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Some Results on Dual Third-Order Jacobsthal Quaternions

Gamaliel Cerda-Morales[a]

[a] *Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Valparaíso, Chile.*

Abstract. Dual Fibonacci and dual Lucas numbers are defined with dual Fibonacci and Lucas quaternions in Nurkan and Güven [14]. In this study, we define the dual third-order Jacobsthal quaternion and the dual third-order Jacobsthal-Lucas quaternion. We derive the relations between the dual third-order Jacobsthal quaternion and dual third-order Jacobsthal-Lucas quaternion which connected the third-order Jacobsthal and third-order Jacobsthal-Lucas numbers. In addition, we give the generating functions, the Binet and Cassini formulas for these new types of quaternions.

1. Introduction

A. F. Horadam [11] defined the Jacobsthal numbers J_n by the recurrence relation

$$J_0 = 0,\ J_1 = 1,\ J_{n+1} = J_n + 2J_{n-1},\ n \geq 1. \tag{1}$$

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation

$$j_0 = 2,\ j_1 = 1,\ j_{n+1} = j_n + 2j_{n-1},\ n \geq 1. \tag{2}$$

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g., [1]). In [5] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [11] is expanded and extended to several identities for some of the higher order cases. In particular, the third-order Jacobsthal numbers $J_n^{(3)}$ and the third-order Jacobsthal-Lucas numbers $j_n^{(3)}$ are defined by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1,\ n \geq 0, \tag{3}$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5,\ n \geq 0, \tag{4}$$

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Email address: gamaliel.cerda.m@mail.pucv.cl (Gamaliel Cerda-Morales)

respectively. Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2,\ \text{and}\ x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{2}{7} \cdot 2^n - \left(\frac{3+2i\sqrt{3}}{21}\right)\omega_1^n - \left(\frac{3-2i\sqrt{3}}{21}\right)\omega_2^n \tag{5}$$

and

$$j_n^{(3)} = \frac{8}{7} \cdot 2^n + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n, \tag{6}$$

respectively.

On the other hand, Horadam [9] introduced the n-th Fibonacci and the n-th Lucas quaternion as follow:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}, \tag{7}$$

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}, \tag{8}$$

respectively. Here F_n and L_n are the n-th Fibonacci and Lucas numbers, respectively. Furthermore, the basis i, j, k satisface the following rules:

$$i^2 = j^2 = k^2 = -1,\ ijk = -1. \tag{9}$$

Note that the rules (9) imply $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. In general, a quaternion is a hyper-complex number and is defined by $q = q_0 + iq_1 + jq_2 + kq_3$, where i, j, k are as in (9). Note that we can write $q = q_0 + u$ where $u = iq_1 + jq_2 + kq_3$. The conjugate of the quaternion q is denoted by $\overline{q} = q_0 - u$. The norm of a quaternion q is defined by $Nr(q) = q\overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$.

Many interesting properties of Fibonacci and Lucas quaternions can be found in [7, 10]. In [8], Halici investigated complex Fibonacci quaternions. In [10] Horadam mentioned the possibility of introducing Pell quaternions and generalized Pell quaternions. In [15], the authors defined the Jacobsthal quaternions and the Jacobsthal-Lucas quaternions. In [3], we defined the third-order Jacobsthal quaternion and third-order Jacobsthal-Lucas quaternion. Furthermore, we investigated some of their identities. In this paper we need some of them.

$$\frac{1}{3}(jQ_n^{(3)} - 4jQ_{n-2}^{(3)}) = \begin{cases} 2 - i - j + 2k & \text{if} \quad n \equiv 0 \pmod 3 \\ -1 - i + 2j - k & \text{if} \quad n \equiv 1 \pmod 3 \\ -1 + 2i - j - k & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{10}$$

$$3JQ_n^{(3)} + jQ_n^{(3)} = 2^{n+1}(1 + 2i + 4j + 8k), \tag{11}$$

$$jQ_n^{(3)} - JQ_{n+2}^{(3)} = \begin{cases} 1 - i + k & \text{if} \quad n \equiv 0 \pmod 3 \\ -1 + j - k & \text{if} \quad n \equiv 1 \pmod 3 \\ i - j & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{12}$$

$$\sum_{s=0}^{n} jQ_s^{(3)} = \begin{cases} jQ_{n+1}^{(3)} + (1 - 4i - 5j - 7k) & \text{if} \quad n \equiv 0 \pmod 3 \\ jQ_{n+1}^{(3)} - 2(1 + 2i + j + 5k) & \text{if} \quad n \equiv 1 \pmod 3 \\ jQ_{n+1}^{(3)} - (2 + i + 5j + 10k) & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{13}$$

$$jQ_n^{(3)} - 4JQ_n^{(3)} = \begin{cases} 2 - 3i + j + 2k & \text{if} \quad n \equiv 0 \pmod 3 \\ -3 + i + 2j - 3k & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 + 2i - 3j + k & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{14}$$

$$jQ_{n+1}^{(3)} + jQ_n^{(3)} = 3JQ_{n+2}^{(3)} \tag{15}$$

and

$$N(JQ_n^{(3)}) = \frac{1}{49}\begin{cases} 340\cdot 2^{2n} - 64\cdot 2^n + 18 & \text{if} \quad n \equiv 0 \pmod 3 \\ 340\cdot 2^{2n} + 68\cdot 2^n + 23 & \text{if} \quad n \equiv 1 \pmod 3 \\ 340\cdot 2^{2n} - 4\cdot 2^n + 15 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}. \tag{16}$$

Here $JQ_n^{(3)}$ and $jQ_n^{(3)}$ are the n-th third-order Jacobsthal quaternion and the n-th third-order Jacobsthal-Lucas quaternion, respectively.

Our main purpose in this paper is on the dual third-order Jacobsthal quaternions as an applied algebra. It is arranged as follows. In the next section we introduce dual third-order Jacobsthal quaternions and we give a brief survey of the basic properties of these quaternions. In the final section we give our results, making some comments on the application of dual quaternions to the dual third-order Jacobsthal quaternions in general quaternion algebra.

