On the Essential Numerical Spectrum of Operators on Banach Spaces

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Abstract. The purpose of this paper is to define and develop a new notion of the essential numerical spectrum $\sigma_{en}(.)$ of an operator on a Banach space X and to study its properties. Our definition is closely related to the essential numerical range $W_e(.)$.

1. Introduction

In 1918, O. Toeplitz [23] and F. Hausdorff [14] introduced the notion of the numerical range of a matrix A acting on a finite dimensional space $X = \mathbb{C}^n$. Afterward, J. R. Giles and G. Josef [13] generalized this notion to operators on Hilbert spaces. G. Lummer in 1961 [17] and F. L. Bauer in 1962 [6] succeeded to generalize these definitions to a more general setting of unbounded operators on a Banach space X as:

$$W(A) := \{\langle Ax, x'\rangle : x \in D(A), x' \in X', \|x\| = \|x'\| = 1 = \langle x, x'\rangle\},$$

where X' is the dual of X. $W(A)$ is called the spatial numerical range.
Due to the lack of an inner product, the numerical range of an operator on a Banach space loses some properties. Using a semigroup approach, M. Adler, W. Dada and A. Radl [1] [10], introduced a new definition, called the numerical spectrum $\sigma_n(A)$, for a closed and densely defined linear operator A on a Banach space X as:

$\sigma_n(A) = \mathbb{C}\backslash\rho_n(A)$, where $\rho_n(A) = \bigcup_{(\theta,\omega)\in M} H_{\theta,\omega}$, $H_{\theta,\omega} := \{e^{i\theta}z : Rez > \omega\}$ and

$$M := \{(\theta, \omega) \in [0, 2\pi) \times \mathbb{R} : H_{\theta,\omega} \subset \rho(A) \text{ and } \qquad \|R(\lambda, A)\| \leq \frac{1}{\text{dist}(\lambda, \partial H_{\theta,\omega})}; \ \forall \lambda \in H_{\theta,\omega}\}.$$

The authors succeeded to prove that $\sigma_n(A)$ remains unitary invariant set and satisfies some properties similar to that of the case of Hilbert space.
The definition of numerical spectrum introduced in [1] is closely related to the numerical range $W(A)$, since in the case of bounded operator on a Hilbert space we have $\sigma_n(A) = cl(W(A))$ and in the case of bounded

operator on a Banach space $\sigma_n(A) = cl(coW(A))$, where cl denotes the closer and co denotes the convex hull.

One of the most important development of the numerical range is the study of the essential numerical range. This notion appeared first in 1968, by J. P. Williams and J. G. Stampfli [20] for a bounded operator A acting on a Hilbert space H as:

$$W_e(A) = \bigcap_{K \in K(H)} cl(W(A + K)),$$

where $K(H)$ denotes the ideal of compact operators on H.
The essential numerical range of a bounded operator on a Hilbert space is a non-empty, closed, convex set. This set is invariant under unitary equivalence and has some properties such as:

1. $W_e(A + K) = W_e(A)$, for all $K \in K(X)$.
2. $W_e(A') = \overline{W_e(A)}$.
3. If $a, b \in \mathbb{C}$, $W_e(aA + bI) = aW_e(A) + b$.
4. If U is a bounded unitary operator on H, then $W_e(UAU') = W_e(A)$.
5. $W_e(I) = \{1\}$, in the case of infinite dimensional Hilbert space.
6. $W_e(A)$ is contained in the close disk of radius $\|\pi(A)\|$ centred around the origin, where $\pi(A) : B(H) \to B(H)/K(H)$ is the canonical quotient map, and $B(H)$ is the Banach space of all bounded linear operators on H.

For more details on the essential numerical range on a Hilbert space, we may refer to [3, 12, 22]. An essential version of the algebraic numerical range of a bounded operator on a Banach space has been started in [7] and recently, using the concept of measure of non-compactness, M. Barraa and V. Müller [5] has succeeded to characterize this set for bounded operators. However, in the works already done, the essential numerical range of unbounded operators on a Banach space is not considered, also some properties are not covered for an unbounded operator on a Banach space.

