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Local Spectral Property of 2×2 Operator Matrices

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Abstract. In this paper we study the local spectral properties of 2×2 operator matrices. In particular, we show that every 2×2 operator matrix with three scalar entries has the single valued extension property. Moreover, we consider the spectral properties of such operator matrices. Finally, we show that some of such operator matrices are decomposable.

1. Introduction

Let $\mathcal{H}$ denote a separable, infinite dimensional, complex Hilbert space, and write $\mathcal{B}(\mathcal{H})$ for the algebra of all bounded, linear operators on $\mathcal{H}$. Let $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, and $\sigma_{su}(T)$ denote the spectrum, the point spectrum, the approximate point spectrum, and the surjective spectrum of T, respectively.

The invariant subspace problem is the question whether every operator in $\mathcal{B}(\mathcal{H})$ has a nontrivial invariant subspace. This problem has been around since the early 1930's when von Neumann raised it, and despite much effort by many mathematicians, it still remains one of intractable problems (see [3] for more details).

The purpose of this note is to study the local spectral properties and the invariant subspace problem for 2×2 operator matrices.

Every operator T in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is associated with a unique 2×2 operator matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{1}$$

where the $T_{ij} \in \mathcal{B}(\mathcal{H})$, $i, j = 1, 2$, and the action of T on an arbitrary vector $(x, y)^t \in \mathcal{H} \oplus \mathcal{H}$ is given by multiplying the matrix (T_{ij}) by the column vector above, yielding

$$T(x \oplus y)^t = (T_{11}x + T_{12}y) \oplus (T_{21}x + T_{22}y)\,.$$

Thus the operators we consider will be given as 2×2 operator matrices, as above. The following problem might be considered by those mathematicians interested in invariant subspaces. We write, as usual, $\mathbb{C}1_{\mathcal{H}}$ for the set of all scalar multiples of the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$.

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Let α, γ, and δ be three different complex scalars, and let $B \notin \mathbb{C}1_{\mathcal{H}}$ be arbitrary in $\mathcal{B}(\mathcal{H})$. Does the operator

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \tag{2}$$

have a nontrivial invariant or hyperinvariant subspace?

This question may look trivial, but we have thus far been unable to solve it positively. However, for the operators

$$S = \begin{pmatrix} 0 & R \\ 1_{\mathcal{H}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}), \tag{3}$$

if R has nice properties and $\sigma(S)$ has nonempty interior in $\mathbb{C}$, then it can be shown that S has a nontrivial invariant subspace. Proposition 1.1 (see below) gives some positive answer. Before that, we recall some basic concepts.

An operator $T \in \mathcal{L}(\mathcal{H})$ has the *single valued extension property (i.e., SVEP)* at $\lambda_0 \in \mathbb{C}$ if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda) f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. The operator T is said to have the single valued extension property if T has the single valued extension property at every $\lambda \in \mathbb{C}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T - \lambda) f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T - \lambda) f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(H)$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well known that

Bishop's property (β) $\Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

Any of the converse implications does not hold, in general (see [9] for more details).

Proposition 1.1. *Let*

$$S = \begin{pmatrix} 0 & R \\ 1_{\mathcal{H}} & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

If R has the Bishop's property (β) (or the SVEP), then S has the Bishop's property (β) (or the SVEP). In addition, if $\sigma(S)$ has nonempty interior in $\mathbb{C}$, then S has a nontrivial invariant subspace.

Proof. Since $S^2 = R \oplus R$ and R has the Bishop's property (β) (or the SVEP), so does S^2. Hence S has the Bishop's property (β) (or the SVEP) from [9]. In addition, if $\sigma(S)$ has nonempty interior in $\mathbb{C}$, the proof follows from [9]. □

2. Main results

In this section we study the local spectral properties of 2×2 operator matrices. In particular, we show that every 2×2 operator matrix with three scalar entries has the single valued extension property. Moreover, we consider the spectral properties of such operator matrices. Finally, we show that some of such operator matrices are decomposable. The assumption in Theorem 2.1 that α, γ, δ are different scalars is made to avoid the same cases with the matrix in Proposition 1.1 by translations and rotations of the matrix in Theorem 2.1 by scalar multiples of $1_{\mathcal{H}}$. We begin with the following theorem for our program.

