Filomat 33:7 (2019), 1855–1863
https://doi.org/10.2298/FIL1907855Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Function Characterizations of Some Spaces in Which Compacta are G_δ

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Abstract. We use real-valued functions to give characterizations of some topological spaces in which compact subsets are (regular) G_δ, such as c-stratifiable spaces, kc-semi-stratifiable spaces. Also, characterizations of some other spaces such as K-semimetrizable spaces, strongly first countable spaces are obtained.

1. Introduction

Throughout, a space always means a Hausdorff topological space. For a space X, we denote by $\mathcal{C}_X$ the family of all compact subsets of X. τ and τ^c denote the topology of X and the family of all closed subsets of X respectively. For a subset A of a space X, we write $\overline{A}$ ($int(A)$) for the closure (interior) of A in X. Also, we use χ_A to denote the characteristic function of A. The set of all positive integers is denoted by $\mathbb{N}$.

A real-valued function f on a space X is called *lower* (*upper*)*semi-continuous* [2] if for any real number r, the set $\{x \in X : f(x) > r\}$ ($\{x \in X : f(x) < r\}$) is open. f is called *k-lower semi-continuous* [15] if for each $K \in \mathcal{C}_X$, f has a minimum value on K. We write $L(X)$ ($U(X)$, $KL(X)$) for the set of all lower (upper, k-lower) semi-continuous functions from X into the unit interval $[0, 1]$. $UKL(X) = U(X) \cap KL(X)$. $C(X)$ is the set of all continuous functions from X into $[0, 1]$. $F(X)$ is the set of all functions from X into $[0, 1]$.

It is known that many classes of spaces such as stratifiable spaces [5, 6], k-semi-stratifiable space [8, 15], countably paracompact spaces [9, 16], monotonically countably paracompact spaces [3] can be characterized with real-valued functions that satisfy certain conditions. In [13], to give characterizations of some generalized metric spaces, the following conditions were introduced.

Let $\mathcal{F} \subset F(X)$. For $x \in X$ and $A \subset X$, denote $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ and $\mathcal{F}(A) = \cup\{f(A) : f \in \mathcal{F}\}$. Consider the following conditions on $\mathcal{F}$.

(B) If $x \notin F \in \tau^c$, then there exists $f \in \mathcal{F}$ such that $f(x) > 0$ and $f(F) = \{0\}$.

(D) For each $x \in X$ and $\mathcal{F}' \subset \mathcal{F}$, if $\mathcal{F}'(x) \subset (a, 1]$ for some $a > 0$, then there exists an open neighborhood V of x such that $\mathcal{F}'(V) \subset (0, 1]$.

(E'') For each $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $\varepsilon > 0$, if $\mathcal{F}'(x) = \{0\}$, then there exists an open neighborhood V of x such that $\mathcal{F}'(V) \subset [0, \varepsilon)$.

(K) For each $K \in \mathcal{C}_X$, $F \in \tau^c$ with $K \cap F = \emptyset$, there exist $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(K) \subset (\frac{1}{m}, 1]$ and $f(F) = \{0\}$.

2010 *Mathematics Subject Classification*. Primary 54C30; Secondary 54C08, 54E20, 54E25, 54E99
Keywords. Real-valued functions, g-functions, c-stratifiable spaces, kc-semi-stratifiable spaces
Received: 15 May 2018; Revised: 03 March 2019; Accepted: 04 March 2019
Communicated by Ljubiša D.R. Kočinac
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(S) If $x \notin F \in \tau^c$, then there exist $f \in \mathcal{F}$, an open neighborhood V of x and $m \in \mathbb{N}$ such that $f(V) \subset (\frac{1}{m}, 1]$ and $f(F) = \{0\}$.

With these conditions, Naimpally and Pareek [13] presented characterizations of a broad class of generalized metric spaces such as first countable spaces, semi-stratifiable spaces, semi-metrizable spaces, developable spaces, stratifiable spaces and γ-spaces. For example, a space X is first countable if and only if there exists a family $\mathcal{F} \subset F(X)$ satisfying (B) and (D). X is stratifiable if and only if there exists a family $\mathcal{F} \subset F(X)$ satisfying (S) and (E'').

In [17], the first author of the present paper introduced another several conditions imposed on real-valued functions. For example.

Let $A, B \subset X$ and f_A a real-valued function on X related to A.

(e_A) $A = f_A^{-1}(0)$.

(m_A) If $A_1 \subset A_2$, then $f_{A_1} \geq f_{A_2}$.

(i_{AB}) If $A \cap B = \emptyset$, then $\inf\{f_A(x) : x \in B\} > 0$.

(i'_{AB}) If $A \cap B = \emptyset$, then there exists an open neighborhood V of B such that $\inf\{f_A(x) : x \in V\} > 0$.

With these conditions, characterizations of some generalized metric spaces were also obtained. For example, a space X is first countable if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying ($e_{\{x\}}$) and ($i_{\{x\}F}$) with $F \in \tau^c$. X is a Nagata space if and only if for each $F \in \tau^c$, there exists $f_F \in C(X)$ satisfying (e_F), (m_F) and ($i_{\{x\}F}$).

A g-function for a space X is a map $g : \mathbb{N} \times X \to \tau$ such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$ and $g(n + 1, x) \subset g(n, x)$. For a subset A of X, denote $g(n, A) = \cup\{g(n, x) : x \in A\}$.

Definition 1.1. A space X is called a c-stratifiable [7] (c-semi-stratifiable [10]) space if there is a g-function g for X such that for each $K \in C_X$, $\bigcap_{n\in\mathbb{N}} \overline{g(n, K)} = K$ ($\bigcap_{n\in\mathbb{N}} g(n, K) = K$).

Definition 1.2. ([12]) A space X is called *kc-semi-stratifiable* if there is a g-function for X such that if $K, H \in C_X$ and $K \cap H = \emptyset$, then $K \cap g(m, H) = \emptyset$ for some $m \in \mathbb{N}$.

c-stratifiable (kc-semi-stratifiable, c-semi-stratifiable) spaces are nature generalizations of stratifiable (k-semi-stratifiable, semi-stratifiable) spaces in which compact subsets are (regular) G_δ-sets. The main purpose of this paper is to give characterizations of these spaces with real-valued functions that satisfy some conditions listed above. Moreover, characterizations of some other spaces such as K-semimetrizable spaces, strongly first countable spaces are obtained.

