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Minimal Bases and Minimal Sub-bases for Topological Spaces

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Abstract. This paper investigates minimal bases and minimal sub-bases for topological spaces. First, a necessary and sufficient condition is derived for the existence of minimal base for a general topological space. Then the concept of minimal sub-base for a topological space is proposed and its properties are discussed. Finally, for Alexandroff spaces, some special results with respect to minimal bases and minimal sub-bases are illustrated.

1. Introduction

Base is a basic concept in general topology. R.E. Stong [11] points out that a finite topological space has a unique minimal base. According to [11], for a finite topological space, a base $\mathscr{B}$ is minimal if and only if there is no union reducible element in $\mathscr{B}$. Finite topological spaces are a special case of Alexandroff spaces, which are introduced by P. Alexandroff [1]. And some related results are shown in [2, 4, 6–10]. From [10], an Alexandroff space has a unique minimal base. In addition, F. G. Arenas [2] extended the necessary and sufficient condition about minimal base for a finite topological space to an Alexandroff space. However, for a general topological space, there is no necessary and sufficient condition for the existence of minimal base. This paper presents a necessary and sufficient condition for the existence of minimal base for a general topological space.

Sub-base is also a basic concept in general topology. If $\mathscr{S}$ is a sub-base for a topological space X, then $\mathscr{B} = \{\bigcap_{S\in\mathscr{S}_0} S | \mathscr{S}_0$ is a finite subfamily of $\mathscr{S}\}$ forms a base for the topological space X, and the topology τ of X can be denoted by $\tau = \{\bigcup_{B\in\mathscr{B}'} B | \mathscr{B}' \in \mathscr{B}\}$ [5]. Based on the result above and the fact that there is no union reducible element in a minimal base, a natural question is: is a sub-base $\mathscr{S}$ minimal when there is no intersection reducible element in $\mathscr{S}$? This paper introduces the concept of minimal sub-base for a topological space. And we show that a minimal sub-base of a general topological space does not have finite intersection reducible elements in this paper. Besides, this paper concludes that a sub-base may not be minimal if it does not have intersection reducible elements.

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Since a minimal base does not have union reducible elements, another question is: are there intersection reducible elements in a minimal base? If the answer is yes, then when all the intersection reducible elements are removed, does the minimal base become a minimal sub-base? In this paper, we prove that a minimal base is also a minimal sub-base if there are no finite intersection reducible elements in the minimal base. Furthermore, for an Alexandroff space, this paper shows that a sub-base without intersection reducible elements is minimal.

The remainder of this paper is organized as follows. In Section 2, a necessary and sufficient condition is obtained for the existence of minimal base for a general topological space. Then the definition of minimal sub-base for a topological space is introduced and its properties are investigated. Section 3 discusses some special results with respect to minimal bases and minimal sub-bases for Alexandroff spaces, which are different from general topological spaces. Section 4 has some concluding remarks.

2. Minimal Bases and Minimal Sub-bases

We begin with the following example.

Example 2.1. Let $\mathbb{R}$ be a topological space with a topology $\tau = \{(-a, a)|a > 0\} \cup \{\emptyset, \mathbb{R}\}$, where

$$a = \begin{cases} \frac{1}{n}, & 0 < a < 1; \\ n, & a > 1. \end{cases}$$

It is easy to check that τ is a topology of $\mathbb{R}$ and $\mathscr{B} = \tau \setminus \{\emptyset, \mathbb{R}\}$ is the minimal base for the topological space $(\mathbb{R}, \tau)$. But $(\mathbb{R}, \tau)$ is not an Alexandroff space. The reason is as follows. Let $\{(-a, a)|0 < a < 1\}$ be a subfamily of τ. Then $\bigcap\{(-a, a)|0 < a < 1\} = \{0\}$ and $\{0\}$ is not an open set of the topological space $(\mathbb{R}, \tau)$.

Example 2.1 shows that a topological space with a minimal base may not be an Alexandroff space.

In [11], R. E. Stong gave a necessary and sufficient condition for the existence of minimal base for a finite topological space. And F. G. Arenas [2] extended this necessary and sufficient condition to an Alexandroff space. Combined with Example 2.1, this necessary and sufficient condition holds for a general topological space. Next we present this necessary and sufficient condition.

Lemma 2.2. ([2]) *Let X be an Alexandroff space and $\mathscr{B}$ a family of open sets. Then $\mathscr{B}$ is the minimal base for the topology of X if and only if:*

(1) $\mathscr{B}$ covers X;

(2) If $A, B \in \mathscr{B}$, then there exists a subfamily $\{B_i|i \in I\}$ of $\mathscr{B}$ such that $A \cap B = \bigcup_{i\in I} B_i$;

(3) If a subfamily $\{B_i|i \in I\}$ of $\mathscr{B}$ verifies $\bigcup_{i\in I} B_i \in \mathscr{B}$, then there exists $i_0 \in I$ such that $\bigcup_{i\in I} B_i = B_{i_0}$.

Suppose $\mathscr{B}$ is only a family of non-empty subsets of X. By Conditions (1) and (2) of Lemma 2.2, $\mathscr{B}$ is a base for a topological space, and Condition (3) of Lemma 2.2 states that $\mathscr{B}$ is minimal. But there is no condition to show that the topology generated by $\mathscr{B}$ is an Alexandroff topology. Hence, Lemma 2.2 can be generalized to general topological spaces. The following example further illustrates this point.

