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A New Fuzzy-Valued Integral and Its Convergence Theorems

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Abstract. In this paper, we generalize Kluvánek-Lewis type set-valued integral to fuzzy number measures in Banach spaces. Firstly, we introduce the new fuzzy-valued integral with some natural properties. And then, We establish Vitali type convergence theorem and dominated convergence theorem for this kind of integral.

1. Introduction

Motivated by the applications in several areas of applied science, such as mathematical economics, fuzzy optimal, process control and decision theory, much effort has been devoted to the generalization of classical measure and integral results to the case when outcomes of a random experiment are represented by sets or fuzzy sets, such as the concepts of set-valued measures and integrals (see, e.g., [3, 5, 9, 14, 16, 19]) and fuzzy-valued measures and integrals (see, e.g., [4, 7, 8, 10–13, 17, 20, 21]). Among them, as an extension of the integrals of real-valued functions with respect to vector measures, the integral of real-valued functions with respect to a set-valued measure was first launched by Papageorgiou [9] who considered the bilinear integral of Dinculeanu [2]. After that, Kandilakis [5] introduced an integral of a bounded real-valued measurable function with respect to a set-valued measure using the set of Kluvánek-Knowles type integrals [6]. Wu, Zhang and Wang [16] introduced the set-valued Bartle integral which is a set of Bartle-Dunford-Schwartz type integrals [1] of a real-valued function with respect to measure selections of the given set-valued measure. This type of definition has also been considered by Zhang, Li, Wang and Gao [18]. Recently, Zhou and Shi [19] introduced set-valued Kluvánek-Lewis type integral which is a Pettis type weak integral of real-valued functions with respect to a set-valued measure in Banach spaces. When it comes to integrals of real-valued functions with respect to fuzzy-valued measures, only some approaches can be distinguished. The integral of real-valued functions with respect to a fuzzy number measure in $\mathbb{R}$ was first introduced by Stojaković [11]. Since then, based on Papageorgiou's set-valued integral [9], the integral of real-valued functions with respect to a fuzzy number measure was introduced by M. Stojaković and Z. Stojaković [12], where the fuzzy number measure takes on values in the family of fuzzy sets with compact convex α-levels

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in finite dimensional Banach spaces. Based on the set-valued Bartle integral [16], Park [10] introduced generalized fuzzy number-valued Bartle integral which is the integration of real-valued functions with respect to a generalized fuzzy number measure, where the generalized fuzzy number measure takes on values in the family of fuzzy sets with weakly compact and convex α-levels in an infinite dimensional Banach space. Zhou and Shi [21] introduced an integral of real-valued functions with respect to a generalized fuzzy number measure which is different from that of Park and main results are extensions of Zhang, Li, Wang and Gao's results [18].

In the present paper, we introduce and study a new fuzzy-valued integral of real-valued functions with respect to a generalized fuzzy number measure. The result is an extension of set-valued Kluvánek-Lewis type integral. We investigate some properties and establish convergence theorems for this kind of integral.

The paper is structured as follows. In Section 2, we state some basic concepts and preliminary results which will be used in the sequel. In Section 3, we first introduce the new integral with some natural properties. And then, we give Vitali type convergence theorem and dominated convergence theorem for this kind of integral. Finally, we give an example to illustrate the feasibility of our results.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathscr{A}, \mu)$ be a complete finite measure space where Ω is a nonempty set, $\mathscr{A}$ is a σ-algebra of subsets of Ω and μ is a measure. Let $(\mathcal{X}, \|\cdot\|)$ be a real separable Banach space with its dual space $\mathcal{X}^*$ and $\mathbb{R}$ the set of reals. Let

$\mathscr{P}_0(\mathcal{X}) = \{A \subset \mathcal{X} : A$ is a nonempty subset of $\mathcal{X}\}$,

$\mathscr{P}_{b(f)(c)}(\mathcal{X}) = \{A \in \mathscr{P}_0(\mathcal{X}) : A$ is bounded (closed) (convex)$\}$,

$\mathscr{P}_{wkc}(\mathcal{X}) = \{A \in \mathscr{P}_0(\mathcal{X}) : A$ is weakly compact and convex$\}$.

For $A, B \in \mathscr{P}_f(\mathcal{X})$, the Hausdorff metric d_H of A and B is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}.$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$. For $A \subset \mathcal{X}$, the number $|A|$ is defined by $|A| = d_H(A,\{0\}) = \sup_{x\in A}\|x\|$. Note that $(\mathscr{P}_{wkc}(\mathcal{X}), d_H)$ is a complete metric space. We shall denote by $\sigma(\cdot, A)$ the support function of the set $A \subset \mathcal{X}$ defined by

$$\sigma(x^*, A) = \sup_{x\in A} x^*(x), \;\; x^* \in \mathcal{X}^*.$$

The support function satisfies the properties: $\sigma(x^*, A+B) = \sigma(x^*, A) + \sigma(x^*, B)$ and $\sigma(x^*, \lambda A) = \lambda\sigma(x^*, A)$ for all $A, B \in \mathscr{P}_0(\mathcal{X})$, $x^* \in \mathcal{X}^*$ and $\lambda \geq 0$. In particular,

$$H(A,B) = \sup_{\|x^*\|\leq 1} |\sigma(x^*, A) - \sigma(x^*, B)|$$

whenever A, B are two convex sets.

