Čebyšev's Type Inequalities and Power Inequalities
for the Berezin Number of Operators

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Abstract. We give operator analogues of some classical inequalities, including Čebyšev type inequality for synchronous and convex functions of selfadjoint operators in Reproducing Kernel Hilbert Spaces (RKHSs). We obtain some Berezin number inequalities for the product of operators. Also, we prove the Berezin number inequality for the commutator of two operators.

1. Introduction

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $p = (p_1, \ldots, p_n)$ with $P_n = \sum_{i=1}^{n} p_i$ be the real sequences. Then the Čebyšev functional is defined by

$$T_n(p; a, b) := P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i.$$

In 1882-1883, Čebyšev [5, 6] proved that if a and b are monotonic in the same (opposite) sense and p is non-negative, then

$$T_n(p; a, b) \geq (\leq) 0. \tag{1}$$

Hardy et al. [19] in their book in 1934 mentioned the inequality (1) in the more general setting of synchronous sequences, i.e., if a, b are synchronous (asynchronous), this means that

$$\left(a_i - a_j\right)\left(b_i - b_j\right) \geq (\leq) 0,$$

for each $i, j \in \{1, \ldots, n\}$, then the (1) is valid.

For all of Čebyšev inequalities and their valuable applications see [10–12, 24–27] and the references therein.

The functions $f, g : [a, b] \rightarrow \mathbb{R}$ are called synchronous (asynchronous) on the interval $[a, b]$ provided that they hold the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0, \tag{2}$$

for each $t, s \in [a, b]$.

Let A be a selfadjoint linear operator on a complex Hilbert space H. The Gelfand map establishes a *-isometrically isomorphism Φ between the set $C(\mathfrak{G}_p(A))$ of all continuous functions defined on the spectrum of A, denoted by $\mathfrak{G}_p(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [14]:

For any $f, g \in C(\mathfrak{G}_p(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have
(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\overline{f}) = \Phi(f)^*$;
(iii) $\|\Phi(f)\| = \|f\| = \sup_{t \in \mathfrak{G}_p(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathfrak{G}_p(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(\mathfrak{G}_p(A))$ and we call it the continuous functional calculus for a selfadjoint operator A. If A is a selfadjoint operator and f is a real-valued continuous function on $\mathfrak{G}_p(A)$, then $f(t) \geq 0$ for any $t \in \mathfrak{G}_p(A)$ implies that $f(A) \geq 0$. Moreover, if both f and g are real-valued functions on $\mathfrak{G}_p(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \mathfrak{G}_p(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$.

Recall that a reproducing kernel Hilbert space (shorty, RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that:
(a) the evaluation functional $f \rightarrow f(\lambda)$ is continuos for each $\lambda \in \Omega$;
(b) for any $\lambda \in \Omega$ there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then by the classical Riesz representation theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel of the space $\mathcal{H}$. It is well known that (see [2, 28])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n \geq 0}$ of the space $\mathcal{H}(\Omega)$. Let $\widehat{k}_\lambda = \dfrac{k_\lambda}{\|k_\lambda\|}$ denote the normalized reproducing kernel of the space $\mathcal{H}$ (note that by (b), we surely have $k_\lambda \neq 0$). For a bounded linear operator A on the RKHS $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ is defined by the formula (see [3])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \quad (\lambda \in \Omega).$$

The Berezin symbol is a function that is bounded by the numerical radius of the operator.

Berezin set and Berezin number of operator A are defined by respectively (see Karaev [20, 21])

$$Ber(A) := Range\left(\widetilde{A}\right) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \text{ and } ber(A) := \sup \left\{ \left| \widetilde{A}(\lambda) \right| : \lambda \in \Omega \right\}.$$

Recall that $W(A) := \left\{ \langle Af, f \rangle : \|f\|_{\mathcal{H}} = 1 \right\}$ is the numerical range of the operator A and

$$w(A) := \sup \left\{ \left| \langle Af, f \rangle \right| : \|f\|_{\mathcal{H}} = 1 \right\}$$

is the numerical radius of A. It is trivial that

$$Ber(A) \subset W(A) \text{ and } ber(A) \leq w(A) \leq \|A\|$$

for any $A \in \mathcal{B}(\mathcal{H})$. For other recent results on the Berezin set, Berezin number, numerical radius and numerical range see [1, 4, 7–9, 13, 17, 18, 23, 29, 32] and the references therein.

A fundamental inequality for the numerical radius is the power inequality

$$w(A^n) \leq w(A)^n, \ n \geq 1,$$

(see [18]). So, the following questions are natural:

Is it true that the above inequality is also provided for Berezin number of operators? This question has been solved negatively by L. A. Coburu "Berezin transform and Weyl-type unitary operators on the Bergman space", Proc. Amer. Math. Soc, 140 (2012), 3445–3451.