Example 2.3. Reconsider Example 2.1. First, $\bigcup \mathscr{B} = \bigcup(\tau \setminus \{\emptyset, \mathbb{R}\}) = \mathbb{R}$. Next for any subsets $A, B \in \mathscr{B}$, it is obvious that $A \subset B$ or $B \subset A$. Without loss of generality, we assume $A \subset B$. Then there exists a subfamily $\{B_i|B_i \subset A, i \in I\}$ of $\mathscr{B}$ such that $A \cap B = A = \bigcup\{B_i|B_i \subset A, i \in I\}$. If a subfamily $\{B_i|i \in I\}$ of $\mathscr{B}$ satisfies $\bigcup_{i\in I} B_i \in \mathscr{B}$, then by the definition of $\mathscr{B}$, there exists $i_0 \in I$ such that $B_i \subset B_{i_0}$ for any $i \in I$. That is $\bigcup_{i\in I} B_i = B_{i_0}$. The analysis above states that $\mathscr{B}$ satisfies the three conditions of Lemma 2.2. From Example 2.1, the covering $\mathscr{B}$ is a minimal base. But the topological space $(\mathbb{R}, \tau)$ is not an Alexandroff space by Example 2.1.

The following theorem is obtained by a little modification of Lemma 2.2

Theorem 2.4. *Let X be a set and $\mathscr{B}$ a family of non-empty subsets of X. Then $\mathscr{B}$ is the minimal base for a topology of X if and only if:*

(1) $\mathscr{B}$ covers X;

(2) If $A, B \in \mathscr{B}$, then there exists a subfamily $\{B_i|i \in I\}$ of $\mathscr{B}$ such that $A \cap B = \bigcup_{i\in I} B_i$;

(3) If a subfamily $\{B_i|i \in I\}$ of $\mathscr{B}$ satisfies $\bigcup_{i\in I} B_i \in \mathscr{B}$, then there exists $i_0 \in I$ such that $\bigcup_{i\in I} B_i = B_{i_0}$.

Proof. Suppose $\mathscr{B}$ is a minimal base for a topology space X. Then Conditions (1) and (2) must hold. Assume a subfamily $\mathscr{B}_1 = \{B_i | i \in I\}$ of $\mathscr{B}$ satisfies $\bigcup_{i\in I} B_i \in \mathscr{B}$, but $\bigcup_{i\in I} B_i \neq B$ for each subset $B \in \mathscr{B}_1$. Then there exists a subset $B' \in \mathscr{B} \setminus \mathscr{B}_1$ such that $\bigcup_{i\in I} B_i = B'$. That implies that $\mathscr{B} \setminus \{B'\}$ is also a base for the topological space X. It is a contradiction. Hence, Condition (3) is true.

Obviously, Conditions (1) and (2) state that $\mathscr{B}$ is a base for a topological space X. If a subfamily $\mathscr{B}'$ of $\mathscr{B}$ is a base for the topological space X, then for each subset $B \in \mathscr{B}$, there exists a subfamily $\{B_i | i \in I\}$ of $\mathscr{B}'$ such that $\bigcup_{i\in I} B_i = B \in \mathscr{B}$. According to Condition (3), there exists $i_0 \in I$ such that $\bigcup_{i\in I} B_i = B_{i_0}$. So $B = B_{i_0} \in \mathscr{B}'$. That implies $\mathscr{B}' = \mathscr{B}$. Therefore, $\mathscr{B}$ is a minimal base for the topological space X. □

Definition 2.5. Let X be a topological space and $\mathscr{B}$ a family of non-empty subsets of X. For each subset $B \in \mathscr{B}$,

(1) if there exists a subfamily $\{B_i | i \in I\}$ of $\mathscr{B}$ such that $B \notin \{B_i | i \in I\}$ and $\bigcup_{i\in I} B_i = B$, then B is called a union reducible element with respect to $\mathscr{B}$;

(2) if there exists a subfamily $\{B_i | i \in I\}$ of $\mathscr{B}$ such that $B \notin \{B_i | i \in I\}$ and $\bigcap_{i\in I} B_i = B$, then B is called an intersection reducible element with respect to $\mathscr{B}$.

Combined Theorem 2.4 with Definition 2.5, a base $\mathscr{B}$ for a topological space is minimal if and only if there are no union reducible elements in the base $\mathscr{B}$. Then are there intersection reducible elements in a minimal base? The following example given in [10] will be used to answer the question.

Example 2.6. Let $(\mathbb{R}^n, \tau)$ be a topological space with a base $\mathscr{B} = \{\overline{B(0,r)} | r \in \mathbb{R}_+ \cup \{0\}\}$. In fact, $\mathscr{B}$ is a minimal base for the topological space $(\mathbb{R}^n, \tau)$. Let $r = 1$ and n be a natural number, then $\bigcap_{n=1}^{\infty} \overline{B(0, 1 + \frac{1}{n})} = \overline{B(0,1)}$. Because $\overline{B(0,1)} \notin \{\overline{B(0, 1 + \frac{1}{n})} | n \text{ is a natural number}\}$, $\overline{B(0,1)}$ is an intersection reducible element with respect to $\mathscr{B}$.

The analysis above shows that a minimal base does not have union reducible elements and may have intersection reducible elements. When all intersection reducible elements in a minimal base are removed, what is the minimal base? We show that a base $\mathscr{B}$ becomes a minimal sub-base if there are no union and intersection reducible elements in $\mathscr{B}$.

Definition 2.7. Let $\mathscr{S}$ be a sub-base for a topological space X. If the following statement holds, then $\mathscr{S}$ is called a minimal sub-base for the topological space X. The statement is: for any subfamily $\mathscr{S}'$ of $\mathscr{S}$, if $\mathscr{S}'$ is a sub-base for the topological space X, then $\mathscr{S}' = \mathscr{S}$.

We give the following example to illustrate Definition 2.7.