2. Dual Third-Order Jacobsthal Quaternions

The dual number invented by Clifford [4] is of the form $D = a + \varepsilon b$, where $a, b \in \mathbb{R}$ and ε is known as the dual unit and it has the following properties:

$$\varepsilon \neq 0,\ \ 0\varepsilon = \varepsilon 0 = 0,\ \ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \ \varepsilon^2 = 0.$$

The real numbers a and b are called the real and dual parts of the number D, respectively. We notice that the dual numbers are extension of the real numbers. The set of dual numbers can be denoted by $\mathbb{D}$ or $\mathbb{R}[\varepsilon]$. The set of dual numbers is a commutative ring having the εb as divisors of zero, is not a field.

Let $\mathbb{H} = \mathbb{R}[i, j, k]$ the algebra of quaternions. A dual quaternion h can be defined in a similar way to the dual numbers, that is $h = p + \varepsilon q$, where $p, q \in \mathbb{H}$. The addition, multiplication and product with a scalar α can be defined as

$$h_1 + h_2 = (p_1 + \varepsilon q_1) + (p_2 + \varepsilon q_2) = (p_1 + p_2) + \varepsilon(q_1 + q_2),$$
$$h_1 h_2 = (p_1 + \varepsilon q_1)(p_2 + \varepsilon q_2) = p_1 p_2 + \varepsilon(p_1 q_2 + q_1 p_2)$$

and

$$\alpha h_1 = \alpha(p_1 + \varepsilon q_1) = \alpha p_1 + \varepsilon \alpha q_1,$$

respectively, where $p_s, q_s \in \mathbb{H}$ $(s = 1, 2)$. Thus, the set of dual quaternions forms noncommutative but associative algebra over the real numbers. Therefore it can be called as algebra of dual quaternions. Furthermore, we can write

$$h = D_0 + D_1 i + D_2 j + D_3 k,\ D_s \in \mathbb{D}\ (s = 0, 1, 2, 3),$$

where $D_s = a_s + \varepsilon b_s$. Hence any dual quaternion h is constructed from eight real parameters. The dual quaternion h consist of a scalar part $S_h = D_0$ and a vector part $V_h = D_1 i + D_2 j + D_3 k$.

In [14], the authors we defined the dual Fibonacci quaternions and dual Lucas quaternions, they derived the relations between the dual Fibonacci and the classic fibonacci numbers. Motivated by this work, we can define the following type of numbers.

Definition 2.1. *The n-th dual third-order Jacobsthal quaternion and n-th dual third-order Jacobsthal-Lucas quaternions are defined as*

$$JD_n^{(3)} = JQ_n^{(3)} + \varepsilon JQ_{n+1}^{(3)},\ n \geq 0, \tag{17}$$

and

$$jD_n^{(3)} = jQ_n^{(3)} + \varepsilon jQ_{n+1}^{(3)},\ n \geq 0, \tag{18}$$

respectively. Here $JQ_n^{(3)} = J_n^{(3)} + iJ_{n+1}^{(3)} + jJ_{n+2}^{(3)} + kJ_{n+3}^{(3)}$ *is the n-th third-order Jacobsthal quaternion and* $jQ_n^{(3)} = j_n^{(3)} + ij_{n+1}^{(3)} + jj_{n+2}^{(3)} + kj_{n+3}^{(3)}$ *is the n-th third-order Jacobsthal-Lucas quaternion.*

In this sense, if we denote the n-th dual third-order Jacobsthal number $\widehat{J}_n^{(3)} = J_n^{(3)} + \varepsilon J_{n+1}^{(3)}$, the dual third-order Jacobsthal quaternion $JD_n^{(3)}$ can be represented as

$$JD_n^{(3)} = \widehat{J}_n^{(3)} + i\widehat{J}_{n+1}^{(3)} + j\widehat{J}_{n+2}^{(3)} + k\widehat{J}_{n+3}^{(3)}. \tag{19}$$

The conjugate of the dual third-order Jacobsthal quaternion $JD_n^{(3)}$, or briefly $\overline{JD}_n^{(3)}$, is defined by

$$\overline{JD}_n^{(3)} = S_{JD_n^{(3)}} - V_{JD_n^{(3)}} = \widehat{J}_n^{(3)} - i\widehat{J}_{n+1}^{(3)} - j\widehat{J}_{n+2}^{(3)} - k\widehat{J}_{n+3}^{(3)}, \tag{20}$$

where $\left(S_{JD_n^{(3)}}, V_{JD_n^{(3)}}\right) = \left(\widehat{J}_n^{(3)}, i\widehat{J}_{n+1}^{(3)} + j\widehat{J}_{n+2}^{(3)} + k\widehat{J}_{n+3}^{(3)}\right)$.

Also the norm of $JD_n^{(3)}$ can be given as $Nr(JD_n^{(3)}) = JD_n^{(3)} \cdot \overline{JD}_n^{(3)}$. Then, we get

$$Nr(JD_n^{(3)}) = \sum_{s=0}^{3} \left((J_{n+s}^{(3)})^2 + 2\varepsilon J_{n+s}^{(3)} J_{n+s+1}^{(3)} \right). \tag{21}$$

Dual third-order Jacobsthal quaternion $JD_n^{(3)}$ with norm unity can be called the unit dual third-order Jacobsthal quaternion. Now, let $Nr(JD_n^{(3)}) \neq 0$, then the inverse of a dual third-order Jacobsthal quaternion $JD_n^{(3)}$ is also a dual third-order Jacobsthal quaternion, and it can be defined as $(JD_n^{(3)})^{-1} = \frac{\overline{JD}_n^{(3)}}{Nr(JD_n^{(3)})}$.

Now, we use the notation

$$H_n(a,b) = \frac{A\omega_1^n - B\omega_2^n}{\omega_1 - \omega_2} = \begin{cases} a & \text{if } n \equiv 0 \pmod 3 \\ b & \text{if } n \equiv 1 \pmod 3 \\ -(a+b) & \text{if } n \equiv 2 \pmod 3 \end{cases}, \tag{22}$$

where $A = b - a\omega_2$ and $B = b - a\omega_1$, in which ω_1 and ω_2 are the complex conjugate cube roots of unity (i.e. $\omega_1^3 = \omega_2^3 = 1$). Furthermore, note that for all $n \geq 0$ we have

$$H_{n+2}(a,b) = -H_{n+1}(a,b) - H_n(a,b), \tag{23}$$

where $H_0(a,b) = a$ and $H_1(a,b) = b$.