Our goal, in this paper, is to combine the numerical spectrum of an unbounded operator on a Banach space and the essential numerical range to obtain a new notion, called the essential numerical spectrum $\sigma_{en}(A)$ of an unbounded operator A on a Banach space X. By this definition, we generalize the ones existing in the literature and we cover some properties. More precisely, the essential numerical spectrum is a convex set, contains the Schechter essential spectrum, isometric invariant and satisfies all the properties $1 - 5$ similar to the Hilbert space case.

We recall that a scalar λ belongs to the Schechter essential spectrum $\sigma_e(A)$ of a closed operator A if $A - \lambda$ is not Fredholm with index 0. Also we have $\sigma_e(A) = \bigcap_{K \in K(X)} \sigma(A + K)$. For more details, we refer the reader to [19].

An obvious consequence is that the Schechter essential spectrum of a compact operator K on infinite dimensional Banach space is reduced to $\{0\}$, the converse is not always true.
However, we have this equivalence in the case of Hilbert space for the essential numerical range. Using the new concept of essential numerical spectrum, we prove that this equivalence remains valid in the case of infinite dimensional Banach space. More precisely, we prove that, operators of the form $A = \lambda + K, K \in K(X)$ can be identified via $\sigma_{en}(A) = \{\lambda\}$.

Furthermore, we introduce a new notion of essential numerical growth bound and we give some properties.
If A is a bounded operator, we prove that the essential numerical spectral bound are related to the essential numerical radius.

The paper is organized as follows. In Section 2, we introduce the new concept of essential numerical spectrum and develop its properties. In Section 3, we study the essential numerical growth bound and the essential numerical spectral bound. In the last Section, we give an example of the essential numerical spectrum.

2. Essential numerical spectrum

At the beginning of this section, we introduce some notation and preliminary results which will be used throughout this paper.

For a closed and densely defined linear operator $(A, D(A))$ on a Banach space X, $\rho(A)$ denotes the resolvent set and $\sigma(A)$ the spectrum of A. For some fixed $\theta \in [0, 2\pi)$ the rotated operator $A_\theta := e^{-i\theta}A$ with $D(A_\theta) = D(A)$. If A_θ generates a C_0-semigroup, we denote it by $(T_\theta(t))_{t \geq 0}$.

The rotated half plane of $\mathbb{C}_\omega := \{\lambda :\ Re\lambda > \omega\}$, $w \in \mathbb{R}$, by an angle θ will be denoted by

$$H_{\theta,\omega} := e^{i\theta}\mathbb{C}_\omega = \{e^{i\theta}\lambda :\ Re\lambda > \omega\}.$$

We introduce the numerical resolvent set and the numerical spectrum of a given operator.

Definition 2.1. *[1] Let $(A, D(A))$ be a closed and densely defined linear operator on a Banach space X. Then $z \in \mathbb{C}$ belongs to the numerical resolvent set $\rho_n(A)$ of A if there exists an open half plane $H_{\theta,\omega}$ in $\mathbb{C}$ such that $z \in H_{\theta,\omega} \subseteq \rho(A)$ and $\|R(\lambda, A)\| \leq \dfrac{1}{d(\lambda, \partial H_{\theta,\omega})}$, $\forall \lambda \in H$, where $d(\lambda, \partial H_{\theta,\omega})$ is the distance between λ and the boundary of $H_{\theta,\omega}$.*

The complementary set $\sigma_n(A) := \mathbb{C} \backslash \rho_n(A)$ is called the numerical spectrum of A.

This definition of numerical spectrum is closely related to the numerical range $W(A)$ of A. By this definition, the authors in [1] and [10] proved that for a bounded operators A $\sigma_n(A) = cl(W(A))$ in the case of Hilbert space and $\sigma_n(A) = cl(co(W(A)))$ in the case of Banach space. Also they retrieved some properties of the numerical range.