2. The First Kind of Characterizations

In this section, we shall present characterizations of c-stratifiable spaces, kc-semi-stratifiable spaces with conditions (e_A), (m_A) and (i_{AB}) listed in section 1.

Theorem 2.1. *For a space X, the following are equivalent.*

(a) X is a c-stratifiable space.

(b) For each $K \in C_X$, there exist $f_K \in L(X)$, $h_K \in UKL(X)$ with $f_K \leq h_K$ such that f_K, h_K satisfy (e_K) and h_K satisfies (m_K).

(c) For each $K \in C_X$, there exist $f_K \in L(X)$, $h_K \in U(X)$ with $f_K \leq h_K$ such that f_K, h_K satisfy (e_K) and h_K satisfies (m_K).

Proof. (a) ⇒ (b) Let g be the g-function for a c-stratifiable space. For each $K \in C_X$, let

$$f_K = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\overline{g(n,K)}}, \quad h_K = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,K)}.$$

Then $f_K \in L(X), h_K \in U(X)$ and $f_K \leq h_K$. It is clear that if $K_1 \subset K_2$, then $h_{K_1} \geq h_{K_2}$. One readily sees that for each $K \in C_X$, $f_K(x) = 0$ if and only if $x \in K$ if and only if $h_K(x) = 0$. That is, $f_K^{-1}(0) = K = h_K^{-1}(0)$.

To show that $h_K \in KL(X)$. Let $H \in \mathcal{C}_X$.

Case 1. $H \cap K \neq \emptyset$. Choose $x_0 \in K \cap H$. Then $h_K(x_0) = 0$ and thus $h_K(x) \geq h_K(x_0)$ for each $x \in H$.

Case 2. $H \cap K = \emptyset$. Then $H \cap \bigcap_{n\in\mathbb{N}} \overline{g(n,K)} = \emptyset$. Since H is compact, it follows that $H \cap \overline{g(n,K)} = \emptyset$ for some $n \in \mathbb{N}$. Let $m = \min\{n \in \mathbb{N} : H \cap g(n,K) = \emptyset\}$. If $m = 1$, then $H \cap g(1,K) = \emptyset$ from which it follows that $h_K(x) = 1$ for each $x \in H$. If $m > 1$, then $H \cap g(m-1,K) \neq \emptyset$ and $H \cap g(n,K) = \emptyset$ for each $n \geq m$. Choose $x_0 \in H \cap g(m-1,K)$. Then $h_K(x_0) = \frac{1}{2^{m-1}}$. Let $x \in H$ and $k_x = \min\{n \in \mathbb{N} : x \notin g(n,K)\}$. Then $k_x \leq m$. Thus

$$h_K(x) = 1 - \sum_{n=1}^{k_x-1} \frac{1}{2^n} = \frac{1}{2^{k_x-1}} \geq \frac{1}{2^{m-1}} = h_K(x_0).$$

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = \{y \in X : h_{\{x\}}(y) < \frac{1}{n}\}$. Since $h_{\{x\}} \in U(X)$ and $h_{\{x\}}(x) = 0$, it follows that $g(n,x)$ is open and $x \in g(n,x)$. It is clear that $g(n+1,x) \subset g(n,x)$. Thus g is a g-function for X. For each $K \in \mathcal{C}_X$ and $n \in \mathbb{N}$, let $F(n,K) = \{y \in X : f_K(y) \leq \frac{1}{n}\}$. For each $x \in K$ and $y \in g(n,x)$, $f_K(y) \leq h_K(y) \leq h_{\{x\}}(y) < \frac{1}{n}$ which implies that $g(n,x) \subset F(n,K)$ and thus $g(n,K) \subset F(n,K)$. Since $F(n,K)$ is closed, we have that $\overline{g(n,K)} \subset F(n,K)$.

Let $K \in \mathcal{C}_X$. If $x \in \bigcap_{n\in\mathbb{N}} \overline{g(n,K)}$, then $x \in \overline{g(n,K)} \subset F(n,K)$ and thus $f_K(x) \leq \frac{1}{n}$ for each $n \in \mathbb{N}$. It follows that $f_K(x) = 0$. Hence, $x \in K$. This implies that $\bigcap_{n\in\mathbb{N}} \overline{g(n,K)} \subset K$. Since it is clear that $K \subset \bigcap_{n\in\mathbb{N}} \overline{g(n,K)}$, we have that $\bigcap_{n\in\mathbb{N}} \overline{g(n,K)} = K$. By Definition 1.1, X is a c-stratifiable space. □

Theorem 2.2. *For a space X, the following are equivalent.*

(a) X is a c-stratifiable space.

(b) For each $K \in \mathcal{C}_X$, there exists $f_K \in U(X)$ satisfying (e_K), (m_K) and (i'_{KH}) with $H \in \mathcal{C}_X$.

(c) For each $K \in \mathcal{C}_X$, there exists $f_K \in U(X)$ satisfying (e_K), (m_K) and $(i'_{K\{x\}})$.

Proof. (a) $\Rightarrow$ (b) Let g be the g-function for a c-stratifiable space. For each $K \in \mathcal{C}_X$, let

$$f_K = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\overline{g(n,K)}}.$$

Then $f_K \in U(X)$ satisfies (e_K) and (m_K).

