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Ideals in Bounded Equality Algebras

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Abstract. In this paper, the concept of ideal in bounded equality algebras is introduced. With respect to this concepts, some related results are given. In particular, we prove that there is an one-to-one corresponding between congruence relation on an involutive equality algebra and the set of ideals on it. Also, we prove the first isomorphism theorem on equality algebras. Moreover, the notions of prime and Boolean ideals in equality algebras are introduced. Finally, we prove that ideal I of involutive prelinear equality algebra E is a Boolean ideal if and only if $\frac{E}{I}$ is a Boolean algebra.

1. Introduction

Fuzzy type theory (FTT) has been developed by Novák as a fuzzy logic of higher order, the fuzzy version of the classical type theory of the classical logic of higher order. BL-algebras, MTL-algebras, MV-algebras are the best known classes of residuated lattices [3, 4] and since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra [7] by Novák and De Baets. EQ-algebras generalize the residuated lattices that have three binary operations meet, multiplication, fuzzy equality and a unit element. If the product operation in EQ-algebras is replaced by another binary operation smaller or equal than the original product we still obtain an EQ-algebra, and this fact might make it difficult to obtain certain algebraic results. For this reason, equality algebras were introduced by Jenei [5], which the motivation cames from EQ-algebras [7]. These algebras are assumed for a possible algebraic semantics of fuzzy type theory. It was proved [2, 5], that any equality algebra has a corresponding BCK- meet-semilattice and any BCK(D)-meet-semilattice (with distributivity property) has a corresponding equality algebra. Filter theory plays an important role in studying these algebras and some types of filters such as (positive) implicative, fantastic, Boolean and prime filters in equality algebras are introduced by Borzooei et al [1]. This motivates us to introduce the notion of ideals in equality algebras and investigate the relations ideals, and the other types of ideals in bounded equality algebras. In particular, we prove that there is an one-to-one corresponding between congruence relation on an involutive equality algebra and the set of ideals on it. Also, we prove the first isomorphism theorem on involutive equality algebras. Moreover, the notions of prime and Boolean ideals in equality algebras are introduced. Finally, we prove that ideal I of involutive prelinear equality algebra E is a Boolean ideal if and only if $\frac{E}{I}$ is a Boolean algebra.

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2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1. *[5] An algebra* $(E, \wedge, \sim, 1)$ *of the type* $(2, 2, 0)$ *is called an* equality algebra *if it satisfies the following conditions, for all* $x, y, z \in E$:
(E1) $(E, \wedge, 1)$ *is a meet-semilattice with top element 1,*
(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ *implies* $x \sim z \leq y \sim z$ *and* $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called *meet (infimum)* and $\sim$ is an *equality operation*. We write $x \leq y$ if and only if $x \wedge y = x$, for all $x, y \in E$. Also, other two operations are defined, called *implication* and *equivalence operation*, respectively:

$$x \to y = x \sim (x \wedge y) \tag{I}$$

$$x \leftrightarrow y = (x \to y) \wedge (y \to x) \tag{II}$$

An equality algebra $(E, \sim, \wedge, 1)$ is *bounded* if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. In a bounded equality algebra E, we define the negation " $-$ " on E by, $x^- = x \to 0 = x \sim 0$, for all $x \in E$. If $x^{--} = x$, for all $x \in E$, then the bounded equality algebra E is called *involutive*. Equality algebra E is called *prelinear*, if 1 is the unique upper bound of the set $\{x \to y, y \to x\}$, for all $x, y \in E$. A *lattice equality algebra* is an equality algebra which is a lattice.

The following propositions provide some properties of equality algebras.

Proposition 2.1. [5] *Let* $(E, \wedge, \sim, 1)$ *be an equality algebra. Then the following properties hold, for all* $x, y, z \in E$:
$(E8)$ $x \to y = 1$ *if and only if* $x \leq y$,
$(E9)$ $1 \to x = x, x \to 1 = 1, x \to x = 1$,
$(E10)$ $x \leq y \to x$,
$(E11)$ $x \leq (x \to y) \to y$,
$(E12)$ $x \to y \leq (y \to z) \to (x \to z)$,
$(E13)$ $x \sim y \leq x \leftrightarrow y \leq x \to y$,
$(E14)$ $x \to (y \to z) = y \to (x \to z)$.

Proposition 2.2. [8] *Let* $(E, \wedge, \sim, 1)$ *be an equality algebra. Then for all* $x, y, z \in E$, *the following statements hold:*
$(E15)$ $x \leq y$ *implies* $y \to z \leq x \to z, z \to x \leq z \to y$,
$(E16)$ $x \to y = x \to (x \wedge y)$,
$(E17)$ $x \to y \leq (z \to x) \to (z \to y)$,
$(E18)$ $x \to y \leq (x \wedge z) \to (y \wedge z)$.

Proposition 2.3. [8] *Let* E *be a bounded lattice equality algebra. Then, for all* $x, y \in E$, *the following statements hold:*
$(E19)$ $x \leq x^{--}$,
$(E20)$ $(x \vee y)^- = x^- \wedge y^-$.

Theorem 2.1. [8] *Any prelinear equality algebra is a distributive lattice.*

Definition 2.2. *[6] Let* $(E, \sim, \wedge, 1)$ *be an equality algebra and* F *be a non-empty subset of* E.
Then F *is called a* deductive system *or* filter *of* E, *if for all* $x, y \in E$, *we have*
(i) $1 \in F$,
(ii) *if* $x \in F$ *and* $x \leq y$, *then* $y \in F$,
(iii) *if* $x \in F$ *and* $x \sim y \in F$, *then* $y \in F$.