For the reader's convenience we recall the following necessary properties:

Properties 2.2. *[1, 10]*
i) $\sigma_n(A)$ is closed, convex set and it contains the spectrum $\sigma(A)$.

ii) $\sigma_n(\alpha A + \beta) = \alpha \sigma_n(A) + \beta$ for all complex numbers α and β.

iii) $\sigma_n(A) = \sigma_n(U^{-1}AU)$ for all isometric isomorphisms U on X.

iv) $\overline{\sigma_n(A)} = \sigma_n(A')$, where A' is the adjoint of A, acting on a reflexive Banach space.

The authors in [1] characterize the numerical spectrum by the notion of C_0-semigroup.

Proposition 2.3. *Let $(A, D(A))$ be a closed and densely defined linear operator on a Banach space X. For $z \in \mathbb{C}$ the following assertions are equivalent.*

1. *$z \in \rho_n(A)$.*
2. *There exists $\theta \in [0, 2\pi)$ and some $\omega \in \mathbb{R}$ such that $z \in H_{\theta,\omega}$ and $A_\theta := e^{-i\theta}A$ generates a ω-contractive C_0-semigroup $(T_\theta(t))_{t \geq 0}$.*

Now, we present a new concept, called the "essential numerical spectrum" of a linear operator on a Banach space.

Definition 2.4. *Let $(A, D(A))$ be a closed and densely defined linear operator on a Banach space X. The essential numerical spectrum $\sigma_{en}(A)$ of A is defined as*

$$\sigma_{en}(A) = \bigcap_{K \in K(X)} \sigma_n(A + K),$$

where $K(X)$ is the ideal of all compact operators on X.

Remark 2.5. 1. *As a first observation, we note that $\sigma_{en}(A)$ is an intersection of closed convex sets, hence it is closed and convex.*

2. *Since $\sigma(A) \subset \sigma_n(A)$, then $\sigma_e(A) \subset \sigma_{en}(A)$.*

3. *If A is a bounded linear operator on a separable Hilbert space, then there exists a compact operator $K \in K(X)$ such that $\sigma_{en}(A) = \sigma_n(A + K)$.*
   Indeed, since $\sigma_n(A) = cl(W(A))$ and there exists a compact operator $K \in K(X)$ such that $W_e(A) = cl(W(A + K))$(See [8]), we have

$$\sigma_{en}(A) = W_e(A) = cl(W(A + K)) = \sigma_n(A + K).$$

4. *If A is a bounded linear operator on a Banach space X, we have $\sigma_n(A) = cl(co(W(A)))$, then $\sigma_{en}(A)$ coincides with the essential algebraic numerical range*

$$V_e(A) = \bigcap_{K \in K(X)} cl(co(W(A + K)))$$

introduced in [7].
In particular, in the case where $X = l_p$, $1 \leq p < \infty$ and the essential algebraic numerical range of A has no interior points, there exists $K \in K(X)$ such that $\sigma_{en}(A) = \sigma_n(A + K)$.
Indeed, since there exists $K \in K(X)$ such that $V_e(A) = cl(coW((A + K))$ (See [2],[8] for $1 < p < \infty$ and [16] for $p = 1$)

$$\sigma_{en}(A) = co(W_e(A)) = cl(co(W(A + K))) = \sigma_n(A + K).$$

Using Properties 2.2, we can easily prove the following results.