Let $K, H \in \mathcal{C}_X$ and $K \cap H = \emptyset$, then $H \cap \overline{g(m,K)} = \emptyset$ for some $m \in \mathbb{N}$. Let $V = X \setminus \overline{g(m,K)}$. Then V is an open neighborhood of H. For each $x \in V$, $x \notin \overline{g(n,K)} \supset g(n,K)$ for all $n \geq m$. Thus

$$f_K(x) = 1 - \sum_{n=1}^{m-1} \frac{1}{2^n} \chi_{\overline{g(n,K)}}(x) \geq 1 - \sum_{n=1}^{m-1} \frac{1}{2^n} = \frac{1}{2^{m-1}}.$$

This implies that $\inf\{f_K(x) : x \in V\} > 0$.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = \{y \in X : f_{\{x\}}(y) < \frac{1}{n}\}$. Then g is a g-function for X. Let $K \in \mathcal{C}_X$. If $x \notin K$, then by $(i'_{K\{x\}})$, there exist an open neighborhood V of x and $m \in \mathbb{N}$ such that $f_K(y) > \frac{1}{m}$ for each $y \in V$. This implies that $x \notin \overline{\{y \in X : f_K(y) < \frac{1}{m}\}}$. For each $z \in K$, we have that $g(m,z) = \{y \in X : f_{\{z\}}(y) < \frac{1}{m}\} \subset \{y \in X : f_K(y) < \frac{1}{m}\}$. Thus $g(m,K) \subset \{y \in X : f_K(y) < \frac{1}{m}\}$ and so $x \notin \overline{g(m,K)}$. This implies that $\bigcap_{n\in\mathbb{N}} \overline{g(n,K)} \subset K$ and thus $\bigcap_{n\in\mathbb{N}} \overline{g(n,K)} = K$. Therefore, X is a c-stratifiable space. □

Theorem 2.3. *For a space X, the following are equivalent.*

(a) X is a kc-semi-stratifiable space.

(b) For each $K \in \mathcal{C}_X$, there exists $f_K \in UKL(X)$ satisfying (e_K) and (m_K).

(c) for each $K \in \mathcal{C}_X$, there exists $f_K \in U(X)$ satisfying (e_K), (m_K) and (i_{KH}) with $H \in \mathcal{C}_X$.

Proof. (a) ⇒ (b) Let g be the g-function for a kc-semi-stratifiable space. For each $K \in C_X$, let

$$f_K = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,K)}.$$

Then $f_K \in U(X)$ satisfies (m_K). It is clear that $K \subset f_K^{-1}(0)$. If $x \notin K$, then $\{x\} \cap g(n, K) = \emptyset$ for some $n \in \mathbb{N}$. It follows that $f_K(x) > 0$ and thus $f_K^{-1}(0) \subset K$. Consequently, $K = f_K^{-1}(0)$.

With a similar argument to that in the proof of (a) ⇒ (b) of Theorem 2.1, we can show that $f_K \in KL(X)$.

(b) ⇒ (c) Assume (b). It suffices to show that f_K satisfies (i_{KH}). Let $H \in C_X$ and $K \cap H = \emptyset$. Since $f_K \in KL(X)$, there exists $x_0 \in H$ such that $f_K(x) \geq f_K(x_0)$ for each $x \in H$. It follows that $\inf\{f_K(x) : x \in H\} \geq f_K(x_0) > 0$.

(c) ⇒ (a) For each $x \in X$ and $n \in \mathbb{N}$, let $g(n, x) = \{y \in X : f_{\{x\}}(y) < \frac{1}{n}\}$. Then g is a g-function for X. Let $K, H \in C_X$ and $K \cap H = \emptyset$. By (i_{KH}), there exists $m \in \mathbb{N}$ such that $f_K(x) > \frac{1}{m}$ for each $x \in H$. Then for each $y \in K$ and $x \in H$, $f_{\{y\}}(x) \geq f_K(x) > \frac{1}{m}$ from which it follows that $x \notin g(m, y)$. Thus $H \cap g(m, K) = \emptyset$. By Definition 1.2, X is a kc-semi-stratifiable space. □

Analogous to Theorem 2.3, we have the following result for c-semi-stratifiable spaces.

Proposition 2.4. *A space X is c-semi-stratifiable if and only if for each $K \in C_X$, there exists $f_K \in U(X)$ satisfying (e_K) and (m_K).*

3. Another Kind of Characterizations

In this section, we introduce the following conditions (B_K), (K') and (S_K) as generalizations of conditions (B), (K) and (S) listed in section 1 with which we present another several characterizations of c-stratifiable spaces, kc-semi-stratifiable spaces.

Let $\mathcal{F} \subset F(X)$. Consider the following conditions on $\mathcal{F}$.

(B_K) If $x \notin K \in C_X$, then there exists $f \in \mathcal{F}$ such that $f(x) > 0$ and $f(K) = \{0\}$.

(K') For each pair $H, K \in C_X$ with $H \cap K = \emptyset$, there exist $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(H) \subset (\frac{1}{m}, 1]$ and $f(K) = \{0\}$.

(S_K) If $x \notin K \in C_X$, then there exist $f \in \mathcal{F}$, an open neighborhood V of x and $m \in \mathbb{N}$ such that $f(V) \subset (\frac{1}{m}, 1]$ and $f(K) = \{0\}$.

Remark 3.1. By their definitions, it is clear that (K') implies (B_K). We can also show that (S_K) implies (K'). Assume (S_K). Let $H, K \in C_X$ be such that $H \cap K = \emptyset$. Then $x \notin K$ for each $x \in H$. By (S_K), there exist $f_x \in \mathcal{F}$, an open neighborhood V_x of x and $m_x \in \mathbb{N}$ such that $f_x(K) = \{0\}$, $f_x(V_x) \subset (\frac{1}{m_x}, 1]$. Since $H \in C_X$ and $H \subset \cup\{V_x : x \in H\}$, there exists a finite subset A of H such that $H \subset \cup\{V_x : x \in A\}$. Let $f = \max\{f_x : x \in A\}$ and $m = \max\{m_x : x \in A\}$. For each $x \in A$ and $y \in K$, $f_x(y) = 0$ from which it follows that $f(y) = 0$ and so $f(K) = \{0\}$. For each $y \in H$, there is $x \in A$ such that $y \in V_x$. Hence, $f(y) \geq f_x(y) > \frac{1}{m_x} \geq \frac{1}{m}$. This implies that $f(H) \subset (\frac{1}{m}, 1]$.