Proposition 2.4. [2, 6] *Let $(E, \sim, \wedge, 1)$ be an equality algebra and F be a non-empty subset of E. Then F is a filter of E if and only if, for all $x, y \in E$,*
(i) $1 \in F$,
(ii) *if x and $x \to y \in F$, then $y \in F$.*

Definition 2.3. *[1] A proper filter F of E is called a* prime filter *if $x \to y \in F$ or $y \to x \in F$, for all $x, y \in E$.*

Definition 2.4. *[1] Let E be a bounded equality lattice. A filter F of E is called a* Boolean filter *if, for all $x \in E$, $x \vee x^- \in F$.*

Definition 2.5. *[6] Let $(E, \wedge, \sim, 1)$ be an equality algebra. A subset θ of $E \times E$ is called a* congruence relation *of E, if it is an equivalence relation on E and, for all $x, y, z, w \in E$ such that $(x, z), (y, w) \in \theta$, it holds that $(x \wedge y, z \wedge w) \in \theta$ and $(x \sim y, z \sim w) \in \theta$. Denote $Con(E)$, the set of all congruence relations of E.*

From now on, in this paper $(E, \wedge, \sim, 1)$ (or simply E) is an equality algebra, unless otherwise stated.

3. Ideals in Bounded Equality Algebras

In this section we introduce concept of ideals in bounded equality algebras and we give some related results.

Definition 3.1. *Let E be a bounded equality algebra and I a nonempty subset of E. Then I is called an ideal of E, if it satisfies:*
(i) if $x \leq y$ and $y \in I$, then $x \in I$, for every $x, y \in E$,
(ii) for every $x, y \in I$, $x^- \to y \in I$.

Denote $Id(E)$ the set of all ideals of a bounded equality algebra E.

Example 3.1. *[1] Let $(E = \{0, a, b, c, d, 1\}, \leq)$ be a lattice with the following diagram. Define the operations $\sim$ and $\to$ on E by*

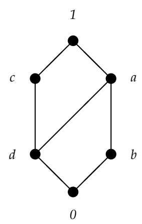

Table 1.

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

Table 2.

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then, by routine calculations, we can see that $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra. Now, let $I = \{0, b\}$. Then I is an ideal of E.

Theorem 3.1. *Let I be a nonempty subset of bounded equality algebra E. Then I is an ideal of E if and only if it satisfies:*
(i) $0 \in I$,
(ii) For every $x, y \in E$, if $x \in I$ and $(x^- \to y^-)^- \in I$, then $y \in I$.

Proof. Let I be an ideal of bounded equality algebras E. Since I is a nonempty subset of E, there exists $x \in E$, such that $x \in I$ and since E is a bounded equality algebra, we have $0 \leq x$. Hence, by Definition 3.1, $0 \in I$. Now, let $x, (x^- \to y^-)^- \in I$, for $x, y \in E$. Then by Definition 3.1, $(x^- \to (x^- \to y^-)^-) \in I$ and by $(E14)$, $((x^- \to y^-) \to x^{--}) \in I$. Since by $(E14)$,

$$\begin{aligned} y \to ((x^- \to y^-) \to x^{--}) &= (x^- \to y^-) \to (y \to x^{--}) \\ &= (x^- \to y^-) \to (x^- \to y^-) \\ &= 1 \end{aligned}$$

By $(E8)$, $y \leq (x^- \to y^-) \to x^{--}$ and since $((x^- \to y^-) \to x^{--}) \in I$, we get that $y \in I$ by Definition 3.1.
Conversely, let $x \leq y$ and $y \in I$, for $x, y \in E$. Then by $(E15)$, $y^- \leq x^-$ and so $(y^- \to x^-)^- = 0 \in I$ and since $y \in I$, by (ii), we get that $x \in I$. Note that by $(E11)$, $x \leq x^{--}$ and if $x^{--} \in I$ and I is an ideal of E, then $x \in I$. Now, let $x \in I$. Then by $(E8)$, $(E9)$ and $(E11)$, $(x^- \to x^{---})^- = 1^- = 0 \in I$, and so $x^{--} \in I$. Now, let $x, y \in I$. Then by $(E11)$, $(E12)$ and $(E15)$,

$$(y^- \to (x^- \to y)^-)^- \leq ((x^- \to y) \to y)^- \leq x^{--}$$

and since $x \in I$, we get that $x^{--} \in I$ and so $(y^- \to (x^- \to y)^-)^- \in I$. Hence, by (ii), $x^- \to y \in I$. □

Lemma 3.1. *In any bounded equality algebra E, for all $x, y \in E$,*
(i) $x^- = x^{---}$.
(ii) $(x^- \to y^-)^{--} = x^- \to y^-$.

Proof. (i) Let $x \in E$. Then by $(E11)$, $x \leq x^{--}$ and so by $(E15)$, $x^{---} \leq x^-$. Now, by $(E14)$,

$$x^- \to x^{---} = x^{--} \to x^{--} = 1$$

So $x^- \leq x^{---}$. Therefore, $x^- = x^{---}$.
(ii) By $(E11)$, $x^- \to y^- \leq (x^- \to y^-)^{--}$, for $x, y \in E$. Now, by $(E14)$,

$$\begin{aligned} (x^- \to y^-)^{--} \to (x^- \to y^-) &= x^- \to ((x^- \to y^-)^{--} \to y^-) \\ &= x^- \to (y \to (x^- \to y^-)^{---}) \\ &= x^- \to (y \to (x^- \to y^-)^-), \text{ by (i)} \\ &= x^- \to ((x^- \to y^-) \to y^-) \\ &= (x^- \to y^-) \to (x^- \to y^-) \\ &= 1 \end{aligned}$$

Hence, $(x^- \to y^-)^{--} \leq (x^- \to y^-)$. Therefore, $(x^- \to y^-)^{--} = x^- \to y^-$ □