Proposition 2.6. *Let $(A, D(A))$ be a closed and densely defined linear operator on a Banach space X, we have the following assertions.*

1. *$\sigma_{en}(A + K) = \sigma_{en}(A)$, for all $K \in K(X)$.*
2. *If X is a reflexive Banach space, then $\overline{\sigma_{en}(A)} = \sigma_{en}(A')$.*
3. *$\sigma_{en}(A) = \sigma_{en}(U^{-1}AU)$ for all isometric isomorphisms U on X.*
4. *$\sigma_{en}(\alpha A + \beta) = \alpha\sigma_{en}(A) + \beta$ for all complex numbers α and β.*

In the sequel, we consider X an infinite-dimensional Banach space. We introduce the definition of the set $N_e(.)$ for an unbounded operator which is already studied by M. Barraa and V. Müller in [5] in the case of a bounded linear operator on a Banach space.

Definition 2.7. *Let $(A, D(A))$ be a linear operator acting on X, we denote by $N_e(A)$ the set of all complex numbers λ with the property that there are nets $(x_\alpha) \subset D(A)$, $(x'_\alpha) \subset X'$ such that*

$$\|x_\alpha\| = \|x'_\alpha\| = \langle x_\alpha, x'_\alpha \rangle = 1 \text{ for all } \alpha,$$

$x_\alpha \to 0$ *weakly and* $\langle Ax_\alpha, x'_\alpha \rangle \to \lambda$.

The following theorem relates our essential numerical spectrum $\sigma_{en}(.)$ to the set $N_e(.)$.

Theorem 2.8. *Let $(A, D(A))$ be a closed and densely defined linear operator on a Banach space X. Then we have*

$$cl(co\{N_e(A)\}) \subset \sigma_{en}(A).$$

Proof. For $\lambda \in co\{N_e(A)\}$, we write λ as

$$\lambda = \sum_{i=1}^{p} \beta_i\lambda_i, \text{ where } 0 \leq \beta_i \leq 1; \ i = 1, ..., p; \ \sum_{i=1}^{p} \beta_i = 1 \text{ and } \lambda_i \in N_e(A).$$

For all $i = 1, .., p$, there exists nets $(x_{i_\alpha}) \subset D(A)$, $(x'_{i_\alpha}) \subset X'$ such that
$\|x_{i_\alpha}\| = \|x'_{i_\alpha}\| = \langle x_{i_\alpha}, x'_{i_\alpha} \rangle = 1, x_{i_\alpha} \to 0$ weakly and $\langle Ax_{i_\alpha}, x'_{i_\alpha} \rangle \to \lambda_i$.

Let $K \in K(X)$, we consider a half plan $H_{\theta,\omega}$ such that $\sigma_n(A + K) \subset \mathbb{C} \setminus H_{\theta,\omega}$ and $e^{-i\theta}(A + K)$ is the generator of a ω-contractive C_0-semigroups. By the Hille-Yosida and Lummer-Phillips Theorems [11], we have

$$Re\, e^{-i\theta} \langle (A + K)x, x' \rangle \le w$$

for all $x \in D(A)$, $x' \in X'$ such that $\|x\| = \|x'\| = \langle x, x' \rangle = 1$.
Hence

$$
\begin{aligned}
Re\, e^{-i\theta} \lambda &= Re\, e^{-i\theta} \sum_{i=1}^{p} \beta_i \lambda_i \\
&= Re\, e^{-i\theta} \sum_{i=1}^{p} \beta_i \lim_{\alpha \to +\infty} \langle Ax_{i_\alpha}, x'_{i_\alpha} \rangle \\
&= \sum_{i=1}^{p} \beta_i (\lim_{\alpha \to +\infty} Re\, e^{-i\theta} \langle (A + K)x_{i_\alpha}, x'_{i_\alpha} \rangle - \lim_{\alpha \to +\infty} Re\, e^{-i\theta} \langle Kx_{i_\alpha}, x'_{i_\alpha} \rangle).
\end{aligned}
$$

Since K is compact and $x_{i_\alpha} \to 0$ weakly, then $\lim_{\alpha \to +\infty} \langle Kx_{i_\alpha}, x'_{i_\alpha} \rangle = 0$. Therefore

$$Re\, e^{-i\theta} \lambda \le \sum_{i=1}^{p} \beta_i \omega = \omega.$$

This implies that $\lambda \in \sigma_n(A + K)$, for all $K \in K(X)$. Hence $cl(co\{N_e(A)\}) \subset \sigma_{en}(A)$, since $\sigma_{en}(A)$ is closed. $\square$

In the following result, we give a particular bound for the essential numerical spectrum of a bounded operator.

