain

$$f' = \frac{1}{c_1 - 2c_2u}. \tag{2.12}$$

Because $f'' \neq 0$, the constant c_2 must be different from zero.

By substituting in (2.10), we obtain a polynomial in u of degree 2, namely

$$8c_2^2c_3u^2 - 8c_1c_2c_3u + 2c_1^2c_3 - 3c_2 = 0. \tag{2.13}$$

It follows that all the coefficients vanish, and if this happens, c_2 must be zero. But this is a contradiction with $f'' \neq 0$.

Let us see what happens if $f'' \neq 0$ at every point, but there exists u_0 such that $f'(u_0) = 0$. As $f' \neq 0$ on $(u_0 - \epsilon, u_0)$ and $(u_0, u_0 + \epsilon)$ for a certain $\epsilon > 0$, we make the same discussion as before to get the contradiction. Analogue for g . Now we analyze the remained cases, namely $f'' = 0$ and $g'' = 0$, respectively.

First let f be an affine function of the form

$$f(u) = Au + B, \tag{2.14}$$

where A and B are constants. Then the minimality condition (2.5) becomes

$$(1 + A^2)g'' + Ag' = Au. \tag{2.15}$$

Obviously, the constant A must be zero and this implies also $g'' = 0$. Hence the minimal translation surface of type I is realized as the plane $\mathcal{F}(u, v) = (u, v, av + b)$, for certain $a, b \in \mathbb{R}$.

If the function g is affine, $g(v) = av + b$, the minimality (2.5) writes as:

$$\left[1 + (a - u)^2\right] f'' + (a - u)f' = 0. \tag{2.16}$$

The general solution of this ODE is found as

$$f(u) = c_1 \left[(u - a) \sqrt{1 + (u - a)^2} - \log \left(\sqrt{1 + (u - a)^2} - (u - a) \right) \right] + c_2,$$

where c_1 and c_2 are constants.

Note that if the constant c_1 vanishes, we find the plane $(u, v, av + b)$. Consequently, we can state the following theorem.

Theorem 2.1. *Type I minimal translation surfaces in the three-dimensional Heisenberg group $H(1, 1)$ are of the following form*

$$\mathcal{F}(u, v) = \left(u, v, c \left[(u - a) \sqrt{1 + (u - a)^2} - \log \left(\sqrt{1 + (u - a)^2} - (u - a) \right) \right] + av + b \right),$$

where a, b and c are constants.

3. Minimal translation surfaces of type II

In this section, we endeavor to obtain all the minimal translation surfaces of type II in $H(1,1)$. In accordance with this purpose, we have to obtain and solve the minimality equation.

Let M be a translation surface of type II parametrized by $\mathcal{F}(u, v) = (u, f(u) + g(v), v)$. The tangent plane of M is spanned by $\{\mathcal{F}_u, \mathcal{F}_v\}$, where

$$\mathcal{F}_u = X + f'(u)Y - uf'(u)\xi \quad \text{and} \quad \mathcal{F}_v = g'(v)Y + (1 - ug'(v))\xi. \tag{3.1}$$

The unit normal to M is computed as

$$N = \frac{1}{\mu} (f'(u)X - (1 - ug'(v))Y + g'(v)\xi), \tag{3.2}$$

where $\mu = [f'(u)^2 + g'(v)^2 + (1 - ug'(v))^2]^{1/2}$.

If we proceed as the previous section, we find that M is minimal if and only if

$$f'' [g'^2 + (1 - ug')^2] + g'' [1 + (1 + u^2)f'^2] + f'g'(1 - ug') = 0. \tag{3.3}$$

Taking the derivative with respect to v under the assumption $g'g'' \neq 0$ we obtain

$$\frac{g'''}{g'g''} [1 + (1 + u^2)f'^2] + \frac{1}{g'} [f' - 2uf''] + 2[(1 + u^2)f'' - uf'] = 0. \tag{3.4}$$

This equation has the following form:

$$A(v)B(u) + C(v)D(u) + E(u) = 0,$$

for (u, v) in a certain 2-dimensional domain. Suppose that $B'(u)$ and $C'(v)$ never vanish. Then, there exist constants c and c_1 such that

$$A(v) = cC(v) + c_1, \quad D(u) = -cB(u), \quad \text{and} \quad E(u) = -c_1B(u).$$

To prove this, take the derivatives with respect to u and v successively to obtain

$$\frac{A'(v)}{C'(v)} = -\frac{D'(u)}{B'(u)}.$$

As the left part depends only on v and the right part depends only on u , they must be a constant, let's call it c . It follows that $A(v) = cC(v) + c_1$ and $-D(u) = cB(u) + c_2$, where $c_1, c_2 \in \mathbb{R}$. Inserting these expressions into the initial equality we get

$$c_1B(u) - c_2C(v) + E(u) = 0, \text{ for all } (u, v).$$

Hence $c_2 = 0$ and $E(u) = -c_1B(u)$.