Definition 3.2. *Let $(E, \sim, \wedge, 1)$ be a bounded equality algebra and X any subset of E. Then the set of complement elements (with respect to X) is denoted by $N(X)$ and is defined by*

$$N(X) = \{x \in E \mid x^- \in X\}$$

Proposition 3.1. *Let I be an ideal of bounded equality algebra E. Then $N(I)$ is a filter of E.*

Proof. Let I be an ideal of bounded equality algebra E. Then by Theorem 3.1, $0 \in I$ and since $1 = 0^-$, we have $1 \in N(I)$. Let $x, x \to y \in N(I)$, for $x, y \in E$. Then $(x \to y)^- \in I$ and $x^- \in I$ and by $(E12)$, $(x \to y) \leq (y \to 0) \to (x \to 0)$ and so by $(E15)$, $(y^- \to x^-)^- \leq (x \to y)^- \in I$. Hence, $(y^- \to x^-)^- \in I$ and since by Lemma 3.1(i), $x^- = x^{---}$, we get that $(y^- \to x^{---})^- \in I$ and so by $(E14)$, $(x^{--} \to y^{--})^- \in I$. Now, since $x^- \in I$ and I is an ideal of E, by Theorem 3.1, we have $y^- \in I$ and so $y \in N(I)$. Therefore, $N(I)$ is a filter of E. □

Proposition 3.2. *Let F be a filter of bounded equality algebra E. Then $N(F)$ is an ideal of E.*

Proof. Let F be a filter of bounded equality algebra E. Since $1 \in F$, we get that $0 = 1^- \in N(F)$. Let $x \in N(F)$ and $(x^- \to y^-)^- \in N(F)$, for $x, y \in E$. Then $(x^- \to y^-)^{--} \in F$ and $x^- \in F$. Since by Lemma 3.1(*ii*), $x^- \to y^- = (x^- \to y^-)^{--}$, then $x^- \to y^- \in F$. Now, since F is a filter of E and $x^- \in F$, we have $y^- \in F$. Hence, $y \in N(F)$. Therefore, $N(F)$ is an ideal of E.
□

Theorem 3.2. *Let I be an ideal of bounded equality algebra E. Then the binary relation $\smile_I$ on E which is defined by*

$$x \smile_I y \text{ if and only if } (x^- \to y^-)^- \in I \text{ and } (y^- \to x^-)^- \in I$$

is a equivalence relation on E.

Proof. Let I be an ideal of E. Since by ($E9$), $(x^- \to x^-)^- = 0 \in I$, we have $x \smile_I x$, for every $x \in E$ and so $\smile_I$ is a reflexive relation. By definition $\smile_I$, $\smile_I$ is a symmetric relation. Now, we prove that $\smile_I$ is transitive relation. Let $x, y, z \in E$, $x \smile_I y$ and $y \smile_I z$. Then $(x^- \to y^-)^- \in I$, $(y^- \to x^-)^- \in I$ and $(y^- \to z^-)^- \in I$, $(z^- \to y^-)^- \in I$. By Lemma 3.1 (*ii*), ($E15$) and ($E17$),

$$\begin{aligned} ((x^- \to y^-)^{--} \to (x^- \to z^-)^{--})^- &= ((x^- \to y^-) \to (x^- \to z^-))^- \\ &\leq (y^- \to z^-)^- \in I \end{aligned}$$

Since I is an ideal of E, we have $((x^- \to y^-)^{--} \to (x^- \to z^-)^{--})^- \in I$ and since $(x^- \to y^-)^- \in I$, by Theorem 3.1, we conclude $(x^- \to z^-)^- \in I$. By similar way, $(z^- \to x^-)^- \in I$. Therefore, $\smile_I$ is a transitive relation and so $\smile_I$ is a equivalence relation on E. □

Note that, if $x \smile_I y$, then $(x^- \to y^-)^- \in I$, $(y^- \to x^-)^- \in I$ and so by ($E12$) and ($E15$),

$$(x^{--} \to y^{--})^- \leq (y^- \to x^-)^-$$

And since $(y^- \to x^-)^- \in I$, we have $(x^{--} \to y^{--})^- \in I$ and by similar way $(y^{--} \to x^{--})^- \in I$. Hence, $x^- \smile_I y^-$. Moreover, by Lemma 3.1 (*i*), $(x^- \to x^{---})^- = 0 \in I$ and $(x^{---} \to x^-)^- = 0 \in I$. Hence, $x \smile_I x^{--}$

Open Problem 3.1. *Is the equivalence relation ($\smile_I$) which is defined in Theorem 3.3, congruence relation?*

Theorem 3.3. *Let I be an ideal of involutive equality algebra E. Then the binary relation $\sim_I$ on E which is defined by*

$$x \sim_I y \text{ if and only if } (x^- \sim y^-)^- \in I$$

is a congruence relation on E.

Proof. Let I be an ideal of involutive equality algebra E. Since $(x^- \sim x^-)^- = 0 \in I$, we have $x \sim_I x$, for every $x \in E$ and so $\sim_I$ is a reflexive relation. Since by ($E2$), $(x^- \sim y^-)^- = (y^- \sim x^-)^-$, for every $x, y \in E$, we have $x \sim_I y$ if and only if $y \sim_I x$. Hence, $\sim_I$ is a symmetric relation. Let $x, y, z \in E$, $x \sim_I y$ and $y \sim_I z$. Then $(x^- \sim y^-)^- \in I$ and $(y^- \sim z^-)^- \in I$, and so by ($E13$) and ($E15$),

$$\begin{aligned} ((x^- \sim y^-)^{--} \to (x^- \sim z^-)^{--})^- &\leq ((x^- \sim y^-)^{--} \sim (x^- \sim z^-)^{--})^- \\ &= ((x^- \sim y^-) \sim (x^- \sim z^-))^-, \text{ Since E is involutive} \\ &\leq (y^- \sim z^-)^-, \text{ by (E7) and (E15)} \end{aligned}$$