Let $\tilde{u} : \mathcal{X} \to [0,1]$. We denote the α-level of $\tilde{u}$ by $\tilde{u}_\alpha = \{x \in \mathcal{X} : \tilde{u}(x) \geq \alpha\}$ for any $\alpha \in (0,1]$. $\tilde{u}$ is called a generalized fuzzy number if for each $\alpha \in (0,1]$, $\tilde{u}_\alpha \in \mathscr{P}_{wkc}(\mathcal{X})$. Let $\mathscr{F}_{wkc}(\mathcal{X})$ denote the set of all generalized fuzzy numbers on $\mathcal{X}$ (cf. [17]).

For $\tilde{u}, \tilde{v} \in \mathscr{F}_{wkc}(\mathcal{X})$ and $\lambda \in \mathbb{R}$, we define $\tilde{u} + \tilde{v}$ and $\lambda\tilde{u}$ as follows:

$$(\tilde{u} + \tilde{v})(x) = \sup_{x=y+z} \min\{\tilde{u}(y), \tilde{v}(z)\},$$

$$(\lambda\tilde{u})(x) = \begin{cases} \tilde{u}\left(\frac{1}{\lambda}x\right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

Obviously, we have $(\tilde{u}+\tilde{v})_\alpha = \tilde{u}_\alpha + \tilde{v}_\alpha$ and $(\lambda\tilde{u})_\alpha = \lambda\tilde{u}_\alpha$ for each $\alpha \in (0,1]$. Hence $\tilde{u}+\tilde{v}, \lambda\tilde{u} \in \mathscr{F}_{wkc}(\mathcal{X})$ (cf. [10, 17]). In the set $\mathscr{F}_{wkc}(\mathcal{X})$ we can define the metric d_H^∞ by

$$d_H^\infty(\tilde{u},\tilde{v}) = \sup_{\alpha\in(0,1]} d_H(\tilde{u}_\alpha, \tilde{v}_\alpha).$$

$(\mathscr{F}_{wkc}(\mathcal{X}), d_H^\infty)$ is a metric space (cf. [10]). The norm $\|\tilde{u}\|$ of $\tilde{u} \in \mathscr{F}_{wkc}(\mathcal{X})$ is defined by

$$\|\tilde{u}\| = d_H^\infty(\tilde{u},\tilde{0}) = \sup_{\alpha\in(0,1]} |\tilde{u}_\alpha|,$$

where $\tilde{0}$ is indicator function of $\{0\}$.

Theorem 2.1. [15] *If $\tilde{u} \in \mathscr{F}_{wkc}(\mathcal{X})$, then*

(1) $\tilde{u}_\alpha \in \mathscr{P}_{wkc}(\mathcal{X})$ *for all* $\alpha \in (0,1]$;

(2) $\tilde{u}_\alpha \supseteq \tilde{u}_\beta$ *for* $0 < \alpha \le \beta \le 1$;

(3) *if* $\{\alpha_n\}_{n\in\mathbb{N}}$ *is a nondecreasing sequence in* $[0,1]$ *converging to* $\alpha \in (0,1]$, *then* $\tilde{u}_\alpha = \bigcap_{n=1}^\infty \tilde{u}_{\alpha_n}$.

Conversely, if $\{A_\alpha : \alpha \in (0,1]\} \subseteq \mathscr{P}_0(\mathcal{X})$ *satisfies* (1)-(3) *above, then there exists a* $\tilde{u} \in \mathscr{F}_{wkc}(\mathcal{X})$ *such that* $\tilde{u}_\alpha = A_\alpha$ *for each* $\alpha \in (0,1]$.

Definition 2.2. [3] Let $(\Omega, \mathscr{A})$ be a measurable space. The mapping $\mathcal{M} : \mathscr{A} \to \mathscr{P}_0(\mathcal{X})$ is called a set-valued measure if it satisfies the following two conditions:

(1) $\mathcal{M}(\emptyset) = \{0\}$;

(2) if $A_1, A_2, \ldots$ are in $\mathscr{A}$, with $A_i \cap A_j = \emptyset$ for $i \ne j$, then

$$\mathcal{M}\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \mathcal{M}(A_i),$$

where $\sum_{i=1}^\infty \mathcal{M}(A_i) = \{x \in \mathcal{X} : x = \sum_{i=1}^\infty x_i \text{ unc. conv.}),\ x_i \in \mathcal{M}(A_i), i \ge 1\}$.

As for single-valued measures, we have the notion of total variation $|\mathcal{M}|$ of $\mathcal{M}$. For $A \in \mathscr{A}$ we define $|\mathcal{M}|(A) = \sup \sum_{i=1}^n |\mathcal{M}(A_i)|$, where the supremum is taken over all finite measurable partitions $\{A_1, \ldots, A_n\}$ of A. If $|\mathcal{M}|(\Omega) < \infty$, then we say that $\mathcal{M}$ is of bounded variation. We say that $\mathcal{M}$ is μ-continuous if for arbitrary $A \in \mathscr{A}$, $\mu(A) = 0$, then $\mathcal{M}(A) = \{0\}$.