Theorem 3.2. *For a space X, the following are equivalent.*

(a) X is c-stratifiable.

(b) There exists a family $\mathcal{F} \subset L(X)$ satisfying (B_K) and (E'').

(c) There exists a family $\mathcal{F} \subset F(X)$ satisfying (S_K) and (E'').

Proof. (a) ⇒ (b) Let g be the g-function for a c-stratifiable space. For each $x \in X$ and $K \in C_X$, if $x \notin K$, then there exists $m \in \mathbb{N}$ such that $x \notin \overline{g(m, K)}$. Set $n_x(K) = \min\{n \in \mathbb{N} : x \notin \overline{g(n, K)}\}$. For each $K \in C_X$, define a function $f_K \in F(X)$ by letting $f_K(x) = 0$ whenever $x \in K$ and $f_K(x) = \frac{1}{n_x(K)}$ whenever $x \notin K$. It is clear that $K = f_K^{-1}(0)$.

Claim 1. For each $n \in \mathbb{N}$, $x \notin \overline{g(n, K)}$ if and only if $f_K(x) \geq \frac{1}{n}$.

Proof of Claim 1. If $x \notin \overline{g(n,K)}$, then $n_x(K) \le n$ from which it follows that $f_K(x) = \frac{1}{n_x(K)} \ge \frac{1}{n}$. Conversely, if $f_K(x) \ge \frac{1}{n}$, then $f_K(x) = \frac{1}{n_x(K)}$ from which it follows that $n_x(K) \le n$. Thus $x \notin \overline{g(n_x(K),K)} \supset \overline{g(n,K)}$.

Claim 2. For each $K \in C_X$, $f_K \in L(X)$.

Proof of Claim 2. Let $a \in [0,1)$ and $f_K(x) > a$. Then $x \notin K$. Set $O_x = X \setminus \overline{g(n_x(K),K)}$. Then O_x is an open neighborhood of x. For each $y \in O_x$, $y \notin \overline{g(n_x(K),K)}$ from which it follows that $n_y(K) \le n_x(K)$. Thus $f_K(y) = \frac{1}{n_y(K)} \ge \frac{1}{n_x(K)} = f_K(x) > a$. This implies that $f_K \in L(X)$.

Now, let $\mathcal{F} = \{f_K : K \in C_X\}$. It is clear that $\mathcal{F}$ satisfies (B_K). To show that $\mathcal{F}$ satisfies (E''), let $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $\varepsilon > 0$. Then there exist $\mathcal{A} \subset C_X$ and $m \in \mathbb{N}$ such that $\mathcal{F}' = \{f_K : K \in \mathcal{A}\}$ and $\frac{1}{m} < \varepsilon$. Suppose that $\mathcal{F}'(x) = \{0\}$. Then $f_K(x) = 0$ for each $K \in \mathcal{A}$ from which it follows that $x \in \cap\mathcal{A}$. Let $V = g(m, \cap\mathcal{A})$. Then V is an open neighborhood of x. For each $y \in V$ and each $K \in \mathcal{A}$, $y \in \overline{g(m,K)}$ and thus $f_K(y) < \frac{1}{m}$ by Claim 1. It follows that $f_K(V) \subset [0,\varepsilon)$ for each $K \in \mathcal{A}$ and thus $\mathcal{F}'(V) \subset [0,\varepsilon)$.

(b) ⇒ (c) Let $\mathcal{F}$ be the family in (b). Then we only need to show that $\mathcal{F}$ satisfies (S_K). Let $x \notin K \in C_X$. By (B_K), there exists $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(K) = \{0\}$, $f(x) > \frac{1}{m}$. Set $V = \{y \in X : f(y) > \frac{1}{m}\}$. Since $f \in L(X)$, V is an open neighborhood of x. It is clear that $f(V) \subset (\frac{1}{m}, 1]$.

(c) ⇒ (a) Assume (c). For each $x \in X$, let $\mathcal{F}_x = \{f \in \mathcal{F} : f(x) = 0\}$. By (S_K), $\mathcal{F}_x \ne \emptyset$. For each $n \in \mathbb{N}$ and $x \in X$, let $g(n,x) = int(\cap\{f^{-1}([0,\frac{1}{n})) : f \in \mathcal{F}_x\})$. Since $\mathcal{F}_x(x) = \{0\}$, it follows from (E'') that $x \in g(n,x)$. It is clear that $g(n+1,x) \subset g(n,x)$. Thus g is a g-function for X.

Let $K \in C_X$ and $x \notin K$. By (S_K), there exist $f \in \mathcal{F}$, an open neighborhood V of x and $m \in \mathbb{N}$ such that $f(K) = \{0\}$, $f(V) \subset (\frac{1}{m}, 1]$. For each $y \in K$, $f(y) = 0$ which implies that $f \in \mathcal{F}_y$. By the definition of $g(m,y)$, we have that $g(m,y) \subset f^{-1}([0,\frac{1}{m}))$. Thus $V \cap g(m,K) = \emptyset$ from which it follows that $x \notin \overline{g(m,K)}$. Consequently, $\bigcap_{n\in\mathbb{N}} \overline{g(n,K)} \subset K$. By Definition 1.1, X is a c-stratifiable space. □

Theorem 3.3. *For a space X, the following are equivalent.*

(*a*) *X is kc-semi-stratifiable.*

(*b*) *There exists a family* $\mathcal{F} \subset U(X)$ *satisfying* (K') *and* (E'').

(*c*) *There exists a family* $\mathcal{F} \subset F(X)$ *satisfying* (K') *and* (E'').

Proof. (a) ⇒ (b) Let g be the g-function for a kc-semi-stratifiable space. For each $x \in X$ and $K \in C_X$, if $x \notin K$, then there exists $m \in \mathbb{N}$ such that $x \notin g(m,K)$. Set $n_x(K) = \min\{n \in \mathbb{N} : x \notin g(n,K)\}$. For each $K \in C_X$, define a function $f_K \in F(X)$ by letting $f_K(x) = 0$ whenever $x \in K$ and $f_K(x) = \frac{1}{n_x(K)}$ whenever $x \notin K$. Then $K = f_K^{-1}(0)$.