Theorem 2.1. *Suppose that α, γ, and δ are three different scalars and R is any operator in $\mathcal{B}(\mathcal{H})$. Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

If R has the single valued extension property, so does T.

Proof. If $\gamma = 0$, then it is well known that T has the single valued extension property. Hence it suffices to show that T has the single valued extension property when $\gamma \neq 0$. Let G be a domain in $\mathbb{C}$ and let $f = f_1 \oplus f_2$ be analytic $\mathcal{H} \oplus \mathcal{H}$-valued function defined on G, where $f_1 : G \to \mathcal{H}$ and $f_2 : G \to \mathcal{H}$ are analytic functions such that

$$(T - z1_{\mathcal{H}\oplus\mathcal{H}})f(z) = \begin{pmatrix} (\alpha - z)1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & (\delta - z)1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = 0$$

on G. Then we get that

$$\begin{cases} (\alpha - z)f_1(z) + Rf_2(z) = 0 \quad \text{and} \\ \gamma f_1(z) + (\delta - z)f_2(z) = 0. \end{cases} \tag{4}$$

Since

$$\begin{aligned} \|(\gamma R - (z-\alpha)(z-\delta)1_{\mathcal{H}})f_2(z)\| \quad \leq \quad & \|\gamma R f_2(z) - \gamma(z-\alpha)f_1(z)\| \\ & + \|\gamma(z-\alpha)f_1(z) - (z-\delta)(z-\alpha)f_2(z)\|, \end{aligned}$$

we have

$$(\gamma R - (z-\alpha)(z-\delta)1_{\mathcal{H}})f_2(z) = 0.$$

Set $\mu = p(z)$ where $p(z) = (z-\alpha)(z-\delta)$. Choose an open disk G_0 of G such that $G_0 \subset G \backslash \{\alpha, \delta\}$. Consider an analytic map g sending μ onto z. Then

$$(\gamma R - \mu)(f_2 \circ g)(\mu) = 0$$

on $p(G_0)$. Since γR has the single valued extension property, $(f_2 \circ g)(\mu) = 0$ on $p(G_0)$, i.e., $f_2(z) = 0$ on G_0. By identity theorem, $f_2(z) = 0$ on G. Hence $f_1(z) = 0$ on G from (4). Thus T has thethe single valued extension property. □

We note from Theorem 2.1 that every operator similar to T (in Theorem 2.1) has the single valued extension property. As some applications of Theorem 2.1, we get the following corollary.

Corollary 2.2. *Suppose that α, γ, and δ are three different scalars and R is any operator $\in \mathcal{B}(\mathcal{H})$. Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

Then the following statements hold.
(i) If R is hyponormal, then T has the single valued extension property and $\sigma(T) = \cup_{x \in \mathcal{H}\oplus\mathcal{H}} \sigma_T(x)$.
(ii) If R is normal, then T and T^* have the single valued extension property.
(iii) If R has the single valued extension property, then $\sigma(T) = \sigma_{ap}(T^*)$.

Proof. (i) Since R has the single valued extension property, the proof follows from Theorem 2.1 and [2].

(ii) It follows from Theorem 2.1 that T has the single valued extension property. Since T^* is unitarily equivalent to $\begin{pmatrix} \bar{\delta}1_{\mathcal{H}} & R^* \\ \bar{\gamma}1_{\mathcal{H}} & \bar{\alpha}1_{\mathcal{H}} \end{pmatrix}$ and $\begin{pmatrix} \bar{\delta}1_{\mathcal{H}} & R^* \\ \bar{\gamma}1_{\mathcal{H}} & \bar{\alpha}1_{\mathcal{H}} \end{pmatrix}$ has the single valued extension property from Theorem 2.1, T^* has the single valued extension property.

(iii) Since T has the single valued extension property from Theorem 2.1, $\sigma(T) = \sigma_{su}(T)$ from [2]. Since $\sigma_{su}(T) = \sigma_{ap}(T^*)$ and $\sigma_{su}(T) = \cup_{x\in\mathcal{H}\oplus\mathcal{H}}\sigma_T(x)$, $\sigma(T) = \sigma_{ap}(T^*)$. $\square$

We next consider another local spectral property, i.e., the Bishop's property (β).