Definition 2.3. [17] Let $(\Omega, \mathscr{A})$ be a measurable space. The mapping $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ is called a generalized fuzzy number measure if it satisfies the following two conditions:

(1) $\tilde{\mathcal{M}}(\emptyset) = \tilde{0}$;

(2) if $A_1, A_2, \ldots$ are in $\mathscr{A}$, with $A_i \cap A_j = \emptyset$ for $i \ne j$, then

$$\tilde{\mathcal{M}}\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty \tilde{\mathcal{M}}(A_i)$$

where $(\sum_{i=1}^\infty \tilde{\mathcal{M}}(A_i))(x) = \sup\{\bigwedge_{i=1}^\infty \tilde{\mathcal{M}}(A_i)(x_i) : x = \sum_{i=1}^\infty x_i \text{ (unc. conv.)}\}$.

We say that $\tilde{\mathcal{M}}$ is μ-continuous if for arbitrary $A \in \mathscr{A}$, $\mu(A) = 0$, then $\tilde{\mathcal{M}}(A) = \tilde{0}$. For $A \in \mathscr{A}$ we define $|\tilde{\mathcal{M}}|(A) = \sup \sum_{i=1}^n \|\tilde{\mathcal{M}}(A_i)\|$, where the supremum is taken over all finite measurable partitions $\{A_1, \ldots, A_n\}$ of A. $|\tilde{\mathcal{M}}|$ is called the total variation of $\tilde{\mathcal{M}}$. Note that for each generalized fuzzy number measure $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ and $A \in \mathscr{A}$, we have $|\tilde{\mathcal{M}}|(A) = \sup_{\alpha\in(0,1]} |\tilde{\mathcal{M}}_\alpha(A)|$. Therefore, $|\tilde{\mathcal{M}}|$ is a real valued measure. If $|\tilde{\mathcal{M}}|(\Omega) < \infty$, then we say that $\tilde{\mathcal{M}}$ is of bounded variation. If $\tilde{\mathcal{M}}$ is of bounded variation, then for each $\alpha \in (0,1]$, $\tilde{\mathcal{M}}_\alpha$ is of bounded variation. For a real valued function $f : \Omega \to \mathbb{R}$, if f is $|\tilde{\mathcal{M}}|$-integrable, then f is $|\tilde{\mathcal{M}}_\alpha|$-integrable for each $\alpha \in (0,1]$ (cf. [10]).

Theorem 2.4. [17] *The mapping* $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ *is a generalized fuzzy number measure if and only if there exists a family of set-valued measures* $\tilde{\mathcal{M}}_\alpha : \mathscr{A} \to \mathscr{P}_{wkc}(\mathcal{X}), \alpha \in (0,1]$ *satisfying the following three conditions:*

(1) *for arbitrary* $\alpha, \beta \in (0,1]$ *and* $A \in \mathscr{A}$, *if* $\alpha \leq \beta$, *then* $\tilde{\mathcal{M}}_\alpha(A) \supseteq \tilde{\mathcal{M}}_\beta(A)$;
(2) *for arbitrary* $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq (0,1]$ *and* $A \in \mathscr{A}$, *if* $\alpha_n \nearrow \alpha$, *then* $\tilde{\mathcal{M}}_\alpha(A) = \bigcap_{n=1}^{\infty} \tilde{\mathcal{M}}_{\alpha_n}(A)$;
(3) *for arbitrary* $A \in \mathscr{A}$, *we have*

$$\tilde{\mathcal{M}}(A)(x) = \begin{cases} \sup\{\alpha \in (0,1] : x \in \tilde{\mathcal{M}}_\alpha(A)\}, & \textit{if } \{\alpha \in (0,1] : x \in \tilde{\mathcal{M}}_\alpha(A)\} \neq \emptyset; \\ 0, & \textit{if } \{\alpha \in (0,1] : x \in \tilde{\mathcal{M}}_\alpha(A)\} = \emptyset. \end{cases}$$

Note that for a generalized fuzzy number measure $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ the set-valued measure $\tilde{\mathcal{M}}_\alpha : \mathscr{A} \to \mathscr{P}_{wkc}(\mathcal{X})$ is determined by

$$\tilde{\mathcal{M}}_\alpha(A) = \{x \in \mathcal{X} : \tilde{\mathcal{M}}(A)(x) \geq \alpha\},$$

i.e., $\tilde{\mathcal{M}}_\alpha(A) = [\tilde{\mathcal{M}}(A)]_\alpha$ for each $A \in \mathscr{A}$ and $\alpha \in (0,1]$.

Let $L^1(\Omega, \mathbb{R}, |\mathcal{M}|)$ be the space of all functions $f : \Omega \to \mathbb{R}$ which are $\mathscr{A}$-measurable and $|\mathcal{M}|$-integrable.