Claim 1. For each $n \in \mathbb{N}$, $x \notin g(n,K)$ if and only if $f_K(x) \ge \frac{1}{n}$.

Proof of Claim 1. Analogous to the proof of Claim 1 in the proof of (a) ⇒ (b) of Theorem 3.2.

Claim 2. For each $K \in C_X$, $f_K \in U(X)$.

Proof of Claim 2. Let $a > 0$ and $f_K(x) < a$.

Case 1. $x \in K$. Then $f_K(x) = 0$ and thus there is $m \in \mathbb{N}$ such that $\frac{1}{m} < a$. Let $V = g(m,K)$. Then V is an open neighborhood of x and $f_K(y) < \frac{1}{m} < a$ for each $y \in V$.

Case 2. $x \notin K$. Then $f_K(x) = \frac{1}{n_x(K)} < a$. Case 2.1. $n_x(K) = 1$. Let $V = X$. Then $f_K(y) \le 1 < a$ for each $y \in V$. Case 2.2. $n_x(K) > 1$. Let $V = g(n_x(K) - 1, K)$. Then V is an open neighborhood of x. For each $y \in V$, if $y \in K$, then $f_K(y) = 0 < a$. If $y \notin K$, then $n_x(K) - 1 < n_y(K)$ and thus $n_x(K) \le n_y(K)$. It follows that $f_K(y) = \frac{1}{n_y(K)} \le \frac{1}{n_x(K)} < a$.

By the above argument, we see that $f_K \in U(X)$ for each $K \in C_X$.

Now, let $\mathcal{F} = \{f_K : K \in C_X\}$. With a similar argument to the proof of (a) ⇒ (b) of Theorem 3.2, we can show that $\mathcal{F}$ satisfies (E''). To show that $\mathcal{F}$ satisfies (K'), let $H, K \in C_X$ with $H \cap K = \emptyset$. Then $f_K(K) = \{0\}$. By definition 1.2, $H \cap g(m,K) = \emptyset$ for some $m \in \mathbb{N}$. For each $x \in H$, $x \notin g(m,K)$ from which it follows that $f_K(x) \ge \frac{1}{m}$. This implies that $f_K(H) \subset (\frac{1}{m+1}, 1]$.

(b) ⇒ (c) is clear.

(c) ⇒ (a) Assume (c). Define a g-function g for X as that in the proof of (c) ⇒ (a) of Theorem 3.2. Let $H, K \in C_X$ with $H \cap K = \emptyset$. By (K'), there exist $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(H) \subset (\frac{1}{m}, 1]$ and $f(K) = \{0\}$. For each $x \in K$, $f(x) = 0$ which implies that $f \in \mathcal{F}_x$. By the definition of $g(m,x)$, we have that

$g(m, x) \subset f^{-1}([0, \frac{1}{m}))$. Thus $g(m, K) \subset f^{-1}([0, \frac{1}{m}))$ from which it follows that $H \cap g(m, K) = \emptyset$. By Definition 1.2, X is a kc-semi-stratifiable space. □

The proof of the following result for c-semi-stratifiable spaces is similar to the proof of Theorem 3.3.

Proposition 3.4. *For a space X, the following are equivalent.*
(*a*) *X is c-semi-stratifiable.*
(*b*) *There exists a family $\mathcal{F} \subset U(X)$ satisfying (B_K) and (E'').*
(*c*) *There exists a family $\mathcal{F} \subset F(X)$ satisfying (B_K) and (E'').*

4. Some Other Spaces

In this section, we introduce another several conditions such as (D') and (wB) so as to obtain characterizations of some generalized metric spaces other than those in [13]. First, we give a characterization of stratifiable spaces which improves a corresponding result for stratifiable spaces in [13].

Lemma 4.1. ([1]) *A space X is stratifiable if and only if for each $F \in \tau^c$, there exists $f_F \in C(X)$ satisfying (e_F) and (m_F).*

Theorem 4.2. *For a space X, the following are equivalent.*
(*a*) *X is stratifiable.*
(*b*) *There exists a family $\mathcal{F} \subset C(X)$ satisfying (S) and (E'').*
(*c*) *There exists a family $\mathcal{F} \subset C(X)$ satisfying (K) and (E'').*
(*d*) *There exists a family $\mathcal{F} \subset L(X)$ satisfying (B) and (E'').*

Proof. (a) ⇒ (b) Since X is stratifiable, by Lemma 4.1, for each $F \in \tau^c$, there exists $f_F \in C(X)$ satisfying (e_F) and (m_F). Let $\mathcal{F} = \{f_F : F \in \tau^c\}$. To show that $\mathcal{F}$ satisfies (S), let $x \notin F \in \tau^c$. By (e_F), $f_F(F) = \{0\}$ and $f_F(x) > 0$. Then there exists $m \in \mathbb{N}$ such that $f_F(x) > \frac{1}{m}$. Let $V = \{y \in X : f_F(y) > \frac{1}{m}\}$. Then V is an open neighborhood of x and it is clear that $f_F(V) \subset (\frac{1}{m}, 1]$.

To show that $\mathcal{F}$ satisfies (E''), let $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $\varepsilon > 0$. Then there exist $\mathcal{A} \subset \tau^c$ and $m \in \mathbb{N}$ such that $\mathcal{F}' = \{f_F : F \in \mathcal{A}\}$ and $\frac{1}{m} < \varepsilon$. Suppose that $\mathcal{F}'(x) = \{0\}$. Then $f_F(x) = 0$ for each $F \in \mathcal{A}$ from which it follows that $x \in \cap\mathcal{A} \in \tau^c$. Let $V = \{y \in X : f_{\cap\mathcal{A}}(y) < \frac{1}{m}\}$. Then V is an open neighborhood of x. By (m_F), for each $y \in V$ and each $F \in \mathcal{A}$, $f_F(y) \leq f_{\cap\mathcal{A}}(y) < \frac{1}{m}$. This implies that $f_F(V) \subset [0, \frac{1}{m}) \subset [0, \varepsilon)$ for each $F \in \mathcal{A}$ and thus $\mathcal{F}'(V) \subset [0, \varepsilon)$.