Going back to the equation (3.4) we obtain that

$$\frac{g'''}{g'g''} = \frac{c}{g'} + c_1, \tag{3.5}$$

$$2uf'' - f' = c[1 + (1 + u^2)f'^2] \tag{3.6}$$

and

$$2[uf' - (1 + u^2)f''] = c_1[1 + (1 + u^2)f'^2], \tag{3.7}$$

where c and c_1 are constants.

The minimality condition (3.3) may be rewritten as

$$g''[1 + (1 + u^2)f'^2] + g'(f' - 2uf'') + g'^2[(1 + u^2)f'' - uf'] + f'' = 0.$$

By replacing (3.6) and (3.7) in the equation above, we obtain that

$$[1 + (1 + u^2)f'^2](g'' - cg' + \frac{c_1}{2}g'^2) + f'' = 0.$$

Identifying the terms depending on u , respectively those depending on v , we deduce the existence of a constant $k \in \mathbb{R}$ such that

$$g'' - cg' - \frac{c_1}{2}g'^2 = k, \tag{3.8}$$

$$f'' = -k[1 + (1 + u^2)f'^2]. \tag{3.9}$$

Combining (3.6) and (3.7) with (3.9), we find

$$\begin{aligned} f' + (c + 2ku)[1 + (1 + u^2)f'^2] &= 0 \\ 2uf' - [c_1 - 2k(1 + u^2)][1 + (1 + u^2)f'^2] &= 0. \end{aligned}$$

If $f' \neq 0$ (on an interval), these two equalities imply

$$c_1 - 2k(1 + u^2) + 2u(c + 2ku) = 0,$$

which represents a second order polynomial in variable u . It follows that the coefficients vanish. This means that the constants k, c and c_1 must be zero. Hence, $f' = 0$, which is a contradiction. If $f' = 0$ (locally), from (3.3) we obtain that g is an affine function and this contradicts the assumption $g'g'' \neq 0$.

Until now, we worked under the hypothesis $g'g'' \neq 0$. Suppose that $g'' \neq 0$, but there exists v_0 such that $g'(v_0) = 0$ and $g'(v) \neq 0$ for $v \in (v_0 - \epsilon, v_0 + \epsilon) \setminus \{v_0\}$, where ϵ is a small enough positive number. We make similar computations as before in the intervals $(v_0 - \epsilon, v_0)$, respectively $(v_0, v_0 + \epsilon)$ and get the same contradiction.

Let now g be an affine function, namely $g(v) = av + b$, with $a, b \in \mathbb{R}$. Hence, the minimality condition (3.3) becomes

$$[(1 - au)^2 + a^2]f'' + a(1 - au)f' = 0.$$

Obviously, any constant function is a solution of this ODE. The other solutions are obtained as follows:

- if $a = 0$ then f is an affine function;
- if $a \neq 0$ then

$$f(u) = c_1[(au - 1)\bar{f}(u) + a^2 \log(au - 1 + \bar{f}(u))] + c_2, \tag{3.10}$$

where $\bar{f}(u) = \sqrt{a^2 + (au - 1)^2}$ and c_1, c_2 are real constants.

We can state the following.

Theorem 3.1. *The minimal translation surfaces of type II in the three dimensional Heisenberg group $H(1, 1)$ are*

- planes $\mathcal{F}(u, v) = (u, au + b, v)$, with $a, b \in \mathbb{R}$ and
- cylinders parametrized by $\mathcal{F}(u, v) = (u, f(u) + av, v)$, where $a \in \mathbb{R} \setminus \{0\}$, and $f(u)$ is given by (3.10).

4. Minimal translation surfaces of type III

In this section we investigate the minimal translation surfaces of type III in $H(1, 1)$, parametrized by $\mathcal{F}(u, v) = (f(u) + g(v), v, u)$. As usual, consider the natural basis in the tangent plane $T_{(u,v)}M$ as well as the unit normal as follows

$$\begin{cases} \mathcal{F}_u = f'(u)X + \xi, \\ \mathcal{F}_v = g'(v)X + Y - (f(u) + g(v))\xi, \\ N = \frac{1}{\eta} [-X + [g'(v) + f'(u)(f(u) + g(v))]Y + f'(u)\xi], \end{cases} \tag{4.1}$$

where $\eta = [1 + f'(u)^2 + [g'(v) + f'(u)(f(u) + g(v))]^2]^{1/2}$. By providing the necessary elements for the minimality condition (1.10), we obtain

$$f''[1 + g^2 + (f + g)^2] + g''(1 + f^2) + f'^2(f + g) + f'g' = 0. \tag{4.2}$$

Taking successive derivatives with respect to u and v one gets

$$2f'''g'(f + g + g'') + 2f'f''(2g' + g''') + f''g'' = 0. \tag{4.3}$$

Now we assume that $f'''g'$ is different from zero; then (4.3) is equivalent to

$$2f + 2(g + g'') + \frac{2f'f''}{f'''} \left(2 + \frac{g'''}{g'} \right) + \frac{f''}{f'''} \frac{g''}{g'} = 0. \tag{4.4}$$