Proposition 2.3. *Suppose that α, γ, and δ are three different scalars and R is any operator $\in \mathcal{B}(\mathcal{H})$. Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$$

where $\gamma \neq 0$. Let G be an open set in $\mathbb{C}$ and let $f_n = f_n^1 \oplus f_n^2$ be analytic $\mathcal{H}\oplus\mathcal{H}$-valued function defined on G, where $f_n^1 : G \to \mathcal{H}$ and $f_n^2 : G \to \mathcal{H}$ are analytic functions such that

$$\lim_{n\to\infty} \|(S - z1_{\mathcal{H}\oplus\mathcal{H}})f_n(z)\|_K = \lim_{n\to\infty} \|\begin{pmatrix} (\alpha - z)1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & (\delta - z)1_{\mathcal{H}} \end{pmatrix}\begin{pmatrix} f_n^1(z) \\ f_n^2(z) \end{pmatrix}\|_K = 0$$

for every compact subset K of G where $\|f\|_K$ denotes $\sup_{z\in K}\|f(z)\|$ for an $\mathcal{H}\oplus\mathcal{H}$-valued function $f(z)$. If R has the Bishop's property (β), then

$$\lim_{n\to\infty} \|f_n^j(z)\|_{K'} = 0$$

for $j = 1, 2$ where K' is a compact subset of G with $K' \subset K$.

Proof. Let G be an open set in $\mathbb{C}$ and let $f_n = f_n^1 \oplus f_n^2$ be analytic $\mathcal{H}\oplus\mathcal{H}$-valued function defined on G, where $f_n^1 : G \to \mathcal{H}$ and $f_n^2 : G \to \mathcal{H}$ are analytic functions such that

$$\lim_{n\to\infty} \|(S - z1_{\mathcal{H}\oplus\mathcal{H}})f_n(z)\|_K = \lim_{n\to\infty} \|\begin{pmatrix} (\alpha - z)1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & (\delta - z)1_{\mathcal{H}} \end{pmatrix}\begin{pmatrix} f_n^1(z) \\ f_n^2(z) \end{pmatrix}\|_K = 0$$

for every compact subset K of G where $\|f\|_K$ denotes $\sup_{z\in K}\|f(z)\|$ for an $\mathcal{H}\oplus\mathcal{H}$-valued function $f(z)$. Then we get that

$$\begin{cases} \lim_{n\to\infty} \|(\alpha - z)f_n^1(z) + Rf_n^2(z)\|_K = 0 & \text{and} \\ \lim_{n\to\infty} \|\gamma f_n^1(z) - (z - \delta)f_n^2(z)\|_K = 0. \end{cases}$$

Since

$$\begin{aligned} \|(\gamma R - (z-\alpha)(z-\delta)1_{\mathcal{H}})f_n^2(z)\|_K \quad \leq \quad & \|\gamma R f_n^2(z) - \gamma(z-\alpha)f_n^1(z)\|_K \\ & + \|\gamma(z-\alpha)f_n^1(z) - (z-\delta)(z-\alpha)f_n^2(z)\|_K, \end{aligned}$$

we have

$$\lim_{n\to\infty} \|(\gamma R - (z-\alpha)(z-\delta)1_{\mathcal{H}})f_n^2(z)\|_K = 0.$$

Set $\mu = p(z)$ where $p(z) = (z-\alpha)(z-\delta)$. Choose an open subset G_0 of G such that $G_0 \subset G\setminus\{\alpha, \delta\}$. Then for any compact subset K' of G_0, K' is also a compact subset of G. Consider an analytic map g sending μ onto z. Then

$$\lim_{n\to\infty} \|(\gamma R - \mu)(f_n^2 \circ g)(\mu)\|_{p(K')} = 0.$$

Since γR has the Bishop's property (β), $\lim_{n\to\infty}\|(f_n^2 \circ g)(\mu)\|_{p(K')} = 0$, i.e., $\lim_{n\to\infty}\|f_n^2(z)\|_{K'} = 0$. Hence $\lim_{n\to\infty}\|f_n^1(z)\|_{K'} = 0$. $\square$

We next study some spectral properties of of 2×2 operator matrices with three scalar entries.