Example 2.8. Let $X = \mathbb{R} \setminus \mathbb{Z}$ and $\mathscr{S} = \{(n, n+1) | n \in \mathbb{Z}\}$. Obviously, $\mathscr{S}$ is a partition of $\mathbb{R} \setminus \mathbb{Z}$. This implies that $\mathscr{S}$ is a minimal sub-base for a topology of $\mathbb{R} \setminus \mathbb{Z}$.

Example 2.8 shows that $\mathscr{S}$ does not have intersection reducible elements. This property is owned by minimal sub-bases for topological spaces.

Theorem 2.9. *Let X be a set and $\mathscr{S}$ a family of non-empty subsets of X. If $\mathscr{S}$ is a minimal sub-base for a topology of X, then there is no finite intersection reducible element in the minimal sub-base $\mathscr{S}$.*

Proof. By Definition 2.5, we will prove that if a finite subfamily $\{S_i | i \in I\}$ of $\mathscr{S}$ satisfies $\bigcap_{i\in I} S_i \in \mathscr{S}$, then there exists $i_0 \in I$ such that $\bigcap_{i\in I} S_i = S_{i_0}$. Assume a finite subfamily $\mathscr{S}_1 = \{S_i | i \in I\}$ of $\mathscr{S}$ satisfies $\bigcap_{i\in I} S_i \in \mathscr{S}$, but $\bigcap_{i\in I} S_i \neq S$ for each subset $S \in \mathscr{S}_1$. Then there exists a subset $S' \in \mathscr{S} \setminus \mathscr{S}_1$ such that $\bigcap_{i\in I} S_i = S'$. That implies that $\mathscr{S} \setminus \{S'\}$ is also a sub-base for the topology of X. It is a contradiction. Hence, the minimal sub-base $\mathscr{S}$ does not have finite intersection reducible elements. □

From the following theorem, a minimal base is a minimal sub-base when all finite intersection reducible elements in the minimal base are removed. But there is no approach to obtain directly a minimal base from a minimal sub-base.

Theorem 2.10. *Let X be a topological space and $\mathscr{B}$ a minimal base for the topological space X. If there are no finite intersection reducible elements in $\mathscr{B}$, then $\mathscr{B}$ is a minimal sub-base for the topological space X.*

Proof. With no doubt, $\mathscr{B}$ can be viewed as a sub-base for the topological space X. Suppose a subfamily $\mathscr{S}$ of $\mathscr{B}$ is a sub-base for the topological space X. According to the definitions of base and sub-base, $\mathscr{B}$ is a minimal base generated by $\mathscr{S}$. Then for each subset $B \in \mathscr{B}$, there exists a finite subfamily $\{S_i | i \in I\}$ of $\mathscr{S}$ such that $\bigcap_{i\in I} S_i = B \in \mathscr{B}$. There exists $i_0 \in I$ such that $\bigcap_{i\in I} S_i = S_{i_0}$, because the minimal base $\mathscr{B}$ does not have finite intersection reducible elements. Thus, $B = S_{i_0} \in \mathscr{S}$, which means $\mathscr{S} = \mathscr{B}$. Therefore, $\mathscr{B}$ is a minimal sub-base for the topological space X. □

Note that a minimal sub-base $\mathscr{S}$ for a topological space X does not have finite intersection reducible elements. However, $\mathscr{S}$ may have infinite intersection reducible elements.

Example 2.11. Rediscuss Example 2.6. There is no doubt that the minimal base $\mathscr{B} = \{\overline{B(0,r)} | r \in \mathbb{R}_+ \cup \{0\}\}$ is a sub-base for the topological space $(\mathbb{R}^n, \tau)$. Then $\mathscr{B}$ is a minimal sub-base for the topological space $(\mathbb{R}^n, \tau)$, because there is no finite intersection reducible element in the minimal base $\mathscr{B}$. By Example 2.6, $\overline{B(0,1)}$ is an infinite intersection reducible element with respect to $\mathscr{B}$.

Next we give some properties about minimal sub-base for a topological space.

Theorem 2.12. *Let X be a set and Y be a topological space. If $f : X \to Y$ is a surjective map and $\mathscr{S}$ is a minimal sub-base for the topological space Y, then $f^{-1}(\mathscr{S}) = \{f^{-1}(S) | S \in \mathscr{S}\}$ is a minimal sub-base for a topology on X.*

Proof. Since $\mathscr{S}$ is a minimal sub-base of the topological space Y, we get $Y = \bigcup \mathscr{S}$. And we have $X = \bigcup f^{-1}(\mathscr{S})$. Thus, $f^{-1}(\mathscr{S})$ is a sub-base for a topology τ on X. Suppose $f^{-1}(\mathscr{S})$ is not a minimal sub-base for the topological space (X, τ). Then there exists $S_0 \in \mathscr{S}$ such that $f^{-1}(\mathscr{S}) \setminus \{f^{-1}(S_0)\}$ is a sub-base for the topological space (X, τ). Since $\mathscr{S} \setminus \{S_0\}$ is not a sub-base for the topological space Y, there exist an open subset W of Y and $y \in W$ such that for any finite family $\mathscr{F} \subset \mathscr{S} \setminus \{S_0\}$, $y \in \bigcap \mathscr{F} \subset W$ does not hold. Take $x \in f^{-1}(y)$. Since $f^{-1}(W)$ is open in the topological space (X, τ) and $f^{-1}(\mathscr{S}) \setminus \{f^{-1}(S_0)\}$ is a sub-base for (X, τ), there exists a finite family $\mathscr{F} \subset \mathscr{S} \setminus \{S_0\}$ such that $x \in \bigcap\{f^{-1}(F) | F \in \mathscr{F}\} \subset f^{-1}(W)$. Thus, $y \in \bigcap\{F | F \in \mathscr{F}\} = \bigcap \mathscr{F} \subset W$. It is a contradiction. Hance, $f^{-1}(\mathscr{S})$ is a minimal sub-base for a topology on X. □

Based on Theorem 2.12, the following corollary is obtained.