Since $(y^- \sim z^-)^- \in I$, we have $((x^- \sim y^-)^{--} \to (x^- \sim z^-)^{--})^- \in I$ and since $(x^- \sim y^-)^- \in I$, by Theorem 3.1, we get that $(x^- \sim z^-)^- \in I$. Therefore, $\sim_I$ is a transitive relation and so $\sim_I$ is a equivalence relation on E. Now, we prove that $\sim_I$ is compatible with $\sim$ and $\wedge$. Let $x, y, z \in E$, $x \sim_I y$. Then $(x^- \sim y^-)^- \in I$ and by ($E6$) and ($E7$),

$$\begin{aligned} ((x \wedge z)^- \sim (y \wedge z)^-)^- &\leq ((y \wedge z) \sim (x \wedge z))^- \\ &\leq (x \sim y)^- \\ &= (x^{--} \sim y^{--})^-, \text{ Since E is involutive} \\ &\leq (y^- \sim x^-)^- \\ &= (x^- \sim y^-)^- \in I \end{aligned}$$

Hence, $((x \wedge z)^- \sim (y \wedge z)^-)^- \in I$. Therefore, $(x \wedge z) \sim_I (y \wedge z)$.
Moreover, by ($E7$),

$$\begin{array}{rcl} ((x \sim z)^- \sim (y \sim z)^-)^- & \leq & ((y \sim z) \sim (x \sim z))^- \\ & \leq & (x \sim y)^- \\ & = & (x^{--} \sim y^{--})^-, \text{ Since E is involutive} \\ & \leq & (y^- \sim x^-)^- \\ & = & (x^- \sim y^-)^- \in I \end{array}$$

Hence, $((x \sim z)^- \sim (y \sim z)^-)^- \in I$. Therefore, $(x \sim z) \sim_I (y \sim z)$. Thus, $\sim_I$ is a congruence relation on E. □

Proposition 3.3. *Let I be an ideal of involutive equality algebra E. Then*
(i) $[0] = \{x \in E \mid x \sim_I 0\}$ *is an ideal of E. Moreover,* $[0] = I$.
(ii) *If θ is a congruence on E, then* $[0]_\theta = \{x \in E \mid (x, 0) \in \theta\}$ *is an ideal of E.*

Proof. (i) Let I be an ideal of involutive equality algebra E. Then

$$\begin{array}{rcl} [0] & = & \{x \in E \mid x \sim_I 0\} \\ & = & \{x \in E \mid (x^- \sim 0^-)^- \in I\} \\ & = & \{x \in E \mid x^{--} \in I\} \\ & = & \{x \in E \mid x \in I\} \\ & = & I \end{array}$$

Therefore, $[0]$ is an ideal of E.
(ii) Let θ be a congruence on E and $x, y \in E$ such that $x \leq y$ and $y \in [0]_\theta$. Then $(y, 0) \in \theta$ and since θ is a congruence on E, we have $(x, x) \in \theta$ and so $(x \wedge y, x \wedge 0) \in \theta$. Hence, $(x, 0) \in \theta$. Therefore, $x \in [0]_\theta$.
Now, let $x, y \in [0]_\theta$. Then $(x, 0), (y, 0) \in \theta$ and since $(0, 0) \in \theta$ and θ is a congruence on E, we get that $(x \sim 0, 0 \sim 0) \in \theta$. Hence, $(x^-, 1) \in \theta$ and so $(x^- \wedge y, 1 \wedge y) \in \theta$. Therefore, $(x^- \wedge y, y) \in \theta$. Moreover, since $(x^-, 1) \in \theta$ and $(x^- \wedge y, y) \in \theta$, we conclude $(x^- \sim y, 1 \sim y) \in \theta$ and $(x^- \sim (x^- \wedge y), x^- \sim y) \in \theta$. Hence, $(x^- \sim (x^- \wedge y), y) \in \theta$ and since $(y, 0) \in \theta$, we have $(x^- \sim (x^- \wedge y), 0) \in \theta$ and so $x^- \to y = x^- \sim (x^- \wedge y) \in [0]_\theta$. Therefore, $[0]_\theta$ is an ideal of E. □

Theorem 3.4. *Let E be an involutive equality algebra E. Then there is a one-to-one correspondence between $Id(E)$ and $Con(E)$.*