Definition 2.5. [19] Let $\mathcal{M} : \mathscr{A} \to \mathscr{P}_{wkc}(\mathcal{X})$ be a set-valued measure and $f : \Omega \to \mathbb{R}$ be an element of $L^1(\Omega, \mathbb{R}, |\mathcal{M}|)$. f is said to be Kluvánek-Lewis integrable with respect to $\mathcal{M}$ (for shortly, (KL) $\mathcal{M}$-integrable) if

(1) f is $\sigma(x^*, \mathcal{M}(\cdot))$-integrable for each $x^* \in \mathcal{X}^*$;
(2) for each $A \in \mathscr{A}$, there exists a $W_A \in \mathscr{P}_{wkc}(\mathcal{X})$ such that

$$\sigma(x^*, W_A) = \int_A f(\omega)\, \mathrm{d}\sigma(x^*, \mathcal{M}(\omega))$$

for each $x^* \in \mathcal{X}^*$. In the case, we write

$$W_A = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\mathcal{M}(\omega)$$

for each $A \in \mathscr{A}$ and call it set-valued Kluvánek-Lewis integral of f with respect to $\mathcal{M}$ on A.

3. Main results

In the sequel, let $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$ be the space of all functions $f : \Omega \to \mathbb{R}$ which are $\mathscr{A}$-measurable and $|\tilde{\mathcal{M}}|$-integrable. Note that if f is $|\tilde{\mathcal{M}}|$-integrable, then f is $|\tilde{\mathcal{M}}_\alpha|$-integrable for each $\alpha \in (0,1]$. Hence, we naturally generalize the set-valued Kluvánek-Lewis integral to generalized fuzzy number measures as follows:

Definition 3.1. Let $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure and $f : \Omega \to \mathbb{R}$ an element of $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$. f is said to be Kluvánek-Lewis integrable with respect to $\tilde{\mathcal{M}}$ (for shortly, (KL) $\tilde{\mathcal{M}}$-integrable) if f is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and there exists a fuzzy number $\tilde{W}_A \in \mathscr{F}_{wkc}(\mathcal{X})$ such that

$$[\tilde{W}_A]_\alpha = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $A \in \mathscr{A}$ and $\alpha \in (0,1]$. In the case, we write $\tilde{W}_A = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)$ and call it fuzzy-valued Kluvánek-Lewis integral of f with respect to $\tilde{\mathcal{M}}$ on A.

We obtain the following remark from Theorem 2.1 and Definition 3.1.

Remark 3.2. Let $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure and $f : \Omega \to \mathbb{R}$ an element of $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$. Then f is (KL) $\tilde{\mathcal{M}}$-integrable if and only if for each $A \in \mathscr{A}$, f satisfies the following three conditions:

(1) $(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) \in \mathscr{P}_{wkc}(\mathcal{X})$ for all $\alpha \in (0,1]$;
(2) $(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) \supseteq (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\beta(\omega)$ for $0 < \alpha \leq \beta \leq 1$;
(3) if $\{\alpha_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence in $[0,1]$ converging to $\alpha \in (0,1]$, then $(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = \bigcap_{n=1}^{\infty} (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_{\alpha_n}(\omega)$.

Theorem 3.3. *Let $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure. If $f = \sum_{i=1}^{n} \lambda_i \chi_{A_i}$ is a nonnegative simple function, then f is (KL) $\tilde{\mathcal{M}}$-integrable.*

Proof. By Example 5 [19], we know that f is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable for each $\alpha \in (0,1]$. This follows that (KL) $\int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) \in \mathscr{P}_{wkc}(\mathcal{X})$ and

$$\text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = \sum_{i=1}^{n} \lambda_i \tilde{\mathcal{M}}_\alpha(A_i \cap A).$$

By Theorem 2.4, for arbitrary $\alpha, \beta \in (0,1]$ with $\alpha \le \beta$ and $A \in \mathscr{A}$, we have $\tilde{\mathcal{M}}_\alpha(A) \supseteq \tilde{\mathcal{M}}_\beta(A)$, which implies that $\sigma(x^*, \tilde{\mathcal{M}}_\alpha(A)) \ge \sigma(x^*, \tilde{\mathcal{M}}_\beta(A))$ for each $x^* \in \mathcal{X}^*$. It follows that

$$\text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) \supseteq \text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\beta(\omega).$$

Now let $\{a_n\}_{n\in\mathbb{N}}$ be a nondecreasing sequence in $[0,1]$ converging to $\alpha \in (0,1]$. Then we obtain that

$$\begin{aligned}
\bigcap_{n=1}^{\infty} \text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_{\alpha_n}(\omega) &= \bigcap_{n=1}^{\infty} \sum_{i=1}^{n} \lambda_i \tilde{\mathcal{M}}_{\alpha_n}(A_i \cap A) \\
&= \bigcap_{n=1}^{\infty} \left[\sum_{i=1}^{n} \lambda_i \tilde{\mathcal{M}}(A_i \cap A) \right]_{\alpha_n} \\
&= \left[\sum_{i=1}^{n} \lambda_i \tilde{\mathcal{M}}(A_i \cap A) \right]_{\alpha} \\
&= \sum_{i=1}^{n} \lambda_i \tilde{\mathcal{M}}_\alpha(A_i \cap A) \\
&= \text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega).
\end{aligned}$$

Up to now, the family $\{\text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\}_{\alpha\in(0,1]}$ satisfies all the conditions of Remark 3.2. Hence $f = \sum_{i=1}^{n} \lambda_i \chi_{A_i}$ is (KL) $\tilde{\mathcal{M}}$-integrable. This completes the proof. □

In what follows, some properties of the fuzzy-valued Kluvánek-Lewis integral will be given.