For which operator classes, there exists a number $C > 0$ such that

$$ber(A)^n \leq C(ber(A^n)) \text{ for all } n? \tag{P}$$

But, an example with a nonzero nilpotent operator shows that there exists operators for which inequality (P) does not hold. It was given some predictions for the constant C by using Hardy-Hilbert type inequalities [15, 16, 30, 31].

In the present paper, we give some inequalities similar to (P) by using Čebyšev type inequality for synchronous and convex functions of selfadjoint operators in Reproducing Kernel Hilbert Spaces (RKHSs). We obtain some Berezin number inequalities for the product of operators. Further, we consider four operators and prove a Berezin number inequality for the sum of products. We prove an inequality for the Berezin number of the commutator $[A, B] := AB - BA$ of operators A and B.

2. Čebyšev's type inequalities

In the following result, we give an inequality of Čebyšev type for functions of selfadjoint operators acting on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$.

Theorem 2.1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS, $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator with $\mathfrak{G}_p(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \to \mathbb{R}$ are continuous and synchronous on $[m, M]$, then*
(i)

$$\widetilde{f(A)g(A)}(\lambda) + \widetilde{f(A)g(A)}(\eta) \geq \widetilde{f(A)}(\lambda)\widetilde{g(A)}(\eta) + \widetilde{f(A)}(\eta)\widetilde{g(A)}(\lambda)$$

for all $\lambda, \eta \in \Omega$;
(ii) $[ber(f(A))]^2 \leq ber(f^2(A))$; in particular, we have

$$ber(A)^2 \leq ber(A^2).$$

Proof. (*i*) Since f, g are synchronous, we get from (2)

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t) \tag{3}$$

for each $t, s \in [m, M]$.

Applying the functional calculus for the inequality (3), we obtain that

$$f(A)g(A) + f(s)g(s)1_{\mathcal{H}} \geq f(A)g(s) + f(s)g(A)$$

and hence

$$\left\langle (f(A)g(A) + f(s)g(s)1_{\mathcal{H}})\widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle \geq \left\langle (f(A)g(s) + f(s)g(A))\widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle$$

for all $\lambda \in \Omega$ and $s \in [m, M]$.

Apply again the functional calculus for the above inequality, then we get that

$$\left\langle \left(\left\langle f(A) g(A) \widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle 1_{\mathcal{H}} + f(A) g(A) \right) \widehat{k_\eta}, \widehat{k_\eta} \right\rangle$$
$$\leq \left\langle \left(\left\langle f(A) \widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle g(A) + \left\langle g(A) \widehat{k_\lambda}, \widehat{k_\lambda} \right\rangle f(A) \right) \widehat{k_\eta}, \widehat{k_\eta} \right\rangle$$

and therefore

$$\widetilde{f(A) g}(A)(\lambda) + \widetilde{f(A) g}(A)(\eta) \geq \widetilde{f(A)}(\lambda) \widetilde{g(A)}(\eta) + \widetilde{f(A)}(\eta) \widetilde{g(A)}(\lambda) \tag{4}$$

for all $\lambda, \mu \in \Omega$. So we get the (*i*).

(*ii*) In particular, for $g = f$ and $\mu = \eta$ in inequality (4), we obtain

$$\left[\widetilde{f(A)}(\lambda) \right]^2 \leq \widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$. Since $\left[\widetilde{f(A)}(\lambda) \right]^2 \geq 0$ and $\widetilde{f^2(A)}(\lambda) \geq 0$, by taking supremum on the last inequality, we obtain

$$[ber(f(A))]^2 \leq ber\left(f^2(A) \right);$$

in particular, for $f(x) = x$, we get

$$ber(A)^2 \leq ber\left(A^2 \right).$$

$\square$

Theorem 2.2. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS, $A \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator with $\mathfrak{G}_p(A) \subseteq [m, M]$ for some real numbers $m < M$ and $f, g : [m, M] \to \mathbb{R}$ be continuous, synchronous. If f, g are also convex, then*
(*i*)

$$f\left(\widetilde{B}(\eta) \right) g\left(\widetilde{A}(\lambda) \right) + f\left(\widetilde{A}(\lambda) \right) g\left(\widetilde{B}(\eta) \right) \leq \widetilde{f(B) g}(B)(\eta) + \widetilde{f(A)}(\lambda) \widetilde{g(A)}(\lambda)$$

for all $\lambda, \mu \in \Omega$;
(*ii*)

$$[f(ber(A))]^2 \leq \frac{ber\left(f^2(A) \right) + [ber(f(A))]^2}{2}.$$

Proof. (*i*) Since f, g are synchronous and $m \leq \left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \leq M$ for any $\lambda \in \Omega$, then we get

$$\left[f(t) - f\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) \right] \left[g(t) - g\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) \right] \geq 0$$

for all $\lambda \in \Omega$ and $t \in [m, M]$.