Corollary 2.13. *Let Y be a quotient space of a topological space X. Suppose $f : X \to Y$ is a quotient mapping. If a covering $\mathscr{S}$ is a minimal sub-base for the quotient space Y, then $f^{-1}(\mathscr{S}) = \{f^{-1}(S) | S \in \mathscr{S}\}$ is a minimal sub-base for the topological space X.*

Theorem 2.14. *Let X and Y be two topological spaces and $f : X \to Y$ open and one-to-one. If a covering $\mathscr{S}$ is a minimal sub-base for the topological space X, then $f(\mathscr{S}) = \{f(S) | S \in \mathscr{S}\}$ is a minimal sub-base for the subspace $f(X)$ of Y.*

Proof. Obviously, $f(\mathscr{S}) = \{f(S) | S \in \mathscr{S}\}$ is a cover of $f(X)$. Since the mapping $f : X \to Y$ is open and $\mathscr{S}$ is a sub-base for the topological space X, $f(\mathscr{S})$ is a sub-base for the topological space $f(X)$. In what follows, we show that $f(\mathscr{S})$ is a minimal sub-base for $f(X)$. Suppose $f(\mathscr{S})$ is not a minimal sub-base for $f(X)$. Then there exists $S_0 \in \mathscr{S}$ such that $\{f(S) | S \in \mathscr{S} \setminus \{S_0\}\}$ is a sub-base for $f(X)$. Since $\mathscr{S}$ is a minimal sub-base for the topological space X, $\mathscr{S} \setminus \{S_0\}$ is not a sub-base for the topological space X. Thus, there exist an open subset U_0 of X and $x \in U_0$ such that for any finite family $\mathscr{F} \subset \mathscr{S} \setminus \{S_0\}$, $\bigcap \mathscr{F} \not\subset U_0$ if $x \in \bigcap \mathscr{F}$. Because the mapping is open, $f(U_0)$ is an open subset of $f(X)$. Since f is one-to-one, $f(\bigcap \mathscr{F}) \not\subset f(U_0)$ for any finite family $\mathscr{F} \subset \mathscr{S} \setminus \{S_0\}$ with $x \in \bigcap \mathscr{F}$. Thus, $\bigcap\{f(F) | F \in \mathscr{F}\} \not\subset f(U_0)$ for any finite family $\mathscr{F} \subset \mathscr{S} \setminus \{S_0\}$ with $x \in \bigcap \mathscr{F}$. This contradicts that $f(\mathscr{S}) \setminus \{f(S_0)\}$ is a sub-base for $f(X)$. □

Some mapping properties have been presented in the theorems above. Next some operational properties will be proposed.

Theorem 2.15. *Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a family of topological spaces. For each index $\gamma \in \Gamma$, $\pi_\gamma : \Pi_{\gamma\in\Gamma} X_\gamma \to X_\gamma$ is a projective mapping. If for each index $\gamma \in \Gamma$, $\mathscr{S}_\gamma$ is a minimal sub-base for the topological space X_γ, then $\mathscr{S} = \{\pi_\gamma^{-1}(S_\gamma) | S_\gamma \subset \mathscr{S}_\gamma, \gamma \in \Gamma\}$ is a minimal sub-base for the product space $\Pi_{\gamma\in\Gamma} X_\gamma$.*

Proof. It can be seen that $\mathscr{S} = \{\pi_\gamma^{-1}(S_\gamma) | S_\gamma \subset \mathscr{S}_\gamma, \gamma \in \Gamma\}$ is a sub-base for the product space $\Pi_{\gamma\in\Gamma} X_\gamma$. Suppose $\mathscr{S}$ is not a minimal sub-base for the product space $\Pi_{\gamma\in\Gamma} X_\gamma$. Then there exist $\gamma_0 \in \Gamma$ and $S_{\gamma_0} \in \mathscr{S}_{\gamma_0}$ such that $\mathscr{S} \setminus \{\pi_{\gamma_0}^{-1}(S_{\gamma_0})\}$ is a sub-base of the product space $\prod_{\gamma\in\Gamma} X_\gamma$. Thus, $\mathscr{S}_{\gamma_0} \setminus \{S_{\gamma_0}\}$ is a sub-base of the space X_{γ_0}. This contradicts that $\mathscr{S}_{\gamma_0}$ is a minimal sub-base for X_{γ_0}. □

Theorem 2.16. *Let $\{X_\gamma\}_{\gamma\in\Gamma}$ be a family of pairwise disjoint topological spaces. For each index $\gamma \in \Gamma$, if $\mathscr{S}_\gamma$ is a minimal sub-base for a topological space (X_γ, τ_γ), then $\mathscr{S} = \{S_\gamma | S_\gamma \in \mathscr{S}_\gamma, \gamma \in \Gamma\}$ is a minimal sub-base for the sum of the spaces $\{X_\gamma\}_{\gamma\in\Gamma}$.*