Theorem 3.4. *Let $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure and $f, g : \Omega \to \mathbb{R}$ elements of $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$ and $\lambda \ge 0$.*

(1) If f and g are (KL) $\tilde{\mathcal{M}}$-integrable, then $f + g$ is (KL) $\tilde{\mathcal{M}}$-integrable and

$$\text{(KL)} \int_A \{f(\omega) + g(\omega)\}\, \mathrm{d}\tilde{\mathcal{M}}(\omega) = \text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega) + \text{(KL)} \int_A g(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)$$

for each $A \in \mathscr{A}$.

(2) If f is (KL) $\tilde{\mathcal{M}}$-integrable, then λf is (KL) $\tilde{\mathcal{M}}$-integrable and

$$\text{(KL)} \int_A \lambda f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega) = \lambda \text{(KL)} \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)$$

for each $A \in \mathscr{A}$.

Proof. (1) Since f and g are (KL) $\tilde{\mathcal{M}}$-integrable, f and g are (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and there exists fuzzy number (KL) $\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)$ and (KL) $\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$ such that

$$\left[(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega),$$

$$\left[(\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0,1]$ and $A \in \mathscr{A}$. According to Theorem 6 [19], $f+g$ is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and

$$(\mathrm{KL})\int_A \{f(\omega)+g(\omega)\}\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = (\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) + (\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0,1]$ and $A \in \mathscr{A}$. It follows that

$$\begin{aligned}(\mathrm{KL})\int_A \{f(\omega)+g(\omega)\}\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) &= (\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) + (\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\\ &= \left[(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha + \left[(\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha\\ &= \left[(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) + (\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha\end{aligned}$$

for each $\alpha \in (0,1]$ and $A \in \mathscr{A}$. Also, since (KL) $\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)$ and (KL) $\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$, (KL) $\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)$ + (KL) $\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$. Hence $f+g$ is (KL) $\tilde{\mathcal{M}}$-integrable and

$$(\mathrm{KL})\int_A \{f(\omega)+g(\omega)\}\,\mathrm{d}\tilde{\mathcal{M}}(\omega) = (\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) + (\mathrm{KL})\int_A g(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)$$

for each $A \in \mathscr{A}$.

(2) Since f is (KL) $\tilde{\mathcal{M}}$-integrable, f is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and there exists fuzzy number (KL) $\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$ such that

$$\left[(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0,1]$ and $A \in \mathscr{A}$. By Theorem 6 [19], λf is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and

$$(\mathrm{KL})\int_A \lambda f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = \lambda(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0,1]$, $A \in \mathscr{A}$ and $\lambda \geq 0$. This follows that

$$\begin{aligned}(\mathrm{KL})\int_A \lambda f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) &= \lambda(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\\ &= \lambda\left[(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha\\ &= \left[\lambda(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha\end{aligned}$$

for each $\alpha \in (0,1]$, $A \in \mathscr{A}$ and $\lambda \geq 0$. Since (KL) $\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$, λ(KL) $\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$. Therefore, λf is (KL) $\tilde{\mathcal{M}}$-integrable and

$$(\mathrm{KL})\int_A \lambda f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) = \lambda(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)$$

for each $A \in \mathscr{A}$. This completes the proof. □

One of the important properties of set-valued Kluvánek-Lewis integrals is that the indefinite integral of an integrable function is a set-valued measure. We would like to extend the result to fuzzy-valued Kluvánek-Lewis integrals as follows.

Theorem 3.5. *Let $(\Omega, \mathscr{A}, \mu)$ be a nonnegative finite measure space, $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a μ-continuous generalized fuzzy number measure and $f : \Omega \to \mathbb{R}$ be (KL) $\tilde{\mathcal{M}}$-integrable. Then $\tilde{\mathcal{M}}' : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ defined by*

$$\tilde{\mathcal{M}}'(A) = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega), \forall A \in \mathscr{A}$$

is a μ-continuous generalized fuzzy number measure.

Proof. Since f is (KL) $\tilde{\mathcal{M}}$-integrable, f is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and there exists fuzzy number (KL) $\int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$ such that

$$\left[(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0, 1]$ and $A \in \mathscr{A}$. Since $\tilde{\mathcal{M}}$ is μ-continuous, $\tilde{\mathcal{M}}_\alpha$ is μ-continuous for each $\alpha \in (0, 1]$. According to Theorem 7 [19], $\tilde{\mathcal{M}}'_\alpha : \mathscr{A} \to \mathscr{P}_{wkc}(\mathcal{X})$ defined by

$$\tilde{\mathcal{M}}'_\alpha(A) = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = \left[(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = \left[\tilde{\mathcal{M}}'(A)\right]_\alpha, \forall A \in \mathscr{A}$$

is a μ–continuous set-valued measure for each $\alpha \in (0, 1]$.