Proof. Define $\psi : Con(E) \longrightarrow Id(E)$ by $\psi(\theta) = [0]_\theta$, for all $\theta \in Con(E)$. Since by Proposition 3.3 (ii), $[0]_\theta$ is an ideal of E, then $\psi(\theta) \in Id(E)$. Moreover, ψ is well defined. In fact, if $\theta_1 = \theta_2$, the $[0]_{\theta_1} = [0]_{\theta_2}$. If $x \in [0]_{\theta_1}$, then $(x, 0) \in \theta_1 = \theta_2$ and so $(x, 0) \in \theta_2$. Hence, $x \in [0]_{\theta_2}$ and so $[0]_{\theta_1} \subseteq [0]_{\theta_2}$. By similarly, $[0]_{\theta_2} \subseteq [0]_{\theta_1}$. Therefore, $[0]_{\theta_1} = [0]_{\theta_2}$ and so $\psi(\theta_1) = \psi(\theta_2)$. Thus, ψ is a well defined mapping. Now, we prove $\theta = \sim_{[0]_\theta}$, for every $\theta \in Con(E)$. Let $(x, y) \in \theta$. Since θ is a congruence on E, we have $(x^-, y^-) \in \theta$ and so $(x^- \sim y^-, y^- \sim y^-) \in \theta$. Hence, $(x^- \sim y^-, 1) \in \theta$ and so $((x^- \sim y^-)^-, 1^-) \in \theta$. Therefore, $(x^- \sim y^-)^- \in [0]_\theta$ and since by Proposition 3.3(ii), $[0]_\theta$ is an ideal of E, we get taht $x \sim_{[0]_\theta} y$ and so $(x, y) \in \sim_{[0]_\theta}$. Thus, $\theta \subseteq \sim_{[0]_\theta}$. Conversely, if $(x, y) \in \sim_{[0]_\theta}$, then $x \sim_{[0]_\theta} y$ and so $(x^- \sim y^-)^- \in [0]_\theta$. Hence, $((x^- \sim y^-)^-, 0) \in \theta$ and so $((x^- \sim y^-)^{--}, 0^-) \in \theta$ and since E is an involutive equality algebra, we have $((x^- \sim y^-), 1) \in \theta$. Moreover, since θ is a congruence on E, we get taht $((x^- \sim y^-) \sim x^-, 1 \sim x^-) \in \theta$ and so $(((x^- \sim y^-) \sim x^-) \wedge y^-, x^- \wedge y^-) \in \theta$ and since $y^- \leq (y^- \sim x^-) \sim x^-$, we have $(y^-, x^- \wedge y^-) \in \theta$. By similar way, $(x^-, x^- \wedge y^-) \in \theta$ and since θ is a transitive relation, we conclude $(x^-, y^-) \in \theta$ and so $(x^{--}, y^{--}) \in \theta$. Hence, $(x, y) \in \theta$ and so $\sim_{[0]_\theta} \subseteq \theta$. Therefore, $\theta = \sim_{[0]_\theta}$. Now, we prove ψ is an one-to-one mapping. Let $\theta_1, \theta_2 \in Con(E)$ such that $\psi(\theta_1) = \psi(\theta_2)$. Then $[0]_{\theta_1} = [0]_{\theta_2}$ and so $\sim_{[0]_{\theta_1}} = \sim_{[0]_{\theta_2}}$. Hence, $\theta_1 = \theta_2$. Therefore, ψ is an one-to-one mapping. Moreover, if $I \in Id(E)$, then $x \in I$ if and only if $(x^- \sim 0^-)^- = x^{--} = x \in I$ if and only if $x \sim_I 0$ if and only if $(x, 0) \in \sim_I$ if and only if $x \in [0]_{\sim_I}$. Hence, $I = [0]_{\sim_I}$ and since by Theorem 3.3, $\sim_I \in Con(E)$ and $I = [0]_{\sim_I} = \psi(\sim_I)$, we get that ψ is an onto mapping. Therefore, ψ is an one-to-one correspondence between $Id(E)$ and $Con(E)$. □

Let $(E, \wedge, \sim, 1)$ be an involutive equality algebra and I be an ideal of E. Define $\frac{E}{I} = \{[x] \mid x \in E\}$, where $[x] = \{y \in E \mid x \sim_I y\}$. We define the following operations on $\frac{E}{I}$:

$$[x] \wedge [y] := [x \wedge y], \quad [x] \sim [y] := [x \sim y]$$

Theorem 3.5. *Let $(E, \wedge, \sim, 1)$ be an involutive equality algebra and I be an ideal of E. Then $(\frac{E}{I}, \wedge, \sim, [1])$ is an involutive equality algebra which is called quotient equality algebra with respect to I.*

Proof. The proof is straight forward. □

4. Homomorphisms in Equality Algebras

In this section the concept of homomorphism is defined in equality algebras.

Definition 4.1. *Let $(E_1, \wedge_1, \sim, 1_{E_1})$ and $(E_2, \wedge_2, \sim_2, 1_{E_2})$ be two equality algebras. Then a mapping $f : E_1 \longrightarrow E_2$ is called an equality homomorphism, if for all $x, y \in E_1$:*
(i) $f(x \wedge_1 y) = f(x) \wedge_2 f(y)$,
(ii) $f(x \sim_1 y) = f(x) \sim_2 f(y)$.
Moreover, if E_1, E_2 are two bounded equality algebras, then an equality homomorphism f is called bounded, if $f(0_1) = 0_2$. An equality homomorphism f is called equality (epimorphism)monomorphism, if f is an (onto)one-to-one mapping and an equality homomorphism f is called an equality isomorphism, if f is an one-to-one and onto mapping.

Lemma 4.1. *Let $(E_1, \wedge_1, \sim_1, 1_{E_1})$ and $(E_2, \wedge_2, \sim_2, 1_{E_2})$ be two equality algebras and $f : E_1 \longrightarrow E_2$ be an equality homomorphism. Then for all $x, y \in E_1$:*
(i) $f(x \rightarrow_1 y) = f(x) \rightarrow_2 f(y)$.
(ii) $f(x \leftrightarrow_1 y) = f(x) \leftrightarrow_2 f(y)$.
(iii) $f(1_{E_1}) = 1_{E_2}$.
(iv) $f(x^-) = (f(x))^-$, where E_1, E_2 are two bounded equality algebras and f is a bounded equality homomorphism.

Proof. It follows form Definition 4.1. □

Proposition 4.1. *Let $f : E_1 \longrightarrow E_2$ be a bounded equality homomorphism and I be an ideal of equality algebra E_2. Then $f^{-1}(I)$ is an ideal of E_1.*

Proof. Let f is a bounded equality homomorphism. Then $f(0_1) = 0_2$ and since I is an ideal of equality algebra E_2, we get that $0_2 \in I$. Hence, $0_1 \in f^{-1}(I)$. Now, let $(x^- \rightarrow y^-)^- \in f^{-1}(I)$ and $x \in f^{-1}(I)$, for $x, y \in E_1$. Then by Lemma 4.1, $((f(x))^- \rightarrow (f(y))^-)^- \in I$ and $f(x) \in I$. Hence, by Theorem 3.1, $f(y) \in I$ and so $y \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is an ideal of E_1. □

Note that, if $f : E_1 \longrightarrow E_2$ is a bounded equality homomorphism, then by Proposition 4.1, $kerf = \{x \in E \mid f(x) = 0_2\} = f^{-1}(\{0_2\})$, is an ideal of E_1.