Proof. Suppose $(\bigoplus_{\gamma\in\Gamma} X_\gamma, \tau)$ is the sum of the spaces $\{X_\gamma\}_{\gamma\in\Gamma}$. For each $\gamma \in \Gamma$, let $\mathscr{B}_\gamma = \{\bigcap_{S_\gamma\in\mathscr{S}'_\gamma} S_\gamma | \mathscr{S}'_\gamma$ is a finite subfamily of $\mathscr{S}_\gamma\}$ denote the base generated by $\mathscr{S}_\gamma$. Then $\mathscr{B} = \{B_\gamma | B_\gamma \in \mathscr{B}_\gamma, \gamma \in \Gamma\}$ is a base generated by $\mathscr{S}$. For each open set $U \in \tau$ and each point $x \in U$, there exists an index $\gamma \in \Gamma$ such that $x \in X_\gamma$. Then $x \in U \cap X_\gamma$. There exists a subset $\bigcap_{S_\gamma\in\mathscr{S}'_\gamma} S_\gamma \in \mathscr{B}_\gamma \subset \mathscr{B}$ such that $x \in \bigcap_{S_\gamma\in\mathscr{S}'_\gamma} S_\gamma \subset U \cap X_\gamma$, because $U \cap X_\gamma$ is open in X_γ, $x \in \bigcap_{S_\gamma\in\mathscr{S}'_\gamma} S_\gamma \subset U$ means that $\mathscr{B}$ is a base for the topological space $(\bigoplus_{\gamma\in\Gamma} X_\gamma, \tau)$. Hence, $\mathscr{S}$ is a sub-base for the topological space $(\bigoplus_{\gamma\in\Gamma} X_\gamma, \tau)$.

Suppose $\mathscr{S}$ is not a minimal sub-base for the topological space $(\bigoplus_{\gamma\in\Gamma} X_\gamma, \tau)$. Then there exist $\gamma_0 \in \Gamma$ and $S_{\gamma_0} \in \mathscr{S}_{\gamma_0}$ such that $\mathscr{S} \setminus \{S_{\gamma_0}\}$ is a sub-base of the topological space $(\bigoplus_{\gamma\in\Gamma} X_\gamma, \tau)$. Thus, $\mathscr{S}_{\gamma_0} \setminus \{S_\gamma\}$ is a sub-base of X_{γ_0}. It is a contradiction. Hance, $\mathscr{S}$ is a minimal sub-base for the topological space $(\bigoplus_{\gamma\in\Gamma} X_\gamma, \tau)$. □

Let X be a topological space and $Y \subset X$ a topological subspace of X. If $\mathscr{S}$ is a minimal sub-base for the topological space X, then $\mathscr{S}|_Y$ is a sub-base for the topological subspace Y. But $\mathscr{S}|_Y$ may not be a minimal sub-base for the subspace Y. The following example is given to show this point.

Example 2.17. Let $X = \{x_1, x_2, \ldots, x_9\}$ and $\mathscr{S} = \{\{x_1, x_2, x_4, x_5\}, \{x_2, x_3, x_5, x_6\}, \{x_4, x_5, x_7, x_8\}, \{x_5, x_6, x_8, x_9\}\}$ be a sub-base for a topological space X. It is easy to see that $\mathscr{S}$ is a minimal sub-base for the topological space X. Take $Y = \{x_1, x_2, x_4, x_5, x_7\}$. According to the definition of topological subspace, we conclude that $\mathscr{S}|_Y = \{\{x_1, x_2, x_4, x_5\}, \{x_2, x_5\}, \{x_4, x_5, x_7\}, \{x_5\}\}$ is a sub-base for the subspace Y. However, $\mathscr{S}|_Y$ is not a minimal sub-base for the subspace Y, because $\{x_5\}$ is a finite intersection reducible element with respect to $\mathscr{S}|_Y$.

3. Some Special Results in Alexandroff Spaces

From [10], a topological space X is an Alexandroff space if and only if each point in X has a unique minimal open neighborhood. Moreover, for an Alexandroff space, there exists a base $\mathscr{B}$ such that $\mathscr{B}$ is composed of all the minimal open neighborhood of each point in X. With no doubt, $\mathscr{B}$ is the unique minimal base for the Alexandroff space X. By Theorem 2.10, a minimal sub-base can be obtained from a minimal base. Then is the minimal sub-base unique for an Alexandroff space? The answer is no. A simply example is given to illustrate why.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$ and $\mathscr{S} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_1, x_3\}\}$ be a sub-base for a topological space (X, τ). Minimal sub-bases are derived for the topological space (X, τ):

$$\mathscr{S}_1 = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_2, x_4\}, \{x_1, x_3\}\},$$

$$\mathscr{S}_2 = \{\{x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_1, x_3\}\},$$

$$\mathscr{S}_3 = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}\}.$$

In fact, $\mathscr{S}_i$ is a sub-base for a topological space (X, τ_i) for $i = 1, 2, 3$. Then it is easy to check that τ and τ_i are discrete topologies for $i = 1, 2, 3$.

Although minimal sub-bases for an Alexandroff space are not unique, there are some special results which are different from general topological spaces. Let's begin with the minimal open neighborhood.

Suppose $\mathscr{P}$ is a family of subsets of X. Denote the minimal set containing x with respect to $\mathscr{P}$ by $N_{\mathscr{P}}(x) = \bigcap\{U | x \in U \in \mathscr{P}\}$.

Remark 3.2. Let X be a topological space. Suppose $\mathscr{B}$ is a base for the topological space X and $\mathscr{S}$ is a sub-base for the topological space X. One can derive $N_\tau(x) = N_{\mathscr{B}}(x) = N_{\mathscr{S}}(x)$ according to the definitions of base and sub-base. Furthermore, if (X, τ) is an Alexandroff space, then $N_\tau(x) = \bigcap\{U | x \in U \in \tau\}$ is the unique minimal open neighborhood of each point $x \in X$. Simply denote $N_\tau(x)$ by $N(x)$. And $\mathscr{B} = \{N(x) | x \in X\}$ is the unique minimal base for the Alexandroff space X [10].

By Example 2.6, there exist intersection reducible elements in a minimal base. As a special case, if a minimal base is a partition, then it does not have any intersection reducible element. When a minimal base is a partition, what conditions does it meet? There is no doubt that a minimal base is a partition for a discrete topological space. The following theorem gives a necessary and sufficient condition of a minimal base of an Alexandroff space being a partition.