From the Binet formulas (5), (6) and Eq. (22), we have

$$J_n^{(3)} = \frac{1}{7}\left(2^{n+1} - V_n^{(3)}\right) \text{ and } j_n^{(3)} = \frac{1}{7}\left(2^{n+3} + 3V_n^{(3)}\right), \tag{24}$$

where $V_n^{(3)} = H_n(2,-3)$.

Lemma 2.2. *For $n \geq 0$,*

$$49\left(J_n^{(3)} \cdot J_{n+1}^{(3)}\right) = \begin{cases} 2^{2n+3} - 2^{n+1} - 6 & \text{if } n \equiv 0 \pmod 3 \\ 2^{2n+3} + 5 \cdot 2^{n+1} - 3 & \text{if } n \equiv 1 \pmod 3 \\ 2^{2n+3} - 4 \cdot 2^{n+1} + 2 & \text{if } n \equiv 2 \pmod 3 \end{cases}. \tag{25}$$

Proof. To obtain formula (25), it suffices to take the Binet's formula of $J_n^{(3)}$. Let $a = 1 + \frac{2i\sqrt{3}}{3}$ and $b = 1 - \frac{2i\sqrt{3}}{3}$, then

$$\begin{aligned} 49\left(J_n^{(3)} \cdot J_{n+1}^{(3)}\right) &= (2^{n+1} - V_n^{(3)})(2^{n+2} - V_{n+1}^{(3)}) \\ &= 2^{2n+3} - (V_{n+1}^{(3)} + 2V_n^{(3)})2^{n+1} + V_n^{(3)}V_{n+1}^{(3)}, \end{aligned}$$

where $V_n^{(3)} = a\omega_1^n + b\omega_2^n$. It is easy to see that

$$V_n^{(3)} = \begin{cases} 2 & \text{if} \quad n \equiv 0 \pmod 3 \\ -3 & \text{if} \quad n \equiv 1 \pmod 3 \\ 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{26}$$

since ω_1 and ω_2 are the complex conjugate cube roots of unity. Then, if $n \equiv 0 (\text{mod } 3)$ we can write $V_{n+1}^{(3)} + 2V_n^{(3)} = 1$ and $V_n^{(3)} V_{n+1}^{(3)} = -6$. The other identities are clear from equation (26). Thus, the proof is completed. □

Theorem 2.3. *Let $n \geq 0$ integer. Then,*

$$49 \cdot Nr(JD_n^{(3)}) = \begin{cases} t_{\varepsilon,n} + (18 - 26\epsilon) - (32 + 30\epsilon)2^{n+1} & \textit{if} \quad n \equiv 0 \pmod 3 \\ t_{\varepsilon,n} + (23 - 20\epsilon) + (34 + 66\epsilon)2^{n+1} & \textit{if} \quad n \equiv 1 \pmod 3 \\ t_{\varepsilon,n} + (15 - 10\epsilon) - (2 + 36\epsilon)2^{n+1} & \textit{if} \quad n \equiv 2 \pmod 3 \end{cases},$$

where $t_{\varepsilon,n} = 85(1 + 4\epsilon)2^{2(n+1)}$.

Proof. To prove the above equation, we can use the definition

$$Nr(JD_n^{(3)}) = Nr(JQ_n^{(3)}) + 2\epsilon \left(\sum_{s=0}^{3} J_{n+s}^{(3)} J_{n+s+1}^{(3)} \right), \tag{27}$$

where $JQ_n^{(3)}$ is the n-th third-order Jacobsthal quaternion. Moreover, if $n \equiv 0 \pmod 3$, by the lemma 2.2 we have $J_n^{(3)} \cdot J_{n+1}^{(3)} = \frac{1}{49}(2^{2n+3} - 2^{n+1} - 6)$, $J_{n+1}^{(3)} \cdot J_{n+2}^{(3)} = 2^{2n+5} + 5 \cdot 2^{n+2} - 3$ and $J_{n+2}^{(3)} \cdot J_{n+3}^{(3)} = 2^{2n+7} - 4 \cdot 2^{n+3} + 2$, then

$$\begin{aligned} \sum_{s=0}^{3} J_{n+s}^{(3)} J_{n+s+1}^{(3)} &= \frac{1}{49}(2^{2n+3} - 2^{n+1} - 6) + \frac{1}{49}(2^{2n+5} + 5 \cdot 2^{n+2} - 3) \\ &\quad + \frac{1}{49}(2^{2n+7} - 4 \cdot 2^{n+3} + 2) + \frac{1}{49}(2^{2n+9} - 2^{n+4} - 6) \\ &= \frac{1}{49}(85 \cdot 2^{2n+3} - 15 \cdot 2^{n+1} - 13). \end{aligned}$$

Furthermore, $Nr(JD_n^{(3)})$ can be write as

$$\begin{aligned} Nr(JD_n^{(3)}) &= \frac{1}{49}\left[(85 \cdot 2^{2n+2} - 64 \cdot 2^n + 18) + 2\varepsilon(85 \cdot 2^{2n+3} - 15 \cdot 2^{n+1} - 13)\right] \\ &= \frac{1}{49}\left[(85 + 340\varepsilon)2^{2n+2} - (32 + 30\varepsilon)2^{n+1} + (18 - 26\varepsilon)\right]. \end{aligned}$$

The other identities are clear from equation (25). □

By using the equations (17) and (20), we can compute

$$JD_n^{(3)} + \overline{JD}_n^{(3)} = 2(J_n^{(3)} + \varepsilon J_{n+1}^{(3)}) = 2\widehat{J}_n^{(3)}.$$

Then, we obtain the result by using the equation given in Theorem 2.3 as

$$\begin{aligned} (JD_n^{(3)})^2 &= JD_n^{(3)}(2\widehat{J}_n^{(3)} - \overline{JD}_n^{(3)}) \\ &= 2JD_n^{(3)} \cdot \widehat{J}_n^{(3)} - JD_n^{(3)}\overline{JD}_n^{(3)} \\ &= 2JD_n^{(3)} \cdot \widehat{J}_n^{(3)} - Nr(JD_n^{(3)}). \end{aligned}$$