Taking again successive derivatives with respect to u and v , we obtain

$$\left(\frac{f''}{f'''} \right)' \left(\frac{g''}{g'} \right)' + \left(\frac{2f'f''}{f'''} \right)' \left(\frac{g'''}{g'} \right)' = 0. \tag{4.5}$$

A. Suppose that $\frac{f'f''}{f'''}$ and $\frac{g''}{g'}$ are not constant functions. Then, there exist constants c, c_1 and c_2 such that

$$\frac{f''}{f'''} = 2c \frac{f'f''}{f'''} + c_1 \tag{4.6}$$

and

$$\frac{g'''}{g'} = -c \frac{g''}{g'} + c_2. \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.4) we obtain

$$\left[f + (c_2 + 2) \frac{f'f''}{f'''} \right] + \left[(g + g'') + c_1 \frac{g''}{2g'} \right] = 0. \tag{4.8}$$

Hence, there exists another constant $\lambda \in \mathbb{R}$ such that

$$f + (2 + c_2) \frac{f'f''}{f'''} = \lambda \tag{4.9}$$

and

$$2(g + g'') + c_1 \frac{g''}{g'} = -2\lambda. \tag{4.10}$$

A.1 Suppose that $c \neq 0$. Combining (4.6) and (4.9) we obtain

$$f' = \frac{\lambda - f}{x + yf} \tag{4.11}$$

where $x = 2c\lambda + c_1(2 + c_2)$ and $y = -2c$. Moreover, we have

$$f'' = -\frac{(x + \lambda y)(\lambda - f)}{(x + yf)^3}. \tag{4.12}$$

By integrating (4.6) we get

$$f' = cf'^2 + c_1f'' + \alpha, \tag{4.13}$$

where α is a constant.

Now, plugging (4.11) and (4.12) into (4.13) we obtain a polynomial equation of degree three in $f(u)$. Since at least one real root exists, all the coefficients must be zero. Thus, we must have $\alpha = \frac{1}{4c}$ and $x = 2c\lambda$. It follows that $x + \lambda y = 0$, which implies $f'' = 0$ and this contradicts the hypothesis $f'''g' \neq 0$.

A.2 If $c = 0$ we obtain $\frac{f''}{f'''} = c_1$ and $\frac{g''}{g'} = c_2$. Moreover, we have

$$f + (2 + c_2)c_1f' = \lambda, \quad g + g'' + c_1\frac{g''}{g'} = -2\lambda, \quad \lambda \in \mathbb{R}. \tag{4.14}$$

It follows that $f'' + (2 + c_2)c_1f''' = 0$ and hence $c_1f'''(3 + c_2) = 0$. Since c_1 is different from zero and f''' is supposed to be non zero, we must have $c_2 = -3$. Therefore, we get

$$g'' = -3g + \beta, \tag{4.15}$$

for a certain constant β . Now, from the equation (4.14) we obtain

$$f - c_1f' = \lambda \tag{4.16}$$

and

$$g' = c_1 \frac{\beta - 3g}{2g - \beta - 2\lambda}. \tag{4.17}$$

Substituting (4.17) into (4.15) we obtain

$$(\beta - 3g) \left[1 - \frac{c_1^2(\beta + 6\lambda)}{(2g - \beta - 2\lambda)^3} \right] = 0.$$

This implies that g is a constant, which contradicts $f'''g' \neq 0$.

B. If $\frac{f'f''}{f'''} is a constant function, then the equation (4.5) yields$

- either $\frac{f''}{f'''} is constant, case in which f' is also constant and this contradicts $f'''g' \neq 0$;$
- or $\frac{g''}{g'}$ is constant.

Set $\frac{f'f''}{f'''} = a_1$ and $\frac{g''}{g'} = a_2$, where $a_1, a_2 \in \mathbb{R}$. We have also $\frac{g'''}{g'} = a_2^2$. Now, using (4.4), it follows that there exists a constant c such that

$$f + \frac{a_2f''}{2f'''} = c \quad \text{and} \quad g + g'' = -a_1(2 + a_2^2) - c.$$

Multiplying the first equation by f' we immediately obtain $f' = \frac{a_1 a_2}{2(c-f)}$ and hence $f'' = \frac{a_1^2 a_2^2}{4(c-f)^3}$. On the other hand, since $f' f'' = a_1 f'''$, there exists a constant α such that $f'^2 = 2a_1 f'' + \alpha$. We obtain

$$(c - f)a_1^2 a_2^2 = 2a_1^3 a_2^2 + 4\alpha(c - f)^3.$$

Thus, f satisfies a polynomial equation for all u . Therefore, we must have $\alpha = 0$ and $a_2 = 0$ (since a_1 is different from 0). It follows that $f = c$, and this is false.