Remark 4.1. *Given a equality homomorphism $f : E_1 \longrightarrow E_2$, the $Imf = \{f(x) \mid x \in E_1\}$ is sub equality algebra of E_2*

Theorem 4.1. *Let E_1, E_2 be two involutive equality algebras and $f : E_1 \longrightarrow E_2$ be a bounded equality homomorphism. Then*

$$\frac{E_1}{kerf} \cong Imf$$

Proof. Let $\phi : \dfrac{E_1}{kerf} \longrightarrow Imf$ defined by $\phi([x]) := f(x)$, for any $[x] \in \dfrac{E_1}{kerf}$. Then ϕ is a well defined mapping. Since for every $x_1, x_2 \in E_1$, if $[x_1] = [x_2]$, then $x_1 \sim_{kerf} x_2$ and so $((x_1)^- \sim (x_2)^-)^- \in kerf$. Hence, $f(((x_1)^- \sim (x_2)^-)^-) = 0$ and by Lemma 4.1, $((f(x_1))^- \sim (f(x_2))^-)^- = 0$ and so $((f(x_1))^- \sim (f(x_2))^-)^{--} = 1$ and since E_2 is an involutive equality algebra, then $(f(x_1))^- \sim (f(x_2))^- = 1$. Therefore, by $(E13)$, $(f(x_1))^- \to (f(x_2))^- = 1$ and so $(f(x_1))^- \leq (f(x_2))^-$. Now, by $(E15)$, $(f(x_2))^{--} \leq (f(x_1))^{--}$ and so $f(x_2) \leq f(x_1)$. By similar way, $f(x_1) \leq f(x_2)$. Thus, $f(x_1) = f(x_2)$ and so $\phi([x_1]) = \phi([x_2])$. Therefore, ϕ is a well defined mapping. By definition of ϕ, ϕ is an equality homomorphism. Moreover, ϕ is an one-to-one. If $\phi([x_1]) = \phi([x_2])$, then $f(x_1) = f(x_2)$ and so $f(x_1)^- = f(x_2)^-$. Now, by Lemma 4.1, $f(((x_1)^- \sim (x_2)^-)^-) = 0$. Hence, $((x_1)^- \sim (x_2)^-)^- \in kerf$ and so $x_1 \sim_{kerf} x_2$. Therefore, $[x_1] = [x_2]$ and so ϕ is an equality monomorphism. That is ϕ is onto is clear by construction of ϕ. Therefore, ϕ is an equality isomorphism and so

$$\frac{E_1}{kerf} \cong Imf$$

□

5. Prime and Boolean ideals in Equality Algebras

In this section, the notions of prime and Boolean ideals in bounded equality algebras are introduced and some of theirs properties are investigated.

Definition 5.1. *Let $(E, \wedge, \sim, 1)$ be a bounded equality algebra and P be an ideal of E. Then P is called a prime ideal of E, if it satisfies for every $x, y \in E$, $(x \to y)^- \in P$ or $(y \to x)^- \in P$.*

Example 5.1. *[1] Let $(E = \{0, a, b, c, 1\}, \leq)$ be a lattice with the following diagram. Define the operations $\sim$ and $\to$ on E as follows,*

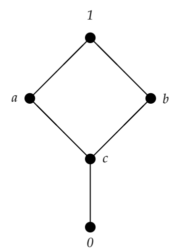

Table 3.

$\sim$	0	c	a	b	1
0	1	0	0	0	0
c	0	1	b	a	c
a	0	b	1	c	a
b	0	a	c	1	b
1	0	c	a	b	1

Table 4.

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then by routine calculations, we can see that $(E, \wedge, \sim, 0, 1)$ is a bounded equality algebra and $I = \{0\}$ and $J = \{0, c\}$ are prime ideals of E.

Proposition 5.1. *Let E be a bounded equality algebra. Then*
(i) If P is a prime ideal of E, then $N(P)$ is a prime filter of E.
(ii) If F is a prime filter of E, then $N(F)$ is a prime ideal of E.

Proof. (i) Let P be a prime ideal of E. Then by Proposition 3.1, $N(P)$ is a filter of E. Now, since $(x \to y)^- \in P$ or $(y \to x)^- \in P$, for every $x, y \in E$, we have $x \to y \in N(P)$ or $y \to x \in N(P)$, for every $x, y \in E$. Therefore, $N(P)$ is a prime filter of E.
(ii) Let F be a prime filter of E. Then by Proposition 3.2, $N(F)$ is an ideal of E. Now, since $x \to y \in F$ or $y \to x \in F$, for every $x, y \in E$ and since by ($E19$), $x \to y \leq (x \to y)^{--}$ and $y \to x \leq (y \to x)^{--}$, we get that $(x \to y)^{--} \in F$ or $(y \to x)^{--} \in F$, for every $x, y \in E$,. Hence, $(x \to y)^- \in N(F)$ or $(y \to x)^- \in N(F)$, for every $x, y \in E$. Therefore, $N(F)$ is a prime ideal of E. □

Theorem 5.1. *Let P be an ideal of involutive equality algebra E. Then P is a prime ideal if and only if $\frac{E}{P}$ is a chain(equivalently $\leq_P$ is totally ordered)*