Theorem 2.4. *Let $n \geq 0$ integer. Then,*

$$3JD_n^{(3)} + jD_n^{(3)} = 2^{n+1}(1+2\varepsilon)(1+2i+4j+8k). \tag{28}$$

Proof. Let $JD_n^{(3)} = JQ_n^{(3)} + \varepsilon JQ_{n+1}^{(3)}$ and $jD_n^{(3)} = jQ_n^{(3)} + \varepsilon jQ_{n+1}^{(3)}$. Then, we have

$$\begin{aligned} 3JD_n^{(3)} + jD_n^{(3)} &= 3(JQ_n^{(3)} + \varepsilon JQ_{n+1}^{(3)}) + (jQ_n^{(3)} + \varepsilon jQ_{n+1}^{(3)}) \\ &= (3JQ_n^{(3)} + jQ_n^{(3)}) + \varepsilon(3JQ_{n+1}^{(3)} + jQ_{n+1}^{(3)}) \end{aligned}$$

Using Eq. (11), we obtain that

$$\begin{aligned} 3JD_n^{(3)} + jD_n^{(3)} &= (2^{n+1}(1+2i+4j+8k)) + \varepsilon(2^{n+2}(1+2i+4j+8k)) \\ &= 2^{n+1}(1+2\varepsilon)(1+2i+4j+8k). \end{aligned}$$

Thus, the proof is completed. □

Theorem 2.5. *Let $n \geq 2$ integer. Then, we have*

$$\frac{1}{3}(jD_n^{(3)} - 4jD_{n-2}^{(3)}) = \begin{cases} (2-\varepsilon) - (1+\varepsilon)i - (1-2\varepsilon)j + (2-\varepsilon)k & \textit{if} \quad n \equiv 0 \pmod 3 \\ -(1+\varepsilon) - (1-2\varepsilon)i + (2-\varepsilon)j - (1+\varepsilon)k & \textit{if} \quad n \equiv 1 \pmod 3 \\ (-1+2\varepsilon) + (2-\varepsilon)i - (1+\varepsilon)j - (1-2\varepsilon)k & \textit{if} \quad n \equiv 2 \pmod 3 \end{cases}. \tag{29}$$

Proof. To prove this theorem, we need the Eq. (10). For definition, we have $jD_n^{(3)} = jQ_n^{(3)} + \varepsilon jQ_{n+1}^{(3)}$ and $jD_{n-2}^{(3)} = jQ_{n-2}^{(3)} + \varepsilon jQ_{n-1}^{(3)}$. Then,

$$\begin{aligned} jD_n^{(3)} - 4jD_{n-2}^{(3)} &= jQ_n^{(3)} + \varepsilon jQ_{n+1}^{(3)} - 4(jQ_{n-2}^{(3)} + \varepsilon jQ_{n-1}^{(3)}) \\ &= (jQ_n^{(3)} - 4jQ_{n-2}^{(3)}) + \varepsilon(jQ_{n+1}^{(3)} - 4jQ_{n-1}^{(3)}). \end{aligned}$$

Using the equation

$$\frac{1}{3}(jQ_n^{(3)} - 4jQ_{n-2}^{(3)}) = \begin{cases} 2-i-j+2k & \text{if} \quad n \equiv 0 \pmod 3 \\ -1-i+2j-k & \text{if} \quad n \equiv 1 \pmod 3 \\ -1+2i-j-k & \text{if} \quad n \equiv 2 \pmod 3 \end{cases},$$

if $n \equiv 0 \pmod 3$ we obtain

$$\begin{aligned} jD_n^{(3)} - 4jD_{n-2}^{(3)} &= 3[(2-i-j+2k) + \varepsilon(-1-i+2j-k)] \\ &= 3[(2-\varepsilon) - (1+\varepsilon)i - (1-2\varepsilon)j + (2-\varepsilon)k]. \end{aligned}$$

In a similar way, the other cases. □

Using the Eqs. (12), (14) and (15) one can easily prove the Theorems 2.6 and 2.7.

Theorem 2.6. *Let $n \geq 0$ integer. Then,*

$$jD_n^{(3)} - 4JD_n^{(3)} = \begin{cases} (2-3\varepsilon) - (3-\varepsilon)i + (1+2\varepsilon)j + (2-3\varepsilon)k & \textit{if} \quad n \equiv 0 \pmod 3 \\ (-3+\varepsilon) + (1+2\varepsilon)i + (2-3\varepsilon)j - (3-\varepsilon)k & \textit{if} \quad n \equiv 1 \pmod 3 \\ (1+2\varepsilon) + (2-3\varepsilon)i - (3-\varepsilon)j + (1+2\varepsilon)k & \textit{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{30}$$

and

$$jD_{n+1}^{(3)} + jD_n^{(3)} = 3JD_{n+2}^{(3)}. \tag{31}$$

Theorem 2.7. *Let $n \geq 0$ integer. Then,*

$$jD_n^{(3)} - JD_{n+2}^{(3)} = \begin{cases} (1-\varepsilon) - i + \varepsilon j + (1-\varepsilon)k & \text{if} \quad n \equiv 0 \pmod 3 \\ -1 + \varepsilon i + (1-\varepsilon)j - k & \text{if} \quad n \equiv 1 \pmod 3 \\ \varepsilon + (1-\varepsilon)i - j + \varepsilon k & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}. \tag{32}$$

The following is a result for the sum of dual third-order Jacobsthal-Lucas quaternions.

Theorem 2.8. *Let $n \geq 0$ integer. Then,*

$$\sum_{s=0}^{n} jD_s^{(3)} = \begin{cases} a_{\varepsilon,n} + b_\varepsilon(1 - 4i - 5j - 7k) & \text{if} \quad n \equiv 0 \pmod 3 \\ a_{\varepsilon,n} - 2b_\varepsilon(1 + 2i + j + 5k) & \text{if} \quad n \equiv 1 \pmod 3 \\ a_{\varepsilon,n} - b_\varepsilon(2 + i + 5j + 10k) & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}, \tag{33}$$

where $a_{\varepsilon,n} = (1+2\varepsilon)jQ_{n+1}^{(3)} - \varepsilon jQ_0^{(3)}$ and $b_\varepsilon = 1 + \varepsilon$.