In order to show that $\tilde{\mathcal{M}}' : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ is a generalized fuzzy number measure, we prove that $\{\tilde{\mathcal{M}}'_\alpha(\cdot)\}_{\alpha\in(0,1]}$ satisfies all conditions of Theorem 2.4. Since (KL) $\int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$ and $\tilde{\mathcal{M}}'_\alpha(A) = [(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)]_\alpha$, $\tilde{\mathcal{M}}'_\alpha$ satisfies conditions (1) and (2) of Theorem 2.4. For arbitrary $A \in \mathscr{A}$, if $\tilde{\mathcal{M}}'(A)(x) > 0$, i.e., $\{\alpha \in (0, 1] : x \in [\tilde{\mathcal{M}}'(A)]_\alpha = \tilde{\mathcal{M}}'_\alpha(A)\} \neq \emptyset$, then

$$\begin{aligned} \tilde{\mathcal{M}}'(A)(x) &= \sup\{\alpha \in (0, 1] : \tilde{\mathcal{M}}'(A)(x) \geq \alpha\} \\ &= \sup\{\alpha \in (0, 1] : x \in [\tilde{\mathcal{M}}'(A)]_\alpha\} \\ &= \sup\{\alpha \in (0, 1] : x \in \mathcal{M}'_\alpha(A)\}. \end{aligned}$$

If $\tilde{\mathcal{M}}'(A)(x) = 0$, then for all $\alpha \in (0, 1]$, $x \notin [\tilde{\mathcal{M}}'(A)]_\alpha = \tilde{\mathcal{M}}'_\alpha(A)$, i.e., $\{\alpha \in (0, 1] : x \in \mathcal{M}'_\alpha(A)\} = \emptyset$. Hence, for any $A \in \mathscr{A}$, $\tilde{\mathcal{M}}'(A)$ satisfies

$$\tilde{\mathcal{M}}'(A)(x) = \begin{cases} \sup\{\alpha \in (0, 1] : x \in \tilde{\mathcal{M}}'_\alpha(A)\}, & \text{if } \{\alpha \in (0, 1] : x \in \tilde{\mathcal{M}}'_\alpha(A)\} \neq \emptyset; \\ 0, & \text{if } \{\alpha \in (0, 1] : x \in \tilde{\mathcal{M}}'_\alpha(A)\} = \emptyset. \end{cases}$$

Up to now, $\{\tilde{\mathcal{M}}'_\alpha(\cdot)\}_{\alpha\in(0,1]}$ satisfies all the conditions of Theorem 2.4. Hence $\tilde{\mathcal{M}}' : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ is a generalized fuzzy number measure. It is easy to see that $\tilde{\mathcal{M}}'$ is absolutely continuous with respect to $\tilde{\mathcal{M}}$, which implies that $\tilde{\mathcal{M}}'$ is μ-continuous. This completes the proof. □

Theorem 3.6. *Let $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure and $f : \Omega \to \mathbb{R}$ an element of $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$. If $f : \Omega \to \mathbb{R}$ is (KL) $\tilde{\mathcal{M}}$-integrable, then for each $A \in \mathscr{A}$, we have*

$$\left\|(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)\right\| \leq \int_A |f(\omega)|\, \mathrm{d}|\tilde{\mathcal{M}}|(\omega).$$

Proof. Since $f : \Omega \to \mathbb{R}$ is (KL) $\tilde{\mathcal{M}}$-integrable, $f : \Omega \to \mathbb{R}$ is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and there exists (KL) $\int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$ such that

$$\left[(\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL}) \int_A f(\omega)\, \mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0,1]$ and $A \in \mathscr{A}$. Then, by Theorem 9 [19], we have

$$\left|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\right| \le \int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}_\alpha|(\omega)$$

for each $A \in \mathscr{A}$ and $\alpha \in (0,1]$. It follows that

$$\begin{aligned}\left\|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right\| &= d_H^\infty\left((\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega),\tilde{0}\right)\\ &= \sup_{\alpha\in(0,1]}\left\|\left[(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha\right\|\\ &= \sup_{\alpha\in(0,1]}\left|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\right|\\ &\le \sup_{\alpha\in(0,1]}\int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}_\alpha|(\omega).\end{aligned}$$

Since for each $A \in \mathscr{A}$, we have $|\tilde{\mathcal{M}}|(A) = \sup_{\alpha\in(0,1]}|\tilde{\mathcal{M}}_\alpha(A)|$, which implies that

$$\sup_{\alpha\in(0,1]}\int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}_\alpha|(\omega) \le \int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega).$$

Up to now, $\|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\| \le \int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega)$. This completes the proof. □