Theorem 3.3. *Let X be an Alexandroff space. For two points $x, y \in X$, define xRy if $N(x) = N(y)$. Then the natural quotient space X/R is a discrete space if and only if the minimal base $\mathscr{B}$ is a partition for the Alexandroff space X.*

Proof. Let $p : X \to X/R$ be a natural quotient mapping.

Suppose X/R is a discrete space. For each point $x \in X$, $\{[x]\}$ is a singleton in X/R and $p^{-1}([x])$ is open in X. So $N(x) \subset p^{-1}([x])$. For each point $y \in p^{-1}([x])$, xRy implies $N(x) = N(y)$, i.e., $y \in N(x)$. Then $p^{-1}([x]) \subset N(x)$, so $p^{-1}([x]) = N(x)$. For any two minimal open neighborhoods $N(x), N(y) \in \mathscr{B}$ and $N(x) \neq N(y)$, if $N(x) \cap N(y) \neq \varnothing$, then there exists a point $z \in X$ such that $z \in N(x) \cap N(y)$. That is $z \in N(x)$ and $z \in N(y)$. So $z \in p^{-1}([x])$ and $z \in p^{-1}([x])$, i.e., xRz and yRz. Then xRy because R is an equivalence relation, which means $N(x) = N(y)$. Therefore, for any elements $N(x), N(y) \in \mathscr{B}$, we have $N(x) = N(y)$ or $N(x) \cap N(y) = \varnothing$, i.e., $\mathscr{B}$ is a partition.

Since $\mathscr{B}$ is a partition, $N(x) = N(y)$ if $N(x) \cap N(y) \neq \varnothing$. Then for each point $y \in N(x)$, $N(x) = N(y)$, that is xRy. Hence, $y \in p^{-1}([x])$ which means $N(x) \subset p^{-1}([x])$. If $y \in p^{-1}([x])$, then xRy which implies $N(x) = N(y)$. So $p^{-1}([x]) \subset N(x)$. Hence, $p^{-1}([x]) = N(x)$ is open in X. Since p is the natural quotient mapping, $\{[x]\}$ is open in X/R. Thus, X/R is a discrete space. □

The following theorem is about locally connected spaces and locally pathwise connected spaces.

Theorem 3.4. *Let X be an Alexandroff space.*

(1) X is a locally connected space if and only if for each point $x \in X$, the minimal open neighborhood of the point x is a connected set;

(2) X is a locally pathwise connected space if and only if for each point $x \in X$, the minimal open neighborhood of the point x is a pathwise connected set.

Proof. (1) Suppose X is a locally connected space. Then for each point $x \in X$ and each neighborhood U of the point x, there exists a connected neighborhood V such that $x \in V \subset U$. This means that the minimal open neighborhood of the point x is a connected set. Suppose for each point $x \in X$, the minimal open neighborhood $N(x)$ of the point x is a connected set. Then for each neighborhood U of the point x, $N(x) \subset U$. Hence, X is a locally connected space.

(2) The proof is similar to (1). □

For an Alexandroff space, based on the uniqueness of minimal base, two theorems about sub-base and minimal sub-base are presented. To this end, a lemma is given.

Lemma 3.5. ([10]) *If τ, τ' are two topologies on X such that X is an Alexandroff space and $N_\tau(x) = N_{\tau'}(x)$ for all $x \in X$, then $\tau = \tau'$.*

Theorem 3.6. *Let (X, τ) be an Alexandroff space. A covering $\mathscr{S}$ is a sub-base for the Alexandroff space (X, τ) if and only if the following conditions hold:*

(1) $\mathscr{S}$ is a sub-base for a topological space X, each point x in the topological space X has the minimal open neighborhood;

(2) $N_{\mathscr{S}}(x) = N(x)$ for each point $x \in X$.

Proof. Suppose $\mathscr{S}$ is a sub-base for a topological space (X, τ'). (X, τ') is an Alexandroff space, because each point x in the topological space (X, τ') has the minimal open neighborhood. Since $N_{\mathscr{S}}(x) = N(x)$ for each point $x \in X$ and (X, τ) is an Alexandroff space, by Lemma 3.5, $\tau = \tau'$. Hence, $\mathscr{S}$ is a sub-base for the Alexandroff space (X, τ).

Suppose $\mathscr{S}$ is a sub-base for the Alexandroff space (X, τ). Then each point x in the Alexandroff space (X, τ) has the minimal open neighborhood. By Remark 3.2, $N_{\mathscr{S}}(x) = N(x)$ for each point $x \in X$. □

Theorem 3.7. *Let $\mathscr{S}$ be a sub-base for an Alexandroff space (X, τ). $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ) if and only if for any covering $\mathscr{S}' \subset \mathscr{S}$, if the following conditions hold, then $\mathscr{S}' = \mathscr{S}$. The conditions are:*

(1) $\mathscr{S}'$ is a sub-base for a topological space X, each point x in the topological space X has the minimal open neighborhood;

(2) $N_{\mathscr{S}'}(x) = N(x)$ for each point $x \in X$.

Proof. Suppose a covering $\mathscr{S}' \subset \mathscr{S}$ is a sub-base for the Alexandroff space (X, τ). By Theorem 3.6, Conditions (1) and (2) hold. So $\mathscr{S}' = \mathscr{S}$. Hence $\mathscr{S}$ is a minimal sub-base for (X, τ).

Suppose $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ). For any covering $\mathscr{S}' \subset \mathscr{S}$, if Conditions (1) and (2) hold, by Theorem 3.6, then $\mathscr{S}'$ is a sub-base for the Alexandroff space (X, τ). Hence, $\mathscr{S}' = \mathscr{S}$ because $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ). □

Section 2 points out that a minimal sub-base may have infinite intersection reducible elements. As we know, the intersection of every family of open sets is open in an Alexandroff space. Then does a minimal sub-base for an Alexandroff space X have infinite intersection reducible elements?