Proof. Using the equation (13), we have

$$\sum_{s=0}^{n} jQ_s^{(3)} = \begin{cases} jQ_{n+1}^{(3)} + (1 - 4i - 5j - 7k) & \text{if} \quad n \equiv 0 \pmod 3 \\ jQ_{n+1}^{(3)} - 2(1 + 2i + j + 5k) & \text{if} \quad n \equiv 1 \pmod 3 \\ jQ_{n+1}^{(3)} - (2 + i + 5j + 10k) & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}.$$

Furthermore, if $n \equiv 0 \pmod 3$, we can write

$$\begin{aligned} \sum_{s=0}^{n} jD_s^{(3)} &= \sum_{s=0}^{n} jQ_s^{(3)} + \varepsilon \sum_{s=0}^{n} jQ_{s+1}^{(3)} \\ &= (1+\varepsilon)\sum_{s=0}^{n} jQ_s^{(3)} + \varepsilon(jQ_{n+1}^{(3)} - jQ_0^{(3)}) \\ &= (1+\varepsilon)\left(jQ_{n+1}^{(3)} + (1 - 4i - 5j - 7k)\right) + \varepsilon(jQ_{n+1}^{(3)} - jQ_0^{(3)}) \\ &= (1+2\varepsilon)jQ_{n+1}^{(3)} - \varepsilon jQ_0^{(3)} + (1+\varepsilon)(1 - 4i - 5j - 7k). \end{aligned}$$

If $n \equiv 1 \pmod 3$, we have $\sum_{s=0}^{n} jQ_s^{(3)} = jQ_{n+1}^{(3)} - 2(1 + 2i + j + 5k)$, then

$$\sum_{s=0}^{n} jD_s^{(3)} = (1+2\varepsilon)jQ_{n+1}^{(3)} - \varepsilon jQ_0^{(3)} - 2(1+\varepsilon)(1 + 2i + j + 5k).$$

The proof is similar to case $n \equiv 2 \pmod 3$. Thus, the proof is completed. □

3. Generating Function for Dual Third-Order Jacobsthal Quaternions

Let $JD_n^{(3)} = JQ_n^{(3)} + \varepsilon JQ_{n+1}^{(3)}$ be the n-th dual third-order Jacobsthal quaternion. Then, the function $G(t) = \sum_{n=0}^{\infty} JD_n^{(3)} t^n$ is called the generating function for the sequence $\{JD_n^{(3)}\}_{n\geq 0}$. In [3], the author found a generating function for third-order Jacobsthal quaternions. In the following theorem, we established the generating function for dual third-order Jacobsthal quaternions.

Theorem 3.1. *The generating function for the dual third-order Jacobsthal quaternion $\{JD_n^{(3)}\}_{n\geq 0}$ is*

$$\sum_{n=0}^{\infty} JD_n^{(3)} t^n = \frac{1}{1 - t - t^2 - 2t^3} \begin{pmatrix} \varepsilon + (1+\varepsilon)i + (1+2\varepsilon)j + (2+5\varepsilon)k + t(1 + \varepsilon i + (1+3\varepsilon)j + (3+4\varepsilon)k) \\ +2t^2(\varepsilon i + (1+\varepsilon)j + (1+2\varepsilon)k) \end{pmatrix}$$

Proof. Assuming that the generating function of the quaternion $\{JD_n^{(3)}\}_{n\geq 0}$ has the form $G(t)=\sum_{n=0}^{\infty} JD_n^{(3)}t^n$, we obtain that

$$\begin{aligned}(1-t-t^2-2t^3)&G(t)\\ &=(JD_0^{(3)}+JD_1^{(3)}t+\cdots)-(JD_0^{(3)}t+JD_1^{(3)}t^2+\cdots)-\cdots\\ &=JD_0^{(3)}+t(JD_1^{(3)}-JD_0^{(3)})+t^2(JD_2^{(3)}-JD_1^{(3)}-JD_0^{(3)}),\end{aligned}$$

since $JD_n^{(3)}=JD_{n-1}^{(3)}+JD_{n-2}^{(3)}+2JD_{n-3}^{(3)}$, $n\geq 3$ and the coefficients of t^n for $n\geq 3$ are equal to zero. In equivalent form is

$$\begin{aligned}(1-t-t^2-2t^3)\sum_{n=0}^{\infty} JD_n^{(3)}t^n &= JD_0^{(3)}+t(JD_1^{(3)}-JD_0^{(3)})+t^2(JD_2^{(3)}-JD_1^{(3)}-JD_0^{(3)})\\ &=(JQ_0^{(3)}+\epsilon JQ_1^{(3)})+t(\epsilon JQ_2^{(3)}+(1-\epsilon)JQ_1^{(3)}-JQ_0^{(3)})\\ &\quad+t^2(\epsilon JQ_3^{(3)}+(1-\epsilon)JQ_2^{(3)}-(1+\epsilon)JQ_1^{(3)}-JQ_0^{(3)}).\end{aligned}$$

Then,

$$\sum_{n=0}^{\infty} JD_n^{(3)}t^n=\frac{1}{1-t-t^2-2t^3}\left(\begin{array}{c} JQ_0^{(3)}+\epsilon JQ_1^{(3)}+t(\epsilon JQ_2^{(3)}+(1-\epsilon)JQ_1^{(3)}-JQ_0^{(3)})\\ +t^2(\epsilon JQ_3^{(3)}+(1-\epsilon)JQ_2^{(3)}-(1+\epsilon)JQ_1^{(3)}-JQ_0^{(3)})\end{array}\right).$$

Thus, the proof is completed. □

The Binet formula for $JD_n^{(3)}$ can be given in the following theorem.