Corollary 3.7. *Let $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure and $f : \Omega \to \mathbb{R}$ an element of $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$. If $f : \Omega \to \mathbb{R}$ is (KL) $\tilde{\mathcal{M}}$-integrable, then*

(1) $\lim_{A\to\emptyset}\|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\| = 0$;

(2) *if $A = \emptyset$, then* $(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) = \tilde{0}$;

(3) *if $f = 0$, then* $(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) = \tilde{0}$ *for each $A \in \mathscr{A}$.*

Proof. (1) By Theorem 3.6, we have

$$\left\|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right\| \le \int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega).$$

It follows that

$$\lim_{A\to\emptyset}\left\|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right\| \le \lim_{A\to\emptyset}\int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega) = 0.$$

This implies that $\lim_{A\to\emptyset} d_H^\infty\left((\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega),\tilde{0}\right) = 0$.

(2) If $A = \emptyset$, then $\int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega) = 0$. Therefore, by Theorem 3.6, we have

$$d_H^\infty\left((\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega),\tilde{0}\right) = \left\|(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right\| \le \int_A |f(\omega)|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega) = 0,$$

which implies that $d_H^\infty\left((\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega),\tilde{0}\right) = 0$. Since $(\mathscr{F}_{wkc}(\mathcal{X}), d_H^\infty)$ is a metric space, $(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) = \tilde{0}$.

(3) If $f = 0$, then

$$(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = \{0\}$$

for each $A \in \mathscr{A}$ and $\alpha \in (0,1]$. It follows that $(\mathrm{KL})\int_A f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) = \tilde{0}$ for each $A \in \mathscr{A}$. This completes the proof. □

By virtue of classical Vitali type convergence theorem, we obtain Vitali type convergence theorem for fuzzy-valued Kluvánek-Lewis integral as follows:

Theorem 3.8. *Let* $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ *be a generalized fuzzy number measure and* $f, f_n : \Omega \to \mathbb{R}, n \in \mathbb{N}$, *elements of* $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$. *If* f *and* f_n, $n \in \mathbb{N}$, *are (KL)* $\tilde{\mathcal{M}}$*-integrable such that* $\{f_n\}_{n\in\mathbb{N}}$ *is uniformly integrable with respect to* $|\tilde{\mathcal{M}}|$ *and* $\lim_{n\to\infty} f_n = f$ $|\tilde{\mathcal{M}}|$*-a.e. Then*

$$\lim_{n\to\infty} d_H^{\infty}\left((\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega), (\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right) = 0.$$

Proof. Since f and f_n $(n \in \mathbb{N})$ are (KL) $\tilde{\mathcal{M}}$-integrable, f and f_n $(n \in \mathbb{N})$ are (KL) $\tilde{\mathcal{M}}_\alpha$-integrable and there exists (KL) $\int_\Omega f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)$ and (KL) $\int_\Omega f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega) \in \mathscr{F}_{wkc}(\mathcal{X})$ $(n \in \mathbb{N})$ such that

$$\left[(\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

and

$$\left[(\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha = (\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)$$

for each $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. By properties of support functions and definition of set-valued Kluvánek-Lewis integral, we have

$$\begin{aligned}
& d_H\left((\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega), (\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\right) \\
&= \sup_{\|x^*\|\le 1}\left|\sigma\left(x^*, (\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\right) - \sigma\left(x^*, (\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\right)\right| \\
&= \sup_{\|x^*\|\le 1}\left|\int_{\Omega} f_n(\omega)\,\mathrm{d}\sigma\left(x^*, \tilde{\mathcal{M}}_\alpha(\omega)\right) - \int_{\Omega} f(\omega)\,\mathrm{d}\sigma\left(x^*, \tilde{\mathcal{M}}_\alpha(\omega)\right)\right| \\
&\le \sup_{\|x^*\|\le 1}\int_{\Omega}\left|f_n(\omega) - f(\omega)\right|\,\mathrm{d}\left|\sigma\left(x^*, \tilde{\mathcal{M}}_\alpha(\cdot)\right)\right|(\omega) \\
&\le \int_{\Omega}\left|f_n(\omega) - f(\omega)\right|\,\mathrm{d}|\tilde{\mathcal{M}}_\alpha|(\omega) \\
&\le \int_{\Omega}\left|f_n(\omega) - f(\omega)\right|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega)
\end{aligned}$$

for each $\alpha \in (0, 1]$. Since $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable with respect to $|\tilde{\mathcal{M}}|$ and $\lim_{n\to\infty} f_n = f$ $|\tilde{\mathcal{M}}|$-a.e., according to classical Vitali's convergence theorem, we have

$$\int_{\Omega}\left|f_n(\omega) - f(\omega)\right|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega) \to 0$$

as $n \to \infty$. Then we we can conclude that

$$\begin{aligned}
& d_H^{\infty}\left((\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega), (\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right) \\
&= \sup_{\alpha\in(0,1]} d_H\left(\left[(\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha, \left[(\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}(\omega)\right]_\alpha\right) \\
&= \sup_{\alpha\in(0,1]} d_H\left((\mathrm{KL})\int_{\Omega} f_n(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega), (\mathrm{KL})\int_{\Omega} f(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega)\right) \\
&\le \int_{\Omega}\left|f_n(\omega) - f(\omega)\right|\,\mathrm{d}|\tilde{\mathcal{M}}|(\omega) \to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Similarly, we can obtain the dominated convergence theorem for fuzzy-valued Kluvánek-Lewis integral as follows:

Theorem 3.9. *Let $\mathcal{M} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathcal{X})$ be a generalized fuzzy number measure and $f, f_n : \Omega \to \mathbb{R}\,(n \in \mathbb{N})$ elements of $L^1(\Omega, \mathbb{R}, |\tilde{\mathcal{M}}|)$. If f and $f_n : \Omega \to \mathbb{R}\,(n \in \mathbb{N})$ are (KL) $\tilde{\mathcal{M}}$-integrable such that $\lim_{n\to\infty} f_n = f$ $|\tilde{\mathcal{M}}|$-a.e. If there exists a nonnegative, $|\tilde{\mathcal{M}}|$-integrable function $g : \Omega \to \mathbb{R}$ such that $|f_n(\omega)| \le g(\omega)$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, then we have*

$$\lim_{n\to\infty} d_H^{\infty}\left(\int_{\Omega} f_n(\omega)\,\mathrm{d}\mathcal{M}(\omega), \int_{\Omega} f(\omega)\,\mathrm{d}\mathcal{M}(\omega)\right) = 0.$$

Proof. Using the same manner as in the proof of Theorem 3.8 and by classical dominated convergence theorem, we can obtain the proof. □

Finally, we give an example to illustrate the feasibility of our results.

Example 3.10. Let $(\Omega, \mathscr{A}, \mu)$ be a complete finite measure space and $\tilde{\mathcal{M}} : \mathscr{A} \to \mathscr{F}_{wkc}(\mathbb{R})$ a generalized fuzzy number measure defined by

$$\tilde{\mathcal{M}}(A)(x) = \begin{cases} \dfrac{x}{\mu(A)}, & x \in [0, \mu(A)] \\ 0, & x \notin [0, \mu(A)] \end{cases}$$

if $\mu(A) \neq 0$ and $\tilde{\mathcal{M}}(A)(x) = \chi_{\{0\}}(x)$ if $\mu(A) = 0$. Let $f = \chi_B$, where χ_B is the characteristic function of measurable set B in $\mathscr{A}$. We can show that f is (KL) $\tilde{\mathcal{M}}$-integrable. Obviously, χ_B is (KL) $\tilde{\mathcal{M}}_\alpha$-integrable for each $\alpha \in (0, 1]$. This follows that (KL) $\int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) \in \mathscr{P}_{wkc}(\mathcal{X})$ and (KL) $\int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) = \tilde{\mathcal{M}}_\alpha(B \cap A)$. For arbitrary $\alpha, \beta \in (0, 1]$ with $\alpha \le \beta$ and $A \in \mathscr{A}$, we have $\tilde{\mathcal{M}}_\alpha(A) \supseteq \tilde{\mathcal{M}}_\beta(A)$. It follows that (KL) $\int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega) \supseteq$ (KL) $\int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\beta(\omega)$. If $\{a_n\}_{n\in\mathbb{N}}$ is a nondecreasing sequence in $[0, 1]$ converging to $\alpha \in (0, 1]$. Then we obtain that

$$\begin{aligned}\bigcap_{n=1}^{\infty} (\mathrm{KL}) \int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_{\alpha_n}(\omega) &= \bigcap_{n=1}^{\infty} \tilde{\mathcal{M}}_{\alpha_n}(B \cap A) = \bigcap_{n=1}^{\infty} \left[\tilde{\mathcal{M}}(B \cap A)\right]_{\alpha_n} = \left[\tilde{\mathcal{M}}(B \cap A)\right]_{\alpha} \\ &= (\mathrm{KL}) \int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}_\alpha(\omega).\end{aligned}$$

Hence $f = \chi_B$ is (KL) $\tilde{\mathcal{M}}$-integrable and

$$\left((\mathrm{KL}) \int_A \chi_B(\omega)\,\mathrm{d}\tilde{\mathcal{M}}\right)(x) = \begin{cases} \sup\{\alpha \in (0, 1] : x \in [\alpha\mu(B \cap A), \mu(B \cap A)]\}, & \text{if } \{\alpha \in (0, 1] : x \in \tilde{\mathcal{M}}_\alpha(B \cap A)\} \neq \emptyset; \\ 0, & \text{if } \{\alpha \in (0, 1] : x \in \tilde{\mathcal{M}}_\alpha(B \cap A)\} = \emptyset. \end{cases}$$

4. Conclusion

In the current paper we introduce a new integral of real-valued functions with respect to a generalized fuzzy number measure. We investigate its properties and establish convergence theorems. In all applications which involve measure and integral, when measurement or data are fuzzy-valued, the structure defined in this paper can be applied. There are several directions for further investigation connected with this topic: specific properties of the integral-Chebyshev type inequalities and Markov type inequalities, application on random case - expectation and conditional expectation, application in economy.

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