Proposition 2.9. *Let A be a bounded linear operator on a Banach space X. Then we have*

$$\sigma_{en}(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \le \|A\|_e\}$$

where $\|A\|_e = \inf_{K \in K(X)} \|A + K\|$.

Proof. Let $\lambda \in \sigma_{en}(A) = \bigcap_{K \in K(X)} \sigma_n(A + K)$, then it follows from Proposition 2.4 in [1] that for all $K \in K(X)$, $|\lambda| \le \|A + K\|$. Hence

$$|\lambda| \le \inf_{K \in K(X)} \|A + K\| = \|A\|_e.$$

$\square$

It is well known that, if K is a compact operator on an infinite dimensional Banach space, then $\sigma_e(K) = \{0\}$. But the converse is not true. In the following, we will use the notion of the essential numerical spectrum to characterize a compact operator. For this, we introduce the measure of non-compactness (see [18], [15] or [5]).

Definition 2.10. *Let A be a bounded linear operator on a Banach space X. We define the seminorm $\|.\|_\mu$ in B(X), called measure of non-compactness, as:*

$$\|A\|_\mu = \inf \{ \|A|_M\| : M \subset X \text{ a subspace of finite codimension}\}.$$

We list some useful properties of the measure of non-compactness.

Lemma 2.11. *Let A be a bounded linear operator on Banach space X, then we have*

1. $\|A\|_\mu = 0$ *if and only if A is a compact.*

2. $\|A\|_\mu$ *is an algebra seminorm in* $B(X)$*, i.e., for all* $T, S \in B(X)$ *and* $\alpha \in \mathbb{C}$*, we have*

$$
\begin{aligned}
\|T + S\|_\mu &\leq \|T\|_\mu + \|S\|_\mu \\
\|TS\|_\mu &\leq \|T\|_\mu.\|S\|_\mu \\
\|\alpha T\|_\mu &= |\alpha|\|T\|_\mu.
\end{aligned}
$$

In the next Theorem, by using the notion of essential numerical spectrum and the measure of non-compactness, we give a characterization of compact operator. This result is a generalization of the result of J. H. Anderson and J. G. Stampfli [3] in the case of Hilbert space and already proved in [7].

Theorem 2.12. *Let K be a bounded linear operator on an infinite dimensional Banach space X, then* $\sigma_{en}(K) = \{0\}$ *if and only if K is compact.*

Proof. Let $K \in K(X)$, it follows from Proposition 2.9 that $\sigma_{en}(K) \subset \{0\}$. On the other hand, since the essential spectrum of a compact operator in an infinite dimensional space is reduced to $\{0\}$, and $\sigma_e(K) \subset \sigma_{en}(K)$, we deduce that $\sigma_{en}(K) = \{0\}$.
For the converse, we consider $K \in B(X)$ such that $\sigma_{en}(K) = \{0\}$, then by Theorem 2.8 we have $cl(co\{N_e(K)\}) \subset \{0\}$.
It follows from [5] that $cl(co\{N_e(K)\}) \neq \emptyset$ and $e^{-1}\|K\|_\mu \leq \max\{|\lambda| : \lambda \in co\{N_e(K)\}\} \leq \|K\|_\mu$. This implies $\|K\|_\mu = 0$. Therefore $K \in K(X)$. $\square$

Corollary 2.13. *Let A be a bounded operator on a infinite dimensional Banach space X.*
$\sigma_{en}(A) = \{\lambda\}$ *if and only if* $A = \lambda I + K$ *where K is compact.*

Proof. The result follows immediately from Theorem 2.12 and Proposition 2.6. $\square$