Proof. Let P be a prime ideal of involutive equality algebra E. Then for every $x, y \in E$, $(x \to y)^- \in P$ or $(y \to x)^- \in P$. Hence, by Lemma 3.1(i), $((x \to y)^{--} \sim 0^-)^- = (x \to y)^- \in P$ or $((y \to x)^{--} \sim 0^-)^- = (y \to x)^- \in P$, for every $x, y \in E$ and so $[(x \to y)^-] = [0]$ or $[(y \to x)^-] = [0]$. Hence, $[(x \to y)^{--}] = [0^-]$ or $[(y \to x)^{--}] = [0^-]$ and since E is an involutive equality algebra, then $[x \to y] = [1]$ or $[y \to x] = [1]$, for every $x, y \in E$ and so $[x] \to [y] = [1]$ or $[y] \to [x] = [1]$, for every $x, y \in E$. Therefore, $[x] \leq_P [y]$ or $[y] \leq_P [x]$, for every $[x], [y] \in \frac{E}{P}$ and so $\frac{E}{P}$ is a chain. Conversely, let $\frac{E}{P}$ be a chain. Then $[x] \leq_P [y]$ or $[y] \leq_P [x]$, for every $[x], [y] \in \frac{E}{P}$ and so $[x \to y] = [1]$ or $[y \to x] = [1]$. Hence, $((x \to y)^- \sim 1^-)^- \in P$ or $((y \to x)^- \sim 1^-)^- \in P$ and so $(x \to y)^{---} \in P$ or $(y \to x)^{---} \in P$, for every $x, y \in E$. Now, since E is an involutive equality algebra, we get that $(x \to y)^- \in P$ or $(y \to x)^- \in P$, for every $x, y \in E$. Therefore, P is a prime ideal of E. □

Theorem 5.2. *Let P be a proper ideal of bounded prelinear equality algebra E. Then P is a prime ideal of E if and only if $x \wedge y \in P$ implies $x \in P$ or $y \in P$, for every $x, y \in E$.*

Proof. Let P be a prime ideal of E. Then for every $x, y \in E$, $(x \to y)^- \in P$ or $(y \to x)^- \in P$. Now, let $x \wedge y \in P$ and $(x \to y)^- \in P$, for $x, y \in E$. Then by ($E12$), ($E15$) and ($E16$),

$$\begin{aligned} ((x \wedge y)^- \to x^-)^- &\leq (x \to x \wedge y)^- \\ &= (x \to y)^- \end{aligned}$$

Since P is an ideal of E, $x \wedge y \in P$ and $(x \to y)^- \in P$, by Theorem 3.1, we conclude $x \in P$. By similar way, if $(y \to x)^- \in P$, then $y \in P$. Conversely, since E is a prelinear equality algebra, we have $(x \to y) \vee (y \to x) = 1$, for every $x, y \in E$ and so by ($E20$), $(x \to y)^- \wedge (y \to x)^- = 0 \in P$, for every $x, y \in E$. Hence, $(x \to y)^- \in P$ or $(y \to x)^- \in P$, for every $x, y \in E$. Therefore, P is a prime ideal of E.
□

Proposition 5.2. *Let $f : E_1 \longrightarrow E_2$ be a bounded equality homomorphism and P be a prime ideal of E_2. Then $f^{-1}(P)$ is a prime ideal of E_1.*

Proof. Let $f : E_1 \longrightarrow E_2$ be a bounded equality homomorphism and P be a prime ideal of E_2. Then by Proposition 4.1, $f^{-1}(P)$ is an ideal of E_1. Now, let $x, y \in E_1$. Then $f(x), f(y) \in E_2$ and since P is a prime ideal of E_2, we get that $(f(x) \to f(y))^- \in P$ or $(f(y) \to f(x))^- \in P$. Hence, by Lemma 4.1, $f((x \to y)^-) \in P$ or $f((y \to x)^-) \in P$ and so $(x \to y)^- \in f^{-1}(P)$ or $(y \to x)^- \in f^{-1}(P)$. Therefore, $f^{-1}(P)$ is a prime ideal of E_1. □

Proposition 5.3. *(Extension property for prime ideals) Let E be a bounded equality algebra and P a prime ideal of E. Then every ideal J of E containing P is also prime.*

Proof. Let P be a prime ideal of E and J an ideal of E such that $P \subseteq J$. Since $(x \to y)^- \in P$ or $(y \to x)^- \in P$, for every $x, y \in E$, we get that $(x \to y)^- \in J$ or $(y \to x)^- \in J$, for every $x, y \in E$. Therefore, J is a prime ideal of E. □

Proposition 5.4. *Let P be a prime ideal of bounded prelinear equality algebra E. Then $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, for any two ideals I and J of E.*

Proof. Let P be a prime ideal of prelinear equality algebra E and I, J be two ideals of E such that $I \cap J \subseteq P$. If $I \nsubseteq P$ and $J \nsubseteq P$, then there exist $x \in I \backslash P$ and $y \in J \backslash P$ and since $x \wedge y \leq x, y$, we have $x \wedge y \in I$ and $x \wedge y \in J$. Hence, $x \wedge y \in I \cap J \subseteq P$ and so $x \wedge y \in P$. Now, by Theorem 5.2, $x \in P$ or $y \in P$, which is a contradiction. Therefore, $I \subseteq P$ or $J \subseteq P$. □

Definition 5.2. *Let I be an ideal of bounded equality algebra E. Then I is called a Boolean ideal of E, if $x \wedge x^- \in I$, for any $x \in E$.*

Example 5.2. *In Example 5.1, $I = \{0\}$ is a Boolean ideal.*

Example 5.3. *In Example 3.1, $I = \{0, d, c\}$ is a Boolean ideal.*

Proposition 5.5. *Let I be an ideal of lattice equality algebra E and F be a filter of E. Then*
(i) I is a Boolean ideal of E if and only if $N(I)$ is a Boolean filter of E.
(ii) If F is a Boolean filter of E, then $N(F)$ is a Boolean ideal of E.