Theorem 2.4. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

If $\gamma = 0$, $\sigma_j(T) = \{\alpha, \delta\}$ *and if* $\gamma \neq 0$,

$$\sigma_j(T) = \Big\{ \frac{\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\gamma\rho}}{2} : \rho \in \sigma_j(R) \Big\}$$

for $j = 0, ap, p$ *where* $\sigma_0(T)$ *denotes* $\sigma(T)$.

Proof. Since

$$\mathbb{C} \backslash \sigma(T) = \{z \in \mathbb{C} : T - z1_{\mathcal{H}\oplus\mathcal{H}} \text{ is invertible.}\}$$

and

$$T - z1_{\mathcal{H}\oplus\mathcal{H}} = \begin{pmatrix} (\alpha - z)1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & (\delta - z)1_{\mathcal{H}} \end{pmatrix},$$

we know from [6] that $T - z1_{\mathcal{H}\oplus\mathcal{H}}$ is invertible if and only if the operator

$$z^2 - (\alpha + \delta)z + \alpha\delta - \gamma R$$

is invertible in $\mathcal{B}(\mathcal{H})$. If $\gamma = 0$, then it is clear that $\sigma(T) = \{\alpha, \delta\}$. If $\gamma \neq 0$, then $R - \frac{z^2 - (\alpha+\delta)z + \alpha\delta}{\gamma}$ is not invertible if and only if $\frac{z^2 - (\alpha+\delta)z + \alpha\delta}{\gamma} \in \sigma(R)$. Hence

$$\sigma(T) = \{z \in \mathbb{C} : \text{there exists } \rho \in \sigma(R) \text{ with } z^2 - (\alpha + \delta)z + \alpha\delta - \gamma\rho = 0\}.$$

Thus

$$\sigma(T) = \Big\{ \frac{\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\gamma\rho}}{2} : \rho \in \sigma(R) \Big\}.$$

To complete the proof, it suffices to find $\sigma_{ap}(T)$. $z \in \sigma_{ap}(T)$ if and only if there exist a sequence $\{x_n\}$, $x_n = (x_n^1 \oplus x_n^2)^t$, with $\|x_n\| = 1$ such that

$$\lim_{n\to\infty} \|(T - z1_{\mathcal{H}\oplus\mathcal{H}})x_n\| = \lim_{n\to\infty} \| \begin{pmatrix} (\alpha - z)1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & (\delta - z)1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix} \| = 0.$$

Then we get that

$$\begin{cases} \lim_{n\to\infty} \|(\alpha - z)x_n^1 + Rx_n^2\| = 0 \quad \text{and} \\ \lim_{n\to\infty} \|\gamma x_n^1 - (\delta - z)x_n^2\| = 0. \end{cases} \tag{5}$$

Since

$$\begin{aligned} \|(\gamma R - (z - \alpha)(z - \delta)1_{\mathcal{H}})x_n^2\| \quad \leq \quad & \|\gamma R x_n^2 - \gamma(z - \alpha)x_n^1\| \\ & + \|\gamma(z - \alpha)x_n^1 - (z - \delta)(z - \alpha)x_n^2\|, \end{aligned}$$

we have

$$\lim_{n\to\infty} \|(\gamma R - (z - \alpha)(z - \delta)1_{\mathcal{H}})x_n^2\| = 0.$$

Since $x_n^2 \neq 0$ from (5) and $\|x_n\| = 1$, $(z - \alpha)(z - \delta) \in \sigma_{ap}(\gamma R)$. Hence

$$\sigma_{ap}(T) = \{z \in \mathbb{C} : \text{there exists } \rho \in \sigma_{ap}(R) \text{ with } z^2 - (\alpha + \delta)z + \alpha\delta - \gamma\rho = 0\}.$$

Thus

$$\sigma_{ap}(T) = \Big\{ \frac{\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\gamma\rho}}{2} : \rho \in \sigma_{ap}(R) \Big\}.$$

So we complete the proof. □

Corollary 2.5. *Suppose that α, γ, and δ are three different scalars and $R \in \mathcal{B}(\mathcal{H})$ has closed range where γ is nonzero. Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

If R is quasinilpotent, then T has a nontrivial hyperinvariant subspace.