3. Essential numerical growth bound, Essential numerical spectral bound

In this section, in analogy to the essential growth bound, the essential spectral bound and the essential radius, we define the essential numerical growth bound $\omega_{n,e}^\theta(.)$, the essential numerical spectral bound $s_{n,e}^\theta(.)$ and we discuss their properties.
It is well known that if A is generator of a C_0-semigroup $(T(t))_{t\geq0}$, its essential growth bound $\omega_e(.)$ is defined as

$$
\omega_e(A) = \inf\{\omega \in \mathbb{R} : \exists M \text{ such that } \|T(t)\|_\mu \leq Me^{t\omega}, \text{ for all } t \geq 0\}.
$$

In [1], the authors defined similar constants related to the numerical spectrum as follows: the numerical growth bound of A_θ is

$$
\omega_n^\theta(A) := \inf\{\omega \in \mathbb{R}; \|T_\theta(t)\| \leq e^{t\omega}, \text{ for all } t \geq 0\}.
$$

We combine the essential growth bound and the numerical growth bound to define the essential numerical growth bound as follows:

Definition 3.1. *Let* $(A, D(A))$ *be a closed and densely defined linear operator on a Banach space X. For* $\theta \in [0, 2\pi[$*, we consider* $A_\theta := e^{-i\theta}A$ *and, in case it exists, the corresponding semigroup* $(T_\theta(t))_{t\geq0}$*. Then we define the essential numerical growth bound of* A_θ *as:*

$$
\omega_{n,e}^\theta(A) := \inf\{\omega \in \mathbb{R} \text{ such that } \|T_\theta(t)\|_\mu \leq e^{t\omega}, \text{ for all } t \geq 0\},
$$

where we set $\inf \emptyset = +\infty$*.*

With the above alternate definition of the essential growth bound, we obtain the following results:

Proposition 3.2. 1. *In contrast to the essential growth bound, the essential numerical growth bound of a closed densely defined linear operator A, if not* $+\infty$*, is attained, i.e.*

$$
\omega_{n,e}^\theta(A) = \min\{\omega \in \mathbb{R} \text{ such that } \|T_\theta(t)\|_\mu \leq e^{t\omega}, \text{ for all } t \geq 0\}
$$

and therefore $\|T_\theta(t)\|_\mu \leq e^{t\omega_{n,e}^\theta(A)}$*, for all* $t \geq 0$*.*

2. $\omega_e(A_\theta) \leq \omega_{n,e}^\theta(A)$, $\forall \theta \in [0, 2\pi)$.
3. $\omega_{n,e}^\theta(A + K) = \omega_{n,e}^\theta(A)$, $\forall K \in K(X)$, $\forall \theta \in [0, 2\pi)$

Proof. 1. Let $(\omega_p)_{p \geq 0}$ be a decreasing sequence with $\omega_p \to \omega_{n,e}^\theta(A)$ as $p \to +\infty$.
Then we have

$$\|T_\theta(t)\|_\mu \leq e^{t\omega_p}, \text{ for all } p \geq 0, t \geq 0.$$

Hence as $p \to +\infty$, $\|T_\theta(t)\|_\mu \leq e^{t\omega_{n,e}^\theta(A)}$, $\forall t \geq 0$.

2. The claim follows immediately from the following inclusion since

$$\{\omega \in \mathbb{R} : \|T_\theta(t)\|_\mu \leq e^{t\omega}, \forall t \geq 0\} \subset \{\omega \in \mathbb{R} : \exists M_\omega | \|T_\theta(t)\|_\mu \leq M_\omega e^{t\omega}, \forall t \geq 0\}.$$

3. It is well known that if A_θ generates a C_0-semigroup $(T_\theta(t))_{t \geq 0}$, then the perturbed operator $(A + K)_\theta$ generates a C_0-semigroup given by:

$$T_{A+K,\theta}(t) = T_\theta(t) + \int_0^t T_\theta(s)KT_{A+K,\theta}(t - s)ds, \ t \geq 0.$$

So, $\|T_\theta(t)\|_\mu = \|T_{A+K,\theta}(t)\|_\mu \leq e^{t\omega}$, this implies that $\omega_{n,e}^\theta(A + K) = \omega_{n,e}^\theta(A)$.