Theorem 3.2. *If $JD_n^{(3)}=JQ_n^{(3)}+\varepsilon JQ_{n+1}^{(3)}$ be the n-th dual third-order Jacobsthal quaternion, then*

$$JD_n^{(3)}=\frac{1}{7}\left[2^{n+1}\underline{\alpha}-\left(1+\frac{2i\sqrt{3}}{3}\right)\omega_1^n\underline{\beta}-\left(1-\frac{2i\sqrt{3}}{3}\right)\omega_2^n\underline{\gamma}\right]=\frac{1}{7}\left[2^{n+1}\underline{\alpha}-VD_n^{(3)}\right], \tag{34}$$

where $\omega_1,\ \omega_2$ are the solutions of the equation $t^2+t+1=0$, and

$$\begin{aligned}\underline{\alpha}&=(1+2i+4j+8k)(1+2\varepsilon),\\ \underline{\beta}&=(1+\omega_1 i+\omega_1^2 j+k)(1+\omega_1\varepsilon),\\ \underline{\gamma}&=(1+\omega_2 i+\omega_2^2 j+k)(1+\omega_2\varepsilon)\end{aligned}$$

and

$$VD_n^{(3)}=VQ_n^{(3)}+\varepsilon VQ_{n+1}^{(3)},\ VQ_n^{(3)}=V_n^{(3)}+V_{n+1}^{(3)}i+V_{n+2}^{(3)}j+V_{n+3}^{(3)}k.$$

Proof. In [3], the author gave the Binet formula for third-order Jacobsthal quaternion by

$$JQ_n^{(3)}=\frac{1}{7}\left[2^{n+1}\alpha-\left(1+\frac{2i\sqrt{3}}{3}\right)\omega_1^n\beta-\left(1-\frac{2i\sqrt{3}}{3}\right)\omega_2^n\gamma\right], \tag{35}$$

where $\omega_1,\ \omega_2$ are the solutions of $t^2+t+1=0$, and $\alpha=1+2i+4j+8k$, $\beta=1+\omega_1 i+\omega_1^2 j+k$ and $\gamma=1+\omega_2 i+\omega_2^2 j+k$. Thus it can be written

$$\begin{aligned}7\cdot JD_n^{(3)}&=7(JQ_n^{(3)}+\epsilon JQ_{n+1}^{(3)})\\ &=\left[2^{n+1}\alpha-(a\omega_1^n\beta+b\omega_2^n\gamma)\right]+\varepsilon\left[2^{n+2}\alpha-(a\omega_1^{n+1}\beta+b\omega_2^{n+1}\gamma)\right]\\ &=2^{n+1}\alpha(1+2\varepsilon)-a\omega_1^n\beta(1+\omega_1\varepsilon)-b\omega_2^n\gamma(1+\omega_2\epsilon),\end{aligned}$$

where $a=1+\frac{2i\sqrt{3}}{3}$ and $b=1-\frac{2i\sqrt{3}}{3}$. Taking $\underline{\alpha}=\alpha(1+2\varepsilon)$, $\underline{\beta}=\beta(1+\omega_1\varepsilon)$ and $\underline{\gamma}=\gamma(1+\omega_2\varepsilon)$ in last equation, then the proof is completed. □

Theorem 3.3. *If* $jD_n^{(3)} = jQ_n^{(3)} + \epsilon jQ_{n+1}^{(3)}$ *be the n-th dual third-order Jacobsthal-Lucas quaternion, then we have*

$$jD_n^{(3)} = \frac{1}{7}\left[2^{n+3}\underline{\alpha} + (3+2i\sqrt{3})\omega_1^n\underline{\beta} + (3-2i\sqrt{3})\omega_2^n\underline{\gamma}\right] = \frac{1}{7}\left[2^{n+3}\underline{\alpha} + 3VD_n^{(3)}\right], \tag{36}$$

where ω_1, ω_2 *are the solutions of the equation* $t^2+t+1=0$; $\underline{\alpha}$, $\underline{\beta}$ *and* $\underline{\gamma}$ *as before.*

Based on the Binet's formulas given in (34) and (36) for the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions, now we give some quadratic identities for these quaternions.

Theorem 3.4. *For every nonnegative integer number n we get*

$$\left(jD_n^{(3)}\right)^2 + 3JD_{n+3}^{(3)} \cdot jD_{n+3}^{(3)} = 4^{n+3}\underline{\alpha}^2 + \frac{3}{7}\left(2^{n+3}(\underline{\alpha}\cdot VD_n^{(3)} - VD_n^{(3)}\cdot\underline{\alpha})\right), \tag{37}$$

where $\underline{\alpha} = \alpha(1+2\epsilon)$, $\alpha = 1+2i+4j+8k$ *and*

$$VD_n^{(3)} = \begin{cases} (2-3\varepsilon)+(-3+\varepsilon)i+(1+2\varepsilon)j+(2-3\varepsilon)k & \text{if } n\equiv 0 \pmod 3 \\ (-3+\varepsilon)+(1+2\varepsilon)i+(2-3\varepsilon)j+(-3+\varepsilon)k & \text{if } n\equiv 1 \pmod 3 \\ (1+2\varepsilon)+(2-3\varepsilon)i+(-3+\varepsilon)j+(1+2\varepsilon)k & \text{if } n\equiv 2 \pmod 3 \end{cases}.$$

Proof. Let $\underline{\alpha} = \alpha(1+2\epsilon)$. Using the relation in (34) and (36) for the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions, the left side of equality (37) can be written as

$$\begin{aligned}\left(jD_n^{(3)}\right)^2 + 3JD_{n+3}^{(3)}\cdot jD_{n+3}^{(3)} &= \left(\frac{1}{7}\left(2^{n+3}\underline{\alpha}+3VD_n^{(3)}\right)\right)^2 + 3\left(\frac{1}{7}\left(2^{n+4}\underline{\alpha} - VD_{n+3}^{(3)}\right)\right)\cdot\left(\frac{1}{7}\left(2^{n+6}\underline{\alpha}+3VD_{n+3}^{(3)}\right)\right)\\ &= \frac{1}{49}\left(2^{2n+6}\underline{\alpha}^2 + 3\cdot 2^{n+3}(\underline{\alpha}\cdot VD_n^{(3)} + VD_n^{(3)}\cdot\underline{\alpha}) + 9\left(VD_n^{(3)}\right)^2\right)\\ &\quad + \frac{3}{49}\left(2^{2n+10}\underline{\alpha}^2 + 2^{n+4}(3\underline{\alpha}\cdot VD_{n+3}^{(3)} - 4VD_{n+3}^{(3)}\cdot\underline{\alpha}) - 3\left(VD_{n+3}^{(3)}\right)^2\right),\end{aligned}$$

where

$$\begin{aligned} VD_n^{(3)} &= \left(1+\frac{2i\sqrt{3}}{3}\right)\omega_1^n\underline{\beta} + \left(1-\frac{2i\sqrt{3}}{3}\right)\omega_2^n\underline{\gamma}\\ &= \begin{cases} (2-3\varepsilon)+(-3+\varepsilon)i+(1+2\varepsilon)j+(2-3\varepsilon)k & \text{if } n\equiv 0 \pmod 3 \\ (-3+\varepsilon)+(1+2\varepsilon)i+(2-3\varepsilon)j+(-3+\varepsilon)k & \text{if } n\equiv 1 \pmod 3 \\ (1+2\varepsilon)+(2-3\varepsilon)i+(-3+\varepsilon)j+(1+2\varepsilon)k & \text{if } n\equiv 2 \pmod 3 \end{cases}.\end{aligned} \tag{38}$$