Proof. Since $\sigma(R) = \{0\}$, $\sigma(T) = \left\{\frac{\alpha+\delta \pm \sqrt{(\alpha-\delta)^2}}{2}\right\}$ consists of two different points from Theorem 2.4. So the proof follows from [12]. □

For an operator T in $\mathcal{B}(\mathcal{H})$, we write $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ for the essential, left essential, and right essential spectra of T, respectively. Recall that $z \in \sigma_{le}(T)$ if and only if there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\{x_n\}$ converges weakly to zero and $\lim_{n\to\infty} \|(T - z)x_n\| = 0$.

Corollary 2.6. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & R \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

If $\gamma = 0$, $\sigma_j(T) = \{\alpha, \delta\}$ and if $\gamma \neq 0$,

$$\sigma_j(T) = \left\{\frac{\alpha + \delta \pm \sqrt{(\alpha-\delta)^2 + 4\gamma\rho}}{2} : \rho \in \sigma_j(R)\right\}$$

for $j = le, re, e$.

Proof. Since $\sigma_{le}(T) \subset \sigma_{ap}(T)$, the proof follows from Theorem 2.4. Since $z \in \sigma_{le}(T)$ if and only if $\bar{z} \in \sigma_{re}(T^*)$ and $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$, we complete the proof from Theorem 2.4. □

We next consider the following lemma, which is very useful for our program.

Lemma 2.7. ([1]) *Let*

$$R = \begin{pmatrix} A & C \\ Z & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

be an operator matrix where range(C) is closed. Then R has the following matrix representation;

$$R = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & 0 & C_1 \\ Z & B_1 & B_2 \end{pmatrix} \tag{6}$$

which maps from $\mathcal{H} \oplus \ker(C) \oplus \ker(C)^\perp$ to $range(C)^\perp \oplus range(C) \oplus \mathcal{H}$ where $C_1 = C|_{\ker(C)^\perp}$, $A_1 = P_{range(C)^\perp}A|_{\mathcal{H}}$, $A_2 = P_{range(C)}A|_{\mathcal{H}}$, B_1 denotes a mapping B from $\ker(C)$ into $\mathcal{H}$, B_2 denotes a mapping B from $N(C)^\perp$ into $\mathcal{K}$, $P_{R(C)^\perp}$ denotes the projection of $\mathcal{H}$ onto $range(C)^\perp$, and $P_{range(C)}$ denotes the projection of $\mathcal{H}$ onto $range(C)$.

If $M = \begin{pmatrix} A & C \\ Z & B \end{pmatrix}$ where $range(C)$ is closed, we denote M by the matrix representation as (6) in Lemma 2.7 for every $Z \in \mathcal{L}(\mathcal{H})$. Since $range(C)$ is closed, $C_1 = C|_{N(C)^\perp} : \ker(C)^\perp \to range(C)$ is invertible. Let $\lambda \in \mathbb{C}$ be given. Using the representation (6) in Lemma 2.7, we write $M - \lambda$ as follows;

$$M - \lambda = \begin{pmatrix} A_1 - \lambda & 0 & 0 \\ A_2 - \lambda & 0 & C_1 \\ Z & B_1 - \lambda & B_2 - \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (B_2-\lambda)C_1^{-1} \end{pmatrix} \begin{pmatrix} B_1-\lambda & \triangle_\lambda & 0 \\ 0 & A_1-\lambda & 0 \\ 0 & 0 & C_1 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix}$$

where $A_1-\lambda = P_{range(C)^\perp}(A-\lambda)|_{\mathcal{H}}$, $A_2-\lambda = P_{range(C)}(A-\lambda)|_{\mathcal{H}}$, $B_1-\lambda = (B-\lambda)|_{\ker(C)}$, $B_2-\lambda = (B-\lambda)|_{\ker(C)^\perp}$ and $\triangle_\lambda = Z-(B_2-\lambda)C_1^{-1}(A_2-\lambda)$.