Proof. (i) Let I be a Boolean ideal of E. Then by Proposition 3.1, $N(I)$ is a filter of E. Since I is a Boolean ideal, then $x^- \wedge (x^-)^- \in I$, for any $x \in E$ and since E is a lattice equality algebra, we get that $(x \vee x^-)^- = x^- \wedge (x^-)^- \in I$. Hence, $x \vee x^- \in N(I)$, for any $x \in E$. Therefore, $N(I)$ is a Boolean filter of E. Conversely, let $N(I)$ be a Boolean filter of E. Then $x \vee x^- \in N(I)$, for any $x \in E$. Hence, $(x \vee x^-)^- \in I$ and so $x^- \wedge (x^-)^- \in I$. Now, since I is an ideal of E and $x \leq x^{--}$, we have $x^- \wedge x \leq x^- \wedge x^{--}$ and so $x^- \wedge x \in I$, for any $x \in E$. Thus, I is a Boolean ideal of E.
(ii) Let F be a Boolean filter of E. Then $x \vee x^- \in F$, for any $x \in E$. By Proposition 3.2, $N(F)$ is an ideal of E and since $x^- \vee x^{--} = (x \wedge x^-)^-$ and $x^- \vee x^{--} \in F$, we have $x \wedge x^- \in N(F)$, for any $x \in E$. Therefore, $N(F)$ is a Boolean ideal of E.
□

Remark 5.1. *If E is an involutive lattice equality algebra and F is a filter of E, then F is a Boolean filter of E if and only if $N(F)$ is a Boolean ideal of E.*

Proposition 5.6. *Let $f : E_1 \longrightarrow E_2$ be a bounded equality homomorphism and I be a Boolean ideal of E_2. Then $f^{-1}(I)$ is a Boolean ideal of E_1.*

Proof. Let $f : E_1 \longrightarrow E_2$ be a bounded equality homomorphism and I be a Boolean ideal of E_2. Then by Proposition 4.1, $f^{-1}(I)$ is an ideal of E_1. Now, let $x \in E_1$. Then $f(x) \in E_2$ and since I is a Boolean ideal of E_2, we get that $f(x) \wedge (f(x))^- \in I$. Hence, by Lemma 4.1, $f(x \wedge x^-) \in I$ and so $x \wedge x^- \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is a Boolean ideal of E_1. □

Theorem 5.3. *Let I be an ideal of involutive prelinear equality algebra E. Then I is a Boolean ideal of E if and only if $\frac{E}{I}$ is a Boolean algebra.*

Proof. Let I be a Boolean ideal of involutive prelinear equality algebra E. Then by Theorem 2.1, E is a distributive lattice equality algebra and for every $x, y \in E$, $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Hence, by Theorem 3.5, $\frac{E}{I}$ is a distributive lattice equality algebra and for every $[x], [y] \in \frac{E}{I}$,

$$[x] \curlyvee [y] = [x \vee y] = [((x \to y) \to y) \wedge ((y \to x) \to x)] = (([x] \to [y]) \to [y]) \curlywedge (([y] \to [x]) \to [x])$$

And since I is a Boolean ideal of E, we have $x \wedge x^- \in I$, for every $x \in E$. Hence,

$$((x \wedge x^-)^- \sim 0^-)^- = (x \wedge x^-)^{--} = x \wedge x^- \in I$$

and so $[x \wedge x^-] = [0]$. Therefore, $[x] \curlywedge [x]^- = [0]$ and since by Theorem 3.5, $\frac{E}{I}$ is an involutive lattice equality algebra, then $([x] \curlywedge [x]^-)^- = [0]^-$ and so $[x]^- \curlyvee [x] = [1]$. Therefore, $\frac{E}{I}$ is a complemented lattice and so $\frac{E}{I}$ is a Boolean algebra.
Conversely, let $\frac{E}{I}$ be a Boolean algebra. Then $[x] \curlywedge [x]^- = [0]$ and so $[x \wedge x^-] = [0]$. Hence, $x \wedge x^- = (x \wedge x^-)^{--} = ((x \wedge x^-)^- \sim 0^-)^- \in I$ and so $x \wedge x^- \in I$, for any $x \in E$. Therefore, I is a Boolean ideal of E. □

6. Conclusion

The results of this paper are devoted to introduce ideals in bounded equality algebras. We presented a characterization and several important properties ideals. In particular, we proved that is an one-to-one corresponding between congruence relation on an involutive equality algebra and the set of ideals on it. Also, we proved that the first isomorphism theorem on equality algebras. Moreover, the notions of prime and Boolean ideals in equality algebras were introduced. Finally, we proved that ideal I of involutive prelinear equality algebra E is a Boolean ideal if and only if $\frac{E}{I}$ is a Boolean algebra.

References

[1] R. A. Borzooei, F. Zebardast and M. Aaly Kologani, Some type filters in equality algebras, Categoreis and General Algebraic Structurs with Applications 7 (2017) 33–55.
[2] L. C. Ciungu, Internal states on equality algebras, Soft Computing 19(4) (2015) 939–953.
[3] F. Esteva, L. Godo, Monoidal t-normbased logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems 124(3) (2001) 271-288.
[4] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic, vol. 4, Kluwer Academic Publishers, 1998.
[5] S. Jenei, Equality algebras, Studia Logica, 100(6) (2012) 1201–1209.
[6] S. Jenei, L. Kóródi, On the variety of equality algebras, Fuzzy Logic and Technology (2011) 153–155.
[7] V. Novák, B. De Baets, EQ-algebras, Fuzzy Sets and Systems, 160(20) (2009) 2956–2978.
[8] F. Zebardast, R. A. Borzooei, M. Aaly Kologani, Results on Equality algebras, Information Sciences 381 (2017) 270–282.