We conclude that under the hypothesis $f''' g' \neq 0$, the minimality equation (4.2) has no solution. Let us discuss the situation when $f''' g'$ vanishes. We will distinguish the following cases.

Case 1. Let $g(v) = a$ be a constant function. Hence the minimality condition is formed as the second-order nonlinear o.d.e

$$f'' [1 + (f + a)^2] + f'^2 (f + a) = 0. \tag{4.18}$$

Of course, any constant function f is a solution of this equation. We are looking now to non constant solutions. First we observe that

$$\frac{d}{du} \log [f'^2 [1 + (f + a)^2]] = 0.$$

Hence $f' \sqrt{1 + (f + a)^2} = k_1$, where $k_1 \in \mathbb{R}$. Integrating, one gets

$$I(f(u) + a) = k_1 u + k_2,$$

where I is the function defined by

$$I(x) = x \sqrt{1 + x^2} + \log(x + \sqrt{1 + x^2}) \tag{4.19}$$

and k_2 is a constant.

Case 2. Suppose that $f''' = 0$, that is $f(u) = au^2 + bu + c$, with $a, b, c \in \mathbb{R}$. Replacing f into the minimality equation (4.2) we obtain a polynomial equation of degree 4 in u whose coefficients are functions depending on v . As this equation is satisfied for an infinitely many u , its coefficients must be zero. The leading coefficient is $6a^3$. So, $a = 0$. We obtain

$$g''(1 + b^2) + b^2(bu + c + g) + bg' = 0.$$

This is a linear equation in u with infinitely many solutions. Thus $b = 0$ and hence g is an affine function.

All the computations we have made so far can be summarized in the following result.

Theorem 4.1. *The minimal translation surfaces of type III in the three dimensional Heisenberg group $H(1, 1)$ belong to the following list*

$$\mathcal{F}(u, v) = (\mathbf{a}v + \mathbf{b}, v, u)$$

or

$$\mathcal{F}(u, v) = (I^{-1}(k_1 u + k_2), v, u)$$

where I is defined by (4.19) and $\mathbf{a}, \mathbf{b}, k_1$ and k_2 are constants.

Remark 4.2. *In general, surfaces of different types I–III are not isometric. Nevertheless, some isometries of $H(1, 1)$ can carry a surface of type (i) into a surface of type (j).*

Example 1. Consider the map $\phi : H(1, 1) \rightarrow H(1, 1)$, $\phi(x, y, z) = (x, y + b, z + c)$ with $b, c \in \mathbb{R}$ and two translation surfaces in $H(1, 1)$ of type I, respectively type II, as follows:

$$M_1: \mathcal{F}_1(u, v) = (u, v, au + v), (u, v) \in \mathbb{R}^2;$$

$$M_2: \mathcal{F}_2(\bar{u}, \bar{v}) = (\bar{u}, -a\bar{u} + (\bar{v} + b - c), \bar{v}), (\bar{u}, \bar{v}) \in \mathbb{R}^2, a \in \mathbb{R}.$$

We have that ϕ is an isometry of $H(1, 1)$, which maps M_1 into M_2 , that is $\phi(\mathcal{F}_1(u, v)) = \mathcal{F}_2(\bar{u}, \bar{v})$.

Example 2. Consider the map $\phi : H(1, 1) \rightarrow H(1, 1)$, $\phi(x, y, z) = (x + a, y, z + ay)$ with $a \in \mathbb{R} \setminus \{0\}$ and two translation surfaces in $H(1, 1)$ of type I, respectively type III, as follows:

$$M_1: \mathcal{F}_1(u, v) = (u, v, u - v), (u, v) \in \mathbb{R}^2;$$

$$M_3: \mathcal{F}_3(\bar{u}, \bar{v}) = ((1 - a)\bar{u} + (\bar{v} + a), \bar{u}, \bar{v}), (\bar{u}, \bar{v}) \in \mathbb{R}^2.$$

We have that ϕ is an isometry of $H(1, 1)$, which maps M_1 into M_3 , that is $\phi(\mathcal{F}_1(u, v)) = \mathcal{F}_3(\bar{u}, \bar{v})$.

For more details on the geometry of the Heisenberg group see, e.g. [I] and [IKOS].

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