Note that $VD_n^{(3)} = VD_{n+3}^{(3)}$ for all $n \geq 0$, which can be simplified as

$$\left(jD_n^{(3)}\right)^2 + 3JD_{n+3}^{(3)}\cdot jD_{n+3}^{(3)} = 2^{2n+6}\underline{\alpha}^2 + \frac{3}{7}\left(2^{n+3}(\underline{\alpha}\cdot VD_n^{(3)} - VD_n^{(3)}\cdot\underline{\alpha})\right).$$

Thus, we get the required result in (37). □

Theorem 3.5. *For every nonnegative integer number n we get*

$$\left(jD_n^{(3)}\right)^2 - 9\left(JD_n^{(3)}\right)^2 = \frac{2^{n+1}}{7}(2\underline{\alpha}^2 + 3(\underline{\alpha}\cdot VD_n^{(3)} + VD_n^{(3)}\cdot\underline{\alpha})), \tag{39}$$

where $\underline{\alpha} = \alpha(1+2\epsilon)$ *and* $VD_n^{(3)}$ *as in (38).*

The proofs of quadratic identities for the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions in this theorem are similar to the proof of the identity (37) of Theorem 3.4, and are omitted here.

Using the notation in Eq. (34), we investigated a type of identities for the dual third-order Jacobsthal quaternions.

Theorem 3.6. *For $m \geq n \geq 0$ integers:*

$$JD_m^{(3)}JD_{n+1}^{(3)} - JD_{m+1}^{(3)}JD_n^{(3)} = \frac{1}{7}\left(2^{m+1}UD_{n+1}^{(3)} - 2^{n+1}UD_{m+1}^{(3)} + WD_{m-n}^{(3)} \right) \tag{40}$$

and

$$\left(JD_{n+1}^{(3)}\right)^2 - JD_{n+2}^{(3)}JD_n^{(3)} = \frac{1}{7}\left(2^{n+1}(2\underline{\alpha}UD_{n+1}^{(3)} - UD_{n+2}^{(3)}\underline{\alpha}) + WD_1^{(3)} \right), \tag{41}$$

where $UD_n^{(3)} = UQ_n^{(3)} + \varepsilon UQ_{n+1}^{(3)}$, $UQ_n^{(3)} = U_n^{(3)} + U_{n+1}^{(3)}i + U_{n+2}^{(3)}j + U_{n+3}^{(3)}k$ *and* $U_n^{(3)} = H_n(0,1)$.

Proof. For $m \geq n$:

$$\begin{aligned} JD_m^{(3)}JD_{n+1}^{(3)} - JD_{m+1}^{(3)}JD_n^{(3)} &= \frac{1}{49}\left(\begin{array}{c} (2^{m+1}\underline{\alpha} - VD_m^{(3)})(2^{n+2}\underline{\alpha} - VD_{n+1}^{(3)}) \\ -(2^{m+2}\underline{\alpha} - VD_{m+1}^{(3)})(2^{n+1}\underline{\alpha} - VD_n^{(3)}) \end{array} \right) \\ &= \frac{1}{49}\left(\begin{array}{c} -2^{m+1}\underline{\alpha}VD_{n+1}^{(3)} - 2^{n+2}VD_m^{(3)}\underline{\alpha} + 2^{m+2}\underline{\alpha}VD_n^{(3)} + 2^{n+1}VD_{m+1}^{(3)}\underline{\alpha} \\ +VD_m^{(3)}VD_{n+1}^{(3)} - VD_{m+1}^{(3)}VD_n^{(3)} \end{array} \right) \\ &= \frac{1}{7}\left(2^{m+1}\underline{\alpha}UD_{n+1}^{(3)} - 2^{n+1}UD_{m+1}^{(3)}\underline{\alpha} + WD_{m-n}^{(3)} \right), \end{aligned} \tag{42}$$

where $UD_{n+1}^{(3)} = \frac{1}{7}\left(2VD_n^{(3)} - VD_{n+1}^{(3)}\right)$, $UD_n^{(3)} = UQ_n^{(3)} + \varepsilon UQ_{n+1}^{(3)}$ and $(\omega_1 - \omega_2)WD_{m-n}^{(3)} = 7(\omega_1^{m-n}\underline{\beta\gamma} - \omega_2^{m-n}\underline{\gamma\beta})$. Furthermore, if $m = n+1$ in Eq. (42), we obtain for $n \geq 0$,

$$\left(JD_{n+1}^{(3)}\right)^2 - JD_{n+2}^{(3)}JD_n^{(3)} = \frac{1}{7}\left(2^{n+1}(2\underline{\alpha}UD_{n+1}^{(3)} - UD_{n+2}^{(3)}\underline{\alpha}) + WD_1^{(3)} \right). \tag{43}$$

□

4. Matrix Representation of Dual Third-Order Jacobsthal Quaternions

The matrix method is very useful method in order to obtain some identities for special sequences. For example, using matrix methods, the authors obtained some identities for various special sequences (see [2, 13]). In this case, the generating matrix of the sequence $\{JD_n^{(3)}\}_{n\geq 0}$ is given by

$$M^n = \left[\begin{array}{ccc} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]^n = \left[\begin{array}{ccc} J_{n+1}^{(3)} & J_n^{(3)} + 2J_{n-1}^{(3)} & 2J_n^{(3)} \\ J_n^{(3)} & J_{n-1}^{(3)} + 2J_{n-2}^{(3)} & 2J_{n-1}^{(3)} \\ J_{n-1}^{(3)} & J_{n-2}^{(3)} + 2J_{n-3}^{(3)} & 2J_{n-2}^{(3)} \end{array} \right], \tag{44}$$

for all $n \geq 0$. We define for convenience $J_{-1}^{(3)} = 0$, $J_{-2}^{(3)} = \frac{1}{2}$ and $J_{-3}^{(3)} = -\frac{1}{4}$.