Note that

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (B_2-\lambda)C_1^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ C_1^{-1}(A_2-\lambda) & 0 & I \end{pmatrix} \text{ are invertible.}$$

As some applications of Lemma 2.7, we get the following theorem.

Theorem 2.8. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & B \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H}\oplus\mathcal{H}).$$

If B has closed range, then both T and T^* are decomposable, and have the Bishop's property (β), the Dunford's property (C), and the single valued extension property.

Proof. Since T is decomposable if and only if both T and T^* have the Bishop's property (β), it suffices to show that T and T^* have the Bishop's property (β). If $\gamma = 0$, it is well known that T has the Bishop's property (β). Let $\gamma \neq 0$. Since B ha closed range, $B_1 = B|_{\ker(B)^\perp} : \ker(B)^\perp \to range(B)$ is invertible. Hence for any $z \in \mathbb{C}$ $S-z$ has the following matrix representation from Lemma 2.7;

$$\begin{aligned} & S - z1_{\mathcal{H}\oplus\ker(B)\oplus\ker(B)^\perp} \\ = & \begin{pmatrix} \alpha_1 - z & 0 & 0 \\ \alpha_2 - z & 0 & B_1 \\ \gamma & \delta_1 - z & \delta_2 - z \end{pmatrix} \\ = & \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (\delta_2-z)B_1^{-1} \end{pmatrix} \begin{pmatrix} \delta_1-z & \triangle_z & 0 \\ 0 & \alpha_1-z & 0 \\ 0 & 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ B_1^{-1}(\alpha_2-z) & 0 & I \end{pmatrix} \end{aligned}$$

where $\alpha_1 - z = P_{range(B)^\perp}(\alpha-z)|_{\mathcal{H}}$, $\alpha_2 - z = P_{range(B)}((\alpha-z)|_{\mathcal{H}}$, $\delta_1 - z = (\delta-z)|_{\ker(B)}$, $\delta_2-z = (\delta-z)|_{\ker(B)^\perp}$ and $\triangle_z = \gamma - (\delta_2-z)B_1^{-1}(\alpha_2-z)$. Note that

$$\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (\delta_2-z)B_1^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ B_1^{-1}(\alpha_2-z) & 0 & I \end{pmatrix} \text{ are invertible.} \tag{7}$$

Let D be an open set in $\mathbb{C}$ and let $f_n = f_n^1 \oplus f_n^2 \oplus f_n^3$ be an analytic $\mathcal{H}\oplus\ker(B)\oplus\ker(B)^\perp$-valued function defined on D, where $f_n^1 : D \to \mathcal{H}$, $f_n^2 : D \to \ker(B)$, and $f_n^2 : D \to \ker(B)^\perp$ are analytic functions such that

$$\begin{aligned} & \lim_{n\to\infty} \|(S - z1_{\mathcal{H}\oplus\ker(B)\oplus\ker(B)^\perp})f_n(z)\|_K \\ = & \lim_{n\to\infty} \left\| \begin{pmatrix} \alpha_1 - z & 0 & 0 \\ \alpha_2 - z & 0 & B_1 \\ \gamma & \delta_1 - z & \delta_2 - z \end{pmatrix} \begin{pmatrix} f_n^1(z) \\ f_n^2(z) \\ f_n^3(z) \end{pmatrix} \right\|_K = 0 \end{aligned}$$

for every compact subset K of D where $\|f\|_K$ denotes $\sup_{z\in K}\|f(z)\|$ for an $\mathcal{H}\oplus\ker(B)\oplus\ker(B)^\perp$-valued function $f(z)$. Since $\begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & (\delta_2-z)B_1^{-1} \end{pmatrix}$ is invertible, it follows from (7) that

$$\lim_{n\to\infty} \left\| \begin{pmatrix} \delta_1-z & \triangle_z & 0 \\ 0 & \alpha_1-z & 0 \\ 0 & 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ B_1^{-1}(\alpha_2-z) & 0 & I \end{pmatrix} \begin{pmatrix} f_n^1(z) \\ f_n^2(z) \\ f_n^3(z) \end{pmatrix} \right\|_K = 0$$