Theorem 3.8. *Let X be an Alexandroff space and $\mathscr{S}$ a family of open sets of X. Then $\mathscr{S}$ is a minimal sub-base for the topology of X if and only if:*

(1) $\mathscr{S}$ is a sub-base for the Alexandroff space X;

(2) If a subfamily $\{S_i | i \in I\}$ of $\mathscr{S}$ satisfies $\bigcap_{i\in I} S_i \in \mathscr{S}$, then there exists $i_0 \in I$ such that $\bigcap_{i\in I} S_i = S_{i_0}$.

Proof. Suppose $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ). Assume a subfamily $\mathscr{S}_1 = \{S_i | i \in I\}$ of $\mathscr{S}$ satisfies $\bigcap_{i\in I} S_i \in \mathscr{S}$, but $\bigcap_{i\in I} S_i \notin \mathscr{S}_1$, i.e., $\bigcap_{i\in I} S_i \in \mathscr{S} \setminus \mathscr{S}_1$. Then there exists a subset $S' \in \mathscr{S} \setminus \mathscr{S}_1$ such that $\bigcap_{i\in I} S_i = S'$. For each point $x \in S'$,

$$\begin{aligned} N_{\mathscr{S}}(x) &= \bigcap\{S \in \mathscr{S} | x \in S\} \\ &= (\bigcap\{S \in \mathscr{S} \setminus \{S'\} | x \in S\}) \cap S' \\ &= (\bigcap\{S \in \mathscr{S} \setminus \{S'\} | x \in S\}) \cap (\bigcap_{i\in I} S_i) \\ &= \bigcap\{S \in \mathscr{S} \setminus \{S'\} | x \in S\} \\ &= N_{\mathscr{S}\setminus\{S'\}}(x). \end{aligned}$$

Thus, for each point $x \in X$, $N_{\mathscr{S}}(x) = N_{\mathscr{S}\setminus\{S'\}}(x)$. In addition, $\mathscr{S} \setminus \{S'\}$ is a sub-base for an Alexandroff space (X, τ'). According to Theorem 3.6, $\mathscr{S} \setminus \{S'\}$ is a sub-base for the Alexandroff space (X, τ). It is a contradiction. Hence, condition (2) holds.

We only need to prove that sub-base $\mathscr{S}$ is minimal. Suppose a subfamily $\mathscr{S}'$ of $\mathscr{S}$ is a sub-base for the Alexandroff space (X, τ), but $\mathscr{S}' \neq \mathscr{S}$. Then there exists an open set $S' \in \mathscr{S}$ such that $S' \notin \mathscr{S}'$. According to Condition (2), for any subfamily $\{S_i | i \in I\}$ of $\mathscr{S} \setminus \{S'\}$, $\bigcap_{i\in I} S_i \neq S'$. Thus, there exists a point $x \in S'$ such that $N_{\mathscr{S}'}(x) \neq N_{\mathscr{S}}(x)$. It is a contradiction. Hence, $\mathscr{S}' = \mathscr{S}$. By the definition of minimal sub-base, $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ). □

Since an Alexandroff space is a special topological space, some operational properties about minimal sub-base for an Alexandroff space should hold.

Proposition 3.9. *Let $\mathscr{S}$ be a sub-base for a topological space (X, τ) and Y a topological space. For two points $x_1, x_2 \in X$, define $x_1 R x_2$ if $N_{\mathscr{S}}(x_1) = N_{\mathscr{S}}(x_2)$. Define a mapping $p : X \to X/R$ a natural quotient mapping. Suppose a mapping $g : X/R \to Y$ is a bijection. If a mapping $f : X \to Y$ satisfies $f = g \circ p$, then the following statements hold:*
(1) For each subset $S \in \mathscr{S}$, $x_1 \in S$ implies $x_2 \in S$ for any points $x_1, x_2 \in X$ satisfying $f(x_1) = f(x_2)$;
(2) For any subsets $S_1, S_2 \in \mathscr{S}$, $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$;
(3) $f(N_{\mathscr{S}}(x)) = \bigcap\{f(S) | x \in S \in \mathscr{S}\}$ for each point $x \in X$;
(4) For each subset $S \in \mathscr{S}$, $f^{-1}(f(S)) = S$;
(5) $f^{-1}(f(N_{\mathscr{S}}(x))) = N_{\mathscr{S}}(x)$ for each point $x \in X$.

Proof. (1) First, we prove that $f(x_1) = f(x_2)$ implies $N_{\mathscr{S}}(x_1) = N_{\mathscr{S}}(x_2)$ for any points $x_1, x_2 \in X$. Since f satisfies $f = g \circ p$, $f(x_1) = f(x_2)$ implies $g(p(x_1)) = g \circ p(x_1) = g \circ p(x_2) = g(p(x_2))$ for any points $x_1, x_2 \in X$. Then $p(x_1) = p(x_2)$ because g is a bijection, i.e., $[x_1] = [x_2]$. So $x_1 R x_2$, which means $N_{\mathscr{S}}(x_1) = N_{\mathscr{S}}(x_2)$. Next, for each subset $S \in \mathscr{S}$, $x_1 \in S$ means $x_1 \in N_{\mathscr{S}}(x_1) \subset S$. $N_{\mathscr{S}}(x_1) = N_{\mathscr{S}}(x_2)$ because $f(x_1) = f(x_2)$. This means $N_{\mathscr{S}}(x_2) \subset S$. That is $x_2 \in S$.