(b) ⇒ (c) We only need to show that (S) implies (K). This can be done with a similar argument to that in Remark 3.1.

(c) ⇒ (d) is clear.

(d) ⇒ (a) For each $x \in X$, let $\mathcal{F}_x = \{f \in \mathcal{F} : f(x) = 0\}$. By (B), $\mathcal{F}_x \neq \emptyset$. For each $n \in \mathbb{N}$ and $x \in X$, let $g(n, x) = int(\cap\{f^{-1}([0, \frac{1}{n})) : f \in \mathcal{F}_x\})$. Then g is a g-function for X.

Let $F \in \tau^c$ and $x \notin F$. By (B), there exist $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(x) > \frac{1}{m}$ and $f(F) = \{0\}$. Set $V = \{y \in X : f(y) > \frac{1}{m}\}$. Since $f \in L(X)$, V is an open neighborhood of x. For each $y \in V$, $f(y) = 0$ which implies that $f \in \mathcal{F}_y$. By the definition of $g(m, y)$, we have that $g(m, y) \subset f^{-1}([0, \frac{1}{m}))$. Thus $V \cap g(m, F) = \emptyset$ from which it follows that $x \notin \overline{g(m, F)}$. Consequently, $\bigcap_{n \in \mathbb{N}} \overline{g(n, F)} \subset F$. Therefore, X is a stratifiable space. □

A function $d : X \times X \to [0, \infty)$ is called a symmetric on X if (1) $d(x, y) = 0$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$ for all $x, y \in X$. A space X is called semi-metrizable [14] if there is a symmetric on X such that for each $x \in X$, $\{B(x, r) : r > 0\}$ is a neighborhood base of x, where $B(x, r) = \{y \in X : d(x, y) < r\}$. X is called K-semimetrizable [11] if there is a semi-metric d on X such that $d(H, K) > 0$ for every disjoint pair H, K of nonempty compact subsets of X. It was shown that [13] X is semi-metrizable space if and only if there exists a family $\mathcal{F} \subset F(X)$ satisfying (B), (D) and (E''). As for K-semimetrizable spaces, we have the following.

Theorem 4.3. *A space X is K-semimetrizable if and only if there exists a family $\mathcal{F} \subset F(X)$ satisfying (B), (D), (E'') and (K').*

Proof. Let d be a K-semimetric on X which is bounded by 1. For each $F \in \tau^c$, define a function $f_F \in F(X)$ by letting $f_F(x) = d(x, F)$ for each $x \in X$. It is clear that $F = f_F^{-1}(0)$. Let $\mathcal{F} = \{f_F : F \in \tau^c\}$. Then $\mathcal{F} \subset F(X)$ satisfies (B).

Let $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and suppose that $\mathcal{F}'(x) \subset (a, 1]$ for some $a > 0$. Then there exist $\mathcal{A} \subset \tau^c$ and $n \in \mathbb{N}$ such that $\mathcal{F}' = \{f_F : F \in \mathcal{A}\}$ and $\frac{1}{n} < a$. Thus $f_F(x) > \frac{1}{n}$ for each $F \in \mathcal{A}$. Let $V = int(B(x, \frac{1}{n}))$. Then V is an open neighborhood of x. For each $y \in V$ and $F \in \mathcal{A}$, $d(x, y) < \frac{1}{n}$ from which it follows that $y \notin F$ (If $y \in F$, then $f_F(x) = d(x, F) \leq d(x, y) < \frac{1}{n}$, a contradiction) and so $f_F(y) > 0$. This implies that $\mathcal{F}'(V) \subset (0, 1]$. Hence, $\mathcal{F}$ satisfies (D).

Let $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $\varepsilon > 0$. Then there exist $\mathcal{A} \subset \tau^c$ and $m \in \mathbb{N}$ such that $\mathcal{F}' = \{f_F : F \in \mathcal{A}\}$ and $\frac{1}{m} < \varepsilon$. Suppose that $\mathcal{F}'(x) = \{0\}$. Then $x \in F$ for each $F \in \mathcal{A}$. Let $V = int(B(x, \frac{1}{m}))$. Then V is an open neighborhood of x. For each $y \in V$ and each $F \in \mathcal{A}$, $f_F(y) = d(y, F) \leq d(x, y) < \frac{1}{m}$. This implies that $f_F(V) \subset [0, \frac{1}{m}) \subset [0, \varepsilon)$ for each $F \in \mathcal{A}$ and thus $\mathcal{F}'(V) \subset [0, \varepsilon)$. Hence, $\mathcal{F}$ satisfies (E'').

Now, let $K, H \in \mathcal{C}_X$ and $K \cap H = \emptyset$. Then $d(K, H) > 0$ from which it follows that there exists $m \in \mathbb{N}$ such that $d(x, y) > \frac{1}{m}$ for each $x \in K$ and $y \in H$. Thus $f_K(y) = d(y, K) \geq \frac{1}{m}$ for each $y \in H$ which implies that $f_K(H) \subset (\frac{1}{m+1}, 1]$. Clearly, $f_K(K) = \{f_K(x) : x \in K\} = \{0\}$. This shows that $\mathcal{F}$ satisfies (K').

Conversely, for each $x \in X$, let $\mathcal{F}_x = \{f \in \mathcal{F} : f(x) = 0\}$. For each $n \in \mathbb{N}$ and $x \in X$, let $h(n, x) = int(\cap\{f^{-1}([0, \frac{1}{n})) : f \in \mathcal{F}_x\})$. By (E''), h is a g-function for X. For each $n \in \mathbb{N}$ and $x \in X$, let $\mathcal{G}_{nx} = \{f \in \mathcal{F} : f(x) \geq \frac{1}{n}\}$. Then let $e(n, x) = X$ whenever $\mathcal{G}_{nx} = \emptyset$ and $e(n, x) = int(\cap\{f^{-1}((0, 1]) : f \in \mathcal{G}_{nx}\})$ otherwise. Then $x \in e(n, x)$ by (D). Let $g(n, x) = h(n, x) \cap \cap_{i \leq n} e(i, x)$. Then g is a g-function for X.