$\square$

We recall that the essential growth bound is also defined by

$$\omega_e(A_\theta) = \inf_{t>0} \frac{1}{t} \log \|T_\theta(t)\|_\mu = \lim_{t\to\infty} \frac{1}{t} \log \|T_\theta(t)\|_\mu,$$

(see [4] or [9]). Analogously, we prove the following results for the essential numerical growth bound.

Proposition 3.3. *Let $(A, D(A))$ be a closed and densely defined linear operator on a Banach space X and $(T_\theta(t))_{t \geq 0}$ be the semigroup generated by A_θ for $\theta \in [0, 2\pi)$. Then*

$$\omega_{n,e}^\theta(A) =^{(1)} \sup_{t>0} \frac{1}{t} \log \|T_\theta(t)\|_\mu =^{(2)} \lim_{t\to 0} \frac{1}{t} \log \|T_\theta(t)\|_\mu.$$

Proof. To show (1), we note that it follows from Proposition 3.2, that $\|T_\theta(t)\|_\mu \leq e^{t\omega_{n,e}^\theta(A)}$ for all $t \geq 0$. Thus

$$\sup_{t>0} \frac{1}{t} \log \|T_\theta(t)\|_\mu \leq \omega_{n,e}^\theta(A), \ \forall \ \theta \in [0, 2\pi).$$

On the other hand, assume that $v = \sup_{t>0} \frac{1}{t} \log \|T_\theta(t)\|_\mu < \omega_{n,e}^\theta(A)$ and take $\phi \in (v, \omega_{n,e}^\theta(A))$, then, for all $t \geq 0$, we have

$$\frac{1}{t} \log \|T_\theta(t)\|_\mu \leq \phi. \tag{1}$$

Since $\phi < \omega_{n,e}^\theta(A)$, there exists $t > 0$ such that $\|T_\theta(t)\|_\mu > e^{t\phi}$, therefore

$$\frac{1}{t} \log \|T_\theta(t)\|_\mu > \phi,$$

and this contradicts inequality (1).

To prove equality (2), we will assume, without loss of generality, that $\omega_{n,e}^\theta(A) = 0$, i.e., $\|T_\theta(t)\|_\mu \leq 1$ for all $t \geq 0$. Since $\|.\|_\mu$ is an algebra seminorm, then the map $t \mapsto \|T_\theta(t)\|_\mu$ is decreasing. Hence,

$$\sup_{t>0} \frac{1}{t} \log \|T_\theta(t)\|_\mu = \lim_{t\to 0} \frac{1}{t} \log \|T_\theta(t)\|_\mu.$$

$\square$

Analogously to the essential spectral bound, we introduce the essential numerical spectral bound associated to $\sigma_{en}(.)$.

Definition 3.4. *For a closed and densely defined linear operator $(A, D(A))$ on a Banach space X and $\theta \in [0, 2\pi)$, we define the essential numerical spectral bound of A as*

$$s_{n,e}^{\theta}(A) := \sup\{Re\lambda; \quad e^{i\theta}\lambda \in \sigma_{en}(A)\}.$$

Note that $s_{n,e}^{\theta}(A)$ can be any real number including $-\infty$ and $+\infty$.
In the case of bounded operator, we consider the essential numerical radius of A

$$r_{n,e}(A) := \sup\{|\lambda|; \quad \lambda \in \sigma_{en}(A)\}.$$

Proposition 3.5. *Let A be a bounded linear operator on a Banach space X. Then $r_{n,e}(A) = \sup\limits_{\theta \in [0,2\pi[} s_{n,e}^{\theta}(A)$*