Now, let us define the following matrix as

$$R = \left[\begin{array}{ccc} JQ_4^{(3)} & JQ_3^{(3)} + 2JQ_2^{(3)} & 2JQ_3^{(3)} \\ JQ_3^{(3)} & JQ_2^{(3)} + 2JQ_1^{(3)} & 2JQ_2^{(3)} \\ JQ_2^{(3)} & JQ_1^{(3)} + 2JQ_0^{(3)} & 2JQ_1^{(3)} \end{array} \right]. \tag{45}$$

This matrix can be called the *third-order Jacobsthal quaternion matrix*. Then, we can give the next theorem by the dual third-order Jacobsthal quaternions.

Theorem 4.1. *If $JD_n^{(3)}$ be the n-th dual third-order Jacobsthal quaternion. Then, for $n \geq 0$:*

$$R \cdot M^n \cdot S_\varepsilon = \left[\begin{array}{ccc} JD_{n+4}^{(3)} & JD_{n+3}^{(3)} + 2JD_{n+2}^{(3)} & 2JD_{n+3}^{(3)} \\ JD_{n+3}^{(3)} & JD_{n+2}^{(3)} + 2JD_{n+1}^{(3)} & 2JD_{n+2}^{(3)} \\ JD_{n+2}^{(3)} & JD_{n+1}^{(3)} + 2JD_n^{(3)} & 2JD_{n+1}^{(3)} \end{array} \right], \tag{46}$$

where $S_\varepsilon = \begin{bmatrix} 1+\varepsilon & \varepsilon & 2\varepsilon \\ \varepsilon & 1 & 0 \\ 0 & \varepsilon & 1 \end{bmatrix}$.

Proof. (By induction on n) If $n = 0$, then the result is obvious. Now, we suppose it is true for $n = t$, that is

$$R \cdot M^t \cdot S_\varepsilon = \begin{bmatrix} JD^{(3)}_{t+4} & JD^{(3)}_{t+3} + 2JD^{(3)}_{t+2} & 2JD^{(3)}_{t+3} \\ JD^{(3)}_{t+3} & JD^{(3)}_{t+2} + 2JD^{(3)}_{t+1} & 2JD^{(3)}_{t+2} \\ JD^{(3)}_{t+2} & JD^{(3)}_{t+1} + 2JD^{(3)}_{t} & 2JD^{(3)}_{t+1} \end{bmatrix}.$$

Using the definition (3), for $t \geq 0$, we have $JD^{(3)}_{t+3} = JD^{(3)}_{t+2} + JD^{(3)}_{t+1} + 2JD^{(3)}_{t}$. Then, by induction hypothesis and $MS_\varepsilon = S_\varepsilon M$ we get

$$\begin{aligned} R \cdot M^{t+1} \cdot S_\varepsilon &= \left(\left(R \cdot M^t \right) \cdot S_\varepsilon \right) \cdot M \\ &= \begin{bmatrix} JD^{(3)}_{t+4} & JD^{(3)}_{t+3} + 2JD^{(3)}_{t+2} & 2JD^{(3)}_{t+3} \\ JD^{(3)}_{t+3} & JD^{(3)}_{t+2} + 2JD^{(3)}_{t+1} & 2JD^{(3)}_{t+2} \\ JD^{(3)}_{t+2} & JD^{(3)}_{t+1} + 2JD^{(3)}_{t} & 2JD^{(3)}_{t+1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} JD^{(3)}_{t+4} + JD^{(3)}_{t+3} + 2JD^{(3)}_{t+2} & JD^{(3)}_{t+4} + 2JD^{(3)}_{t+3} & 2JD^{(3)}_{t+4} \\ JD^{(3)}_{t+3} + JD^{(3)}_{t+2} + 2JD^{(3)}_{t+1} & JD^{(3)}_{t+3} + 2JD^{(3)}_{t+2} & 2JD^{(3)}_{t+3} \\ JD^{(3)}_{t+2} + JD^{(3)}_{t+1} + 2JD^{(3)}_{t} & JD^{(3)}_{t+2} + 2JD^{(3)}_{t+1} & 2JD^{(3)}_{t+2} \end{bmatrix} \\ &= \begin{bmatrix} JD^{(3)}_{t+5} & JD^{(3)}_{t+4} + 2JD^{(3)}_{t+3} & 2JD^{(3)}_{t+4} \\ JD^{(3)}_{t+4} & JD^{(3)}_{t+3} + 2JD^{(3)}_{t+2} & 2JD^{(3)}_{t+3} \\ JD^{(3)}_{t+3} & JD^{(3)}_{t+2} + 2JD^{(3)}_{t+1} & 2JD^{(3)}_{t+2} \end{bmatrix}. \end{aligned}$$

Hence, the equation (46) holds for all $n \geq 0$. □

Corollary 4.2. *For* $n \geq 0$,

$$JD^{(3)}_{n+2} = (1+\varepsilon)JQ^{(3)}_{n+2} + \varepsilon JQ^{(3)}_{n+1} + 2\varepsilon JQ^{(3)}_{n}. \tag{47}$$

Proof. The proof can be easily seen by the coefficient in the third row and first column of the matrix $(R \cdot M^n) \cdot S_\varepsilon$ and the equation (44). □

Corollary 4.3. *For* $n \geq 0$,

$$JD^{(3)}_{n+2} = JQ^{(3)}_2 \widehat{J}^{(3)}_{n+1} + (JQ^{(3)}_1 + JQ^{(3)}_0)\widehat{J}^{(3)}_n + 2JQ^{(3)}_1 \widehat{J}^{(3)}_{n-1}. \tag{48}$$

Proof. The proof can be easily seen by the coefficient in the third row and first column of the matrix $R \cdot (M^n \cdot S_\epsilon)$ and the equations (44) and (45). □

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