for every compact subset K of D. Then we get that

$$\begin{cases} \lim_{n\to\infty} \|(\delta_1 - z)g_n^1(z) + \triangle_z g_n^2(z)\|_K = 0 \\ \lim_{n\to\infty} \|(\alpha_1 - z)g_n^2(z)\|_K = 0 \qquad\qquad \text{and} \\ \lim_{n\to\infty} \|B_1 g_n^3(z)\|_K = 0 \end{cases} \tag{8}$$

where

$$\begin{pmatrix} g_n^1(z) \\ g_n^2(z) \\ g_n^3(z) \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ B_1^{-1}(\alpha_2 - z) & 0 & I \end{pmatrix} \begin{pmatrix} f_n^1(z) \\ f_n^2(z) \\ f_n^3(z) \end{pmatrix}.$$

Since B_1 is invertible, $\lim_{n\to\infty} \|g_n^3(z)\|_K = 0$ from (8). Since α_1 and δ_1 have the Bishop's property (β), we obtain from (8) that $\lim_{n\to\infty} \|g_n^j(z)\|_K = 0$ for $j = 1, 2$. Since $\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ B_1^{-1}(\alpha_2 - z) & 0 & I \end{pmatrix}$ is invertible, it follows that $\lim_{n\to\infty} \|f_n^j(z)\|_K = 0$ for $j = 1, 2, 3$. Hence T has the Bishop's property (β).

It is known from the closed range theorem that B has closed range if and only if B^* has. Since

$$T^* = \begin{pmatrix} \bar{\alpha}1_{\mathcal{H}} & \bar{\gamma}1_{\mathcal{H}} \\ B^* & \bar{\delta}1_{\mathcal{H}} \end{pmatrix}$$

and T^* is unitarily equivalent to $\begin{pmatrix} \bar{\delta}1_{\mathcal{H}} & B^* \\ \bar{\gamma}1_{\mathcal{H}} & \bar{\alpha}1_{\mathcal{H}} \end{pmatrix}$, i.e.,

$$\begin{pmatrix} \bar{\delta}1_{\mathcal{H}} & B^* \\ \bar{\gamma}1_{\mathcal{H}} & \bar{\alpha}1_{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} 0 & 1_{\mathcal{H}} \\ 1_{\mathcal{H}} & 0 \end{pmatrix} \begin{pmatrix} \bar{\alpha}1_{\mathcal{H}} & \bar{\gamma}1_{\mathcal{H}} \\ B^* & \bar{\delta}1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} 0 & 1_{\mathcal{H}} \\ 1_{\mathcal{H}} & 0 \end{pmatrix},$$

T^* has the Bishop's property (β) by similar method with the previous proof. Since T is decomposable if and only if T^* is, we complete the proof. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and finite dimensional null space and is called *lower semi-Fredholm* if it has closed range and its range has finite co-dimension. If $T \in \mathcal{L}(\mathcal{H})$ is either upper or lower semi-Fredholm, then T is called *semi-Fredholm* and *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$i(T) := \alpha(T) - \beta(T).$$

If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called *Fredholm*. Let $\sigma_{uf}(T)$, $\sigma_{lf}(T)$, $\sigma_{sf}(T)$, and $\sigma_f(T)$ denote the upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum, the semi-Fredholm spectrum, and the Fredholm spectrum of T, respectively, defined as

$$\sigma_{uf}(T) = \{z \in \mathbb{C} : T - z \text{ is not upper semi-Fredholm}\},$$

$$\sigma_{lf}(T) = \{z \in \mathbb{C} : T - z \text{ is not lower semi-Fredholm}\},$$

$$\sigma_{sf}(T) = \{z \in \mathbb{C} : T - z \text{ is not semi-Fredholm}\},$$

and

$$\sigma_f(T) = \{z \in \mathbb{C} : T - z \text{ is not Fredholm}\}.$$

Corollary 2.9. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & B \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

Suppose that one of the following statements holds;
(i) there exists a constant $c > 0$ such that for any $x \in \mathcal{H}$ there exists a $y \in \mathcal{H}$ such that $Bx = By$ and $\|y\| \le c\|Bx\|$.
(ii) B is semi-Fredholm.
Then both T and T^* are decomposable, and have the Bishop's property (β), the Dunford's property (C), and the single valued extension property.