Proof. " $\leq$ " Let $\lambda \in \sigma_{en}(A)$, there exist $\theta_0 \in [0, 2\pi[$ and $r > 0$ such that $\lambda = re^{i\theta_0}$. We have $r \in \{Re\lambda; e^{i\theta_0}\lambda \in \sigma_{en}(A)\}$ and $|\lambda| = r \leq s_{n,e}^{\theta_0}(A) \leq \sup\limits_{\theta \in [0,2\pi[} s_{n,e}^{\theta}(A)$. Then $r_{n,e}(A) \leq \sup\limits_{\theta \in [0,2\pi[} s_{n,e}^{\theta}(A)$.
" $\geq$ " For all $\theta \in [0, 2\pi[$ and for all λ such that $e^{i\theta}\lambda \in \sigma_{en}(A)$, we have
$Re\lambda \leq |\lambda| \leq r_{n,e}(A)$. Then for all $\theta \in [0, 2\pi[$, $s_{n,e}^{\theta}(A) \leq r_{n,e}(A)$. Hence

$$\sup\limits_{\theta \in [0,2\pi[} s_{n,e}^{\theta}(A) \leq r_{n,e}(A).$$

$\square$

4. Example

Let Ω be a locally compact space, we consider the Banach space $X = C_0(\Omega)$, where

$$C_0(\Omega) := \{f \in C(\Omega): \forall \varepsilon > 0 \text{ there exists a compact } K_\varepsilon \subset \Omega \text{ such that } |f(s)| < \varepsilon, \forall s \in \Omega \backslash K_\varepsilon\}.$$

Let $\varphi : \Omega \to \mathbb{C}$ a continuous function verifying the following hypothesis

$$(H): \quad \varphi^{-1}(\{\lambda\}) \text{ is a countable set}, \quad \forall \lambda \in \varphi(\Omega).$$

We consider the multiplication operator

$$M_\varphi f = \varphi f, \text{ for all } f \in D(M_\varphi) = \{f \in X: \varphi f \in X\}.$$

It is well known that the operator $(M_\varphi, D(M_\varphi))$ is a closed and densely defined linear operator and $\sigma(M_\varphi) = cl(\varphi(\Omega))$ (see [11]).
In the following proposition, we describe the essential numerical spectrum of M_φ.

Proposition 4.1.
$$\sigma_{en}(M_\varphi) = cl(co(\sigma_e(M_\varphi)))$$

Proof.
As a first step, since $\sigma(M_\varphi) = \sigma_p(M_\varphi) \cup \sigma_e(M_\varphi)$, we study the point spectrum.
Let $\lambda \in \sigma_p(M_\varphi) \subset \sigma(M_\varphi) = cl(\varphi(\Omega))$, then there exists $f \in X$, $f \neq \widetilde{0}$ such that $M_\varphi f = \lambda f$.
Hence $(\varphi(x) - \lambda)f(x) = 0$ for all $x \in \Omega$.
Since φ satisfy (H) and f is a continuous function in Ω, we obtain $f = \widetilde{0}$, thus contradict the definition of f.
We at once deduce that $\sigma_p(M_\varphi) = \varnothing$ and $\sigma_e(M_\varphi) = \sigma(M_\varphi) = cl(\varphi(\Omega))$.

In the second step, we characterize the essential numerical spectrum.

In [1], Adler et al. prove that $\sigma_n(M_\varphi) = cl(co\varphi(\Omega))$, then we have the following inclusion

$$cl(\varphi(\Omega)) = \sigma_e(M_\varphi) \subset \sigma_{en}(M_\varphi) \subset \sigma_n(M_\varphi) = cl(co\varphi(\Omega)).$$

Since $\sigma_{en}(M_\varphi)$ is a convex set, then we have

$$\sigma_{en}(M_\varphi) = cl(co\varphi(\Omega)) = \sigma_n(M_\varphi).$$

$\square$

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