(2) First, we prove that $f(S_1) \cap f(S_2) = \varnothing$ if $S_1 \cap S_2 = \varnothing$. By contradiction, assume $f(S_1) \cap f(S_2) \neq \varnothing$. Then there exists a point $y \in Y$ such that $y \in f(S_1) \cap f(S_2)$, i.e., $y \in f(S_1)$ and $y \in f(S_2)$. So there exist two points $x_1 \in S_1$ and $x_2 \in S_2$ such that $f(x_1) = f(x_2) = y$. By (1), $x_2 \in S_1$. Then $x_2 \in S_1 \cap S_2$. It is a contradiction. Hence, $f(S_1) \cap f(S_2) = \varnothing$. Next, we prove that if $S_1 \cap S_2 \neq \varnothing$, then $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$. Obviously, $f(S_1 \cap S_2) \subset f(S_1) \cap f(S_2)$ holds. For each point $y \in f(S_1) \cap f(S_2)$, there exist two points $x_1 \in S_1$ and $x_2 \in S_2$ such that $f(x_1) = f(x_2) = y$. By (1), $x_2 \in S_1$. Then $x_2 \in S_1 \cap S_2$, which implies $y = f(x_2) \in f(S_1 \cap S_2)$. So $f(S_1) \cap f(S_2) \subset f(S_1 \cap S_2)$. That is $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$.

(3) The proof is similar to (2).

(4) It is obvious that $S \subset f^{-1}(f(S))$ is always true. For each point $x \in f^{-1}(f(S))$, $f(x) \in f(S)$. Thus, there exists a point $x' \in S$ such that $f(x) = f(x')$. By (1), $x \in S$. So $f^{-1}(f(S)) \subset S$. That is $f^{-1}(f(S)) = S$.

(5) The proof is similar to (4). □

Lemma 3.10. ([3]) *Let X be a topological space and R an equivalence relation. If a mapping $p : X \to X/R$ is a natural quotient mapping, then a mapping g of a quotient space X/R to a topological space Y is continuous if and only if the composition $g \circ p$ is continuous.*

Theorem 3.11. *Let $\mathscr{S}$ be a sub-base for an Alexandroff space (X, τ_X) and Y a topological space. For two points $x_1, x_2 \in X$, define $x_1 R x_2$ if $N_{\mathscr{S}}(x_1) = N_{\mathscr{S}}(x_2)$. Define a mapping $p : X \to X/R$ a natural quotient mapping. Suppose a mapping $g : X/R \to Y$ is a continuous bijection. If a mapping $f : X \to Y$ is an open mapping satisfying $f = g \circ p$ and $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ_X), then $f(\mathscr{S})$ is a minimal sub-base for an Alexandroff space (Y, τ_Y).*

Proof. Because g is a continuous bijection, by Lemma 3.10, f is an open and continuous surjection. And (Y, τ_Y) is an Alexandroff space. Obviously, $f(\mathscr{S})$ is a covering of Y. By Proposition 3.9 (3),

$$f(N_{\mathscr{S}}(x)) = \bigcap\{f(S) | x \in S \in \mathscr{S}\} = \bigcap\{f(S) \in f(\mathscr{S}) | f(x) \in f(S)\} = N_{f(\mathscr{S})}(f(x))$$

for each point $x \in X$. Since $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ_X), by Theorem 3.6, $N_{\mathscr{S}}(x) = N(x)$ for each point $x \in X$ and $N_{\mathscr{S}}(x)$ is open in (X, τ_X). Then $N_{f(\mathscr{S})}(f(x)) = f(N_{\mathscr{S}}(x))$ is open in (Y, τ_Y) because f is an open mapping. It is easy to see that $N_{f(\mathscr{S})}(f(x))$ is the minimal set containing $f(x)$ for each point $x \in X$. So $N_{f(\mathscr{S})}(f(x))$ is the minimal open neighborhood of $f(x)$. Besides,

$$N_{f(\mathscr{S})}(f(x)) = f(N_{\mathscr{S}}(x)) = f(N(x)) = N(f(x))$$

for each point $x \in X$. Hence, $f(\mathscr{S})$ is a sub-base for the Alexandroff space (Y, τ_Y).

For any covering $f(\mathscr{S}') \subset f(\mathscr{S})$, suppose each point y in a topological space Y generated by $f(\mathscr{S}')$ as a sub-base has the minimal open neighborhood. And $N_{f(\mathscr{S}')}(y) = N(y)$ for each point $y \in Y$. Then $\mathscr{S}'$ is a sub-base for the Alexandroff space (X, τ_X). If not, then there exist points $x, x' \in X$ such that the minimal open neighborhood of x does not exist or $N_{\mathscr{S}}(x') \neq N(x')$. It is obvious that there exist two points $y, y' \in Y$ such that $y = f(x)$ and $y' = f(x')$. According to the proof above, we obtain that the minimal open neighborhood of y does not exist or $N_{f(\mathscr{S})}(y') \neq N(y')$. It is a contradiction. So $\mathscr{S}'$ is a sub-base for the Alexandroff space (X, τ_X). Since $\mathscr{S}$ is a minimal sub-base for the Alexandroff space (X, τ_X), by Definition 2.7, $\mathscr{S}' = \mathscr{S}$. Thus, $f(\mathscr{S}') = f(\mathscr{S})$. By Theorem 3.7, $f(\mathscr{S})$ is a minimal sub-base for the Alexandroff space (Y, τ_Y). □

4. Conclusion

For a topological space, the definition of minimal sub-base has been proposed for the first time in this paper. Moreover, the relationship between minimal base and minimal sub-base has been investigated. This paper has re-discussed the necessary and sufficient condition for the existence of minimal base for a topological space. Based on the particularity of Alexandroff space, a necessary and sufficient condition for the existence of minimal sub-base has been derived, and an approach to obtain a minimal sub-base from a covering has been given. Throughout this paper, some criteria of minimal bases and minimal sub-bases for general topological spaces have been provided.

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