Proof. (i) By [11], B has closed range. Hence the proof follows from Theorem 2.8.
(ii) Since B has closed range, the proof follows from Theorem 2.8. □

Recall that a subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be spectral maximal for $T \in \mathcal{B}(\mathcal{H})$ if for every T-invariant subspace $\mathcal{N}$ of $\mathcal{H}$ the inclusion $\sigma(T|_{\mathcal{N}}) \subseteq \sigma(T|_{\mathcal{M}})$ implies $\mathcal{N} \subseteq \mathcal{M}$.

Corollary 2.10. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & B \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

where B has closed range. Then the following statements hold.
(i) If $\sigma(T)$ has nonempty interior in $\mathbb{C}$, then both T and T^* have a nontrivial invariant subspace.
(ii) For all closed subset F of $\sigma(T)$, $\mathcal{H}_T(F)$ is a spectral maximal space of T and $\sigma(T|_{\mathcal{H}_T(F)}) \subset F$.

Proof. (i) Since both T and T^* have the Bishop's property (β) by Theorem 2.8, the proof follows from [9].
(ii) Since T is decomposable from Theorem 2.8, the proof follows from [4]. □

Recall that A and B in $\mathcal{B}(\mathcal{H})$ are norm equivalent if there exist two positive real numbers k_1 and k_2 such that $k_1\|Ax\| \leq \|Bx\| \leq k_2\|Ax\|$ for all $x \in \mathcal{H}$.

Corollary 2.11. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & B \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

where B has closed range. If A and B are norm equivalent, then

$$S = \begin{pmatrix} \alpha 1_{\mathcal{H}} & A \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$$

and S^* are decomposable, and have the Bishop's property (β), the Dunford's property (C), and the single valued extension property.

Proof. Assume A and B are norm equivalent. Since B has closed range, so does A from [11]. Hence it follows from Theorem 2.8 that the proof is completed. □

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be a Weyl operator if it is a Fredholm operator having index 0. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Browder operator if it has both finite ascent and finite decent. Let $\sigma_w(T)$ and $\sigma_b(T)$ denote the Weyl spectrum and the Browder spectrum of T, respectively, defined as

$$\sigma_w(T) = \{z \in \mathbb{C} : T - z \text{ is not Weyl}\} \text{ and}$$

$$\sigma_b(T) = \{z \in \mathbb{C} : T - z \text{ is not Browder}\}.$$

Corollary 2.12. *Let*

$$T = \begin{pmatrix} \alpha 1_{\mathcal{H}} & B \\ \gamma 1_{\mathcal{H}} & \delta 1_{\mathcal{H}} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}).$$

If B has closed range, then the following statements hold.
(i) $\sigma_{ap}(T^*) = \sigma_{ap}(T)^*$, $\sigma_{su}(T^*) = \sigma_{su}(T)^*$, and $\cup_{x \in \mathcal{H}} \sigma_T(x)^* = \sigma(T)^* = \sigma(T^*) = \cup_{x \in \mathcal{H}} \sigma_{T^*}(x)$.
(ii) $\sigma_{sf}(T) = \sigma_{uf}(T) = \sigma_{lf}(T) = \sigma_f(T) = \sigma_w(T) = \sigma_b(T)$.
(iii) T satisfies the Browder's theorem.

Proof. (i) Since both T and T^* have the single valued extension property from Theorem 2.8, we get from [2] that $\sigma_{ap}(T^*) = \sigma(T) = \sigma_{su}(T) = \cup_{x\in\mathcal{H}}\sigma_T(x)$ and $\sigma_{ap}(T) = \sigma(T^*) = \sigma_{su}(T^*) = \cup_{x\in\mathcal{H}}\sigma_{T^*}(x)$.

(ii) The proof follows from [2].

(iii) Since $\sigma_w(T) = \sigma_b(T)$ from Corollary 2.12, we complete the proof. □

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