By (B), for each $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(x) > \frac{1}{m}$ and $f(y) = 0$. Then $e(m, x) \subset f^{-1}((0, 1])$ and thus $y \notin e(m, x) \supset g(m, x)$. Also, $h(m, y) \subset f^{-1}([0, \frac{1}{m}))$ and thus $x \notin h(m, y) \supset g(m, y)$. Let $m(x, y) = \min\{n \in \mathbb{N} : y \notin g(n, x) \textit{ and } x \notin g(n, y)\}$. Define a function $d : X \times X \to [0, \infty)$ by letting $d(x, y) = 0$ whenever $x = y$ and $d(x, y) = \frac{1}{m(x,y)}$ whenever $x \neq y$. It is easy to verify that $y \in B(x, \frac{1}{n})$ if and only if $x \in g(n, y)$ or $y \in g(n, x)$.

Claim 1. $B(x, r)$ is a neighborhood of x for each $x \in X$ and $r > 0$.

Proof of Claim 1. Let $r > 0$ and choose $m \in \mathbb{N}$ such that $\frac{1}{m} < r$. Then $g(m, x) \subset B(x, \frac{1}{m}) \subset B(x, r)$. This implies that $B(x, r)$ is a neighborhood of x.

Claim 2. $\{B(x, r) : r > 0\}$ is a neighborhood base of x for each $x \in X$.

Proof of Claim 2. Let $x \in U \in \tau$. By (B), there exist $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(x) > \frac{1}{m}$ and $f(y) = 0$ for each $y \in X \setminus U$. Then $h(m, y) \subset f^{-1}([0, \frac{1}{m}))$ and thus $x \notin h(m, y) \supset g(m, y)$. Also, $e(m, x) \subset f^{-1}((0, 1])$ and thus $y \notin e(m, x) \supset g(m, x)$. As a result, $y \notin B(x, \frac{1}{m})$. This implies that $B(x, \frac{1}{m}) \subset U$.

By Claim 1 and Claim 2, d is a semi-metric on X.

Now, let $K, H \in \mathcal{C}_X$ and $K \cap H = \emptyset$. By (K'), there exist $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(K) = \{0\}$ and $f(H) \subset (\frac{1}{m}, 1]$. Thus $f(x) = 0$ and $f(y) > \frac{1}{m}$ for each $x \in K$, $y \in H$. It follows that $x \notin e(m, y) \supset g(m, y)$ and $y \notin h(m, x) \supset g(m, x)$. Hence, $d(x, y) = \frac{1}{m(x,y)} \geq \frac{1}{m}$. This implies that $d(K, H) > 0$.

Consequently, X is a K-semimetrizable space. □

A space X is called strongly first countable [4] if there exists a g-function g for X such that for each $x \in X$, $\{g(n, x) : n \in \mathbb{N}\}$ is a neighborhood base of x and if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$. To give a characterization of strongly first countable spaces, we introduce the following condition.

(D') For each $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $n \in \mathbb{N}$, if $\mathcal{F}'(x) \subset [\frac{1}{n}, 1]$, then there exists an open neighborhood V of x such that $\mathcal{F}'(V) \subset [\frac{1}{n}, 1]$.

Theorem 4.4. *A space X is strongly first countable if and only if there exists a family $\mathcal{F}$ satisfying (B) and (D').*

Proof. Let g be the g-function for a strongly first countable space. Let $x \in X$ and $F \in \tau^c$. If $x \notin F$, then there exists $m \in \mathbb{N}$ such that $g(m, x) \cap F = \emptyset$. Set $n_x(F) = \min\{n \in \mathbb{N} : g(n, x) \cap F = \emptyset\}$. For each $F \in \tau^c$, define a function $f_F \in F(X)$ by letting $f_F(x) = 0$ whenever $x \in F$ and $f_F(x) = \frac{1}{n_x(F)}$ whenever $x \notin F$. Then $F = f_F^{-1}(0)$. It is easy to verify that for each $n \in \mathbb{N}$, $g(n,x) \cap F = \emptyset$ if and only if $f_F(x) \geq \frac{1}{n}$.

Let $\mathcal{F} = \{f_F : F \in \tau^c\}$. Then $\mathcal{F}$ satisfies (B). To show that $\mathcal{F}$ satisfies (D'), let $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $n \in \mathbb{N}$. Then there exists $\mathcal{A} \subset \tau^c$ such that $\mathcal{F}' = \{f_F : F \in \mathcal{A}\}$. Suppose that $\mathcal{F}'(x) \subset [\frac{1}{n}, 1]$. Then $f_F(x) \geq \frac{1}{n}$ and thus $g(n,x) \cap F = \emptyset$ for each $F \in \mathcal{A}$. Let $V = g(n,x)$. Then for each $y \in V$, $g(n,y) \subset g(n,x)$ from which it follows that $g(n,y) \cap F = \emptyset$ and thus $f_F(y) \geq \frac{1}{n}$ for each $F \in \mathcal{A}$. This implies that $\mathcal{F}'(V) \subset [\frac{1}{n}, 1]$.

Conversely, for each $n \in \mathbb{N}$ and $x \in X$, let $\mathcal{G}_{nx} = \{f \in \mathcal{F} : f(x) \geq \frac{1}{n}\}$. Then let $h(n,x) = X$ whenever $\mathcal{G}_{nx} = \emptyset$ and $h(n,x) = int(\cap\{f^{-1}([\frac{1}{n}, 1]) : f \in \mathcal{G}_{nx}\})$ otherwise. Then $x \in h(n,x)$ by (D'). Now, for each $n \in \mathbb{N}$ and $x \in X$, let $g(n,x) = \cap_{i \leq n} h(i,x)$. Then g is a g-function for X.

Let $x \in U \in \tau$. By (B), there exists $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(x) \geq \frac{1}{m}$ and $f^{-1}((0,1]) \subset U$. Then $f \in \mathcal{G}_{mx}$ and thus $g(m,x) \subset h(m,x) \subset f^{-1}([\frac{1}{m}, 1]) \subset f^{-1}((0,1]) \subset U$.

Now, let $y \in g(n,x) = \cap_{i \leq n} h(i,x)$. For each $i \leq n$, if $h(i,x) = X$, then $h(i,y) \subset h(i,x)$. If $h(i,x) = int(\cap\{f^{-1}([\frac{1}{i}, 1]) : f \in \mathcal{G}_{ix}\})$, from $y \in h(i,x)$ it follows that $\mathcal{G}_{ix} \subset \mathcal{G}_{iy}$ and thus $h(i,y) \subset h(i,x)$. This implies that $g(n,y) \subset g(n,x)$. Consequently, X is strongly first countable. □

A space X is called an α-spaces [4] if there exists a g-function g for X such that $\{x\} = \bigcap_{n \in \mathbb{N}} g(n,x)$ for each $x \in X$ and if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$.

(wB) If $x \neq a$, then there exists $f \in \mathcal{F}$ such that $f(x) > 0$ and $f(a) = 0$.

Theorem 4.5. *A space X is an α-space if and only if there exists a family $\mathcal{F}$ satisfying (wB) and (D').*

Proof. Let g be the g-function for an α-space. Let $x, a \in X$. If $x \neq a$, then there exists $m \in \mathbb{N}$ such that $a \notin g(m,x)$. Set $n_x(a) = \min\{n \in \mathbb{N} : a \notin g(n,x)\}$. For each $a \in X$, define a function $f_a \in F(X)$ by letting $f_a(x) = 0$ whenever $x = a$ and $f_a(x) = \frac{1}{n_x(a)}$ whenever $x \neq a$. Then $\{a\} = f_a^{-1}(0)$. It is easy to verify that for each $n \in \mathbb{N}$, $a \notin g(n,x)$ if and only if $f_a(x) \geq \frac{1}{n}$.

Let $\mathcal{F} = \{f_a : a \in X\}$. Then $\mathcal{F}$ satisfies (wB). To show that $\mathcal{F}$ satisfies (D'), let $x \in X$, $\mathcal{F}' \subset \mathcal{F}$ and $n \in \mathbb{N}$. Then there exists $A \subset X$ such that $\mathcal{F}' = \{f_a : a \in A\}$. Suppose that $\mathcal{F}'(x) \subset [\frac{1}{n}, 1]$. Then $f_a(x) \geq \frac{1}{n}$ for each $a \in A$. It follows that $a \notin g(n,x)$ for each $a \in A$. Let $V = g(n,x)$. Then for each $y \in V$, $g(n,y) \subset g(n,x)$ from which it follows that $a \notin g(n,y)$ for each $a \in A$. Thus $f_a(y) \geq \frac{1}{n}$ for each $a \in A$. This implies that $\mathcal{F}'(V) \subset [\frac{1}{n}, 1]$.

Conversely, define a g-function g for X as that in the proof of Theorem 4.4.

Let $x, a \in X$ and $x \neq a$. By (wB), there exists $f \in \mathcal{F}$ and $m \in \mathbb{N}$ such that $f(x) \geq \frac{1}{m}$ and $f(a) = 0$. Then $f \in \mathcal{G}_{mx}$ and thus $g(m,x) \subset h(m,x) \subset f^{-1}([\frac{1}{m}, 1])$. It follows that $a \notin g(m,x)$.

With a similar argument to that in the proof of Theorem 4.4, we can show that if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$. Consequently, X is an α-space. □

Acknowledgements

The authors would like to thank the referee for the valuable comments and suggestions.

References

[1] C.J.R. Borges, On stratifiable spaces, Pacific J. Math. 17 (1966) 1–16.
[2] R. Engelking, General Topology, Revised and Completed Edition, Heldermann Verlag, Berlin, 1989.
[3] C. Good, L. Haynes, Monotone versions of countable paracompactness, Topol. Appl. 154 (2007) 734–740.
[4] R.E. Hodel, Spaces defined by sequence of open covers which guarantee that certain sequences have cluster points, Duke Math. J. 39 (1972) 253–263.
[5] Y. Jin, L. Xie, H. Yue, Monotone insertion of semi-continuous functions on stratifiable spaces, Filomat 31 (2017) 575–584.
[6] E. Lane, P. Nyikos, C. Pan, Continuous function characterizations of stratifiable spaces, Acta Math. Hungar. 92 (2001) 219–231.
[7] K. B. Lee, Spaces in which compacta are uniformly regular G_δ, Pacific J. Math. 81 (1979) 435–446.
[8] K. Li, S. Lin, R. Shen, Insertions of k-semi-stratifiable spaces by semi-continuous functions, Stud. Sci. Math. Hung. 48 (2011) 320–330.
[9] J. Mack, Countable paracompactness and weak normality properties, Trans. Amer. Math. Soc. 148 (1970) 265–272.

[10] H. Martin, Metrizability of M-spaces, Canad. J. Math. 4 (1973) 840–841.
[11] H. W. Martin, Local connectedness in developable spaces, Pacific J. Math. 61 (1975) 219–224.
[12] H.W. Martin II, Metrization and submetrization of topological spaces, Ph D thesis, University of Pittsburgh, 1973.
[13] S.A. Naimpally, C.M. Pareek, Characterizations of metric and generalized metric spaces by real valued functions, Q & A in General Topology, 8 (1990) 51–59.
[14] W.A. Wilson, On semi-metric spaces, Amer. J. Math. 53 (1931) 361–373.
[15] P. Yan, E. Yang, Semi-stratifiable spaces and the insertion of semi-continuous functions, J. Math. Anal. Appl. 38 (2007) 429–437.
[16] E. Yang, Properties defined with semi-continuous functions and some related spaces, Houston J. Math. 41 (2015) 1097–1106.
[17] E. Yang, Real-valued functions and some related spaces, Top. Appl. 238 (2018) 76–89.