Using the functional calculus for the above inequality, we obtain that

$$\left\langle \left[f(B) - f\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) \right] \left[g(B) - g\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) \right] \widehat{k_\eta}, \widehat{k_\eta} \right\rangle \geq 0$$

for any $B \in \mathcal{B}(\mathcal{H})$ with $\mathfrak{G}_p(B) \subseteq [m, M]$ and all $\lambda, \mu \in \Omega$. So, we have from the above inequality

$$\left\langle f(B) \widehat{k_\eta}, \widehat{k_\eta} \right\rangle g\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) + f\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) \left\langle g(B) \widehat{k_\eta}, \widehat{k_\eta} \right\rangle$$
$$\leq \left\langle f(B) g(B) \widehat{k_\eta}, \widehat{k_\eta} \right\rangle + f\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right) g\left(\left\langle \widehat{Ak_\lambda}, \widehat{k_\lambda} \right\rangle \right)$$

which is equivalent to

$$\widetilde{f(B)}(\eta)\, g\left(\widetilde{A}(\lambda)\right) + f\left(\widetilde{A}(\lambda)\right) \widetilde{g(B)}(\eta) \leq \widetilde{f(B)\, g}(B)(\eta) + f\left(\widetilde{A}(\lambda)\right) g\left(\widetilde{A}(\lambda)\right)$$

for all $\lambda, \mu \in \Omega$. Since f and g are convex, we have that

$$f\left(\widetilde{B}(\eta)\right) g\left(\widetilde{A}(\lambda)\right) + f\left(\widetilde{A}(\lambda)\right) g\left(\widetilde{B}(\eta)\right) \leq \widetilde{f(B)}(\eta)\, g\left(\widetilde{A}(\lambda)\right) + f\left(\widetilde{A}(\lambda)\right) \widetilde{g(B)}(\eta) \tag{5}$$

and

$$f\left(\widetilde{B)\, g}(B)(\eta) + f\left(\widetilde{A}(\lambda)\right) g\left(\widetilde{A}(\lambda)\right) \leq \widetilde{f(B)\, g}(B)(\eta) + \widetilde{f(A)}(\lambda)\, \widetilde{g(A)}(\lambda) \tag{6}$$

for all $\lambda, \mu \in \Omega$. Then we get from the (5) and (6)

$$f\left(\widetilde{B}(\eta)\right) g\left(\widetilde{A}(\lambda)\right) + f\left(\widetilde{A}(\lambda)\right) g\left(\widetilde{B}(\eta)\right) \leq \widetilde{f(B)\, g}(B)(\eta) + \widetilde{f(A)}(\lambda)\, \widetilde{g(A)}(\lambda)$$

for all $\lambda, \mu \in \Omega$.

(ii) Now by replacing $A = B$, $\lambda = \mu$ and $f = g$ above the inequality

$$2\left[f\left(\widetilde{A}(\lambda)\right)\right]^2 \leq \widetilde{f^2(A)}(\lambda) + \left[\widetilde{f(A)}(\lambda)\right]^2$$

and hence

$$\left[f\left(\widetilde{A}(\lambda)\right)\right]^2 \leq \frac{\widetilde{f^2(A)}(\lambda) + \left[\widetilde{f(A)}(\lambda)\right]^2}{2}$$

for all selfadjoint operator A and $\lambda \in \Omega$. Since $\left[f\left(\widetilde{A}(\lambda)\right)\right]^2 \geq 0$ and $\left[\widetilde{f(A)}(\lambda)\right]^2 \geq 0$, by taking supremum on the last inequality, we obtain that

$$\left[f\left(ber(A)\right)\right]^2 \leq \frac{ber\left(f^2(A)\right) + \left[ber\left(f(A)\right)\right]^2}{2}$$

for all selfadjoint operator A. This proves the theorem. □

3. Inequalities for a product of two operators

In this section, we give some power inequalities for Berezin number of operators. Before giving results, we need the following well-known results (see [19, 22]).

Lemma 3.1. *For $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have:*

(a) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq \left[\alpha a^r + (1-\alpha) b^r\right]^{\frac{1}{r}}$ *for $r \geq 1$;*

(b) $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q} \leq \left(\dfrac{a^{pr}}{p} + \dfrac{b^{qr}}{q}\right)^{\frac{1}{r}}$ *for $r \geq 1$.*

Lemma 3.2 (McCarty inequality). *Let $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$ and let $x \in \mathcal{H}$ be any unit vector. Then*
(a) $\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$ *for $r \geq 1$;*
(b) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ *for $0 < r \leq 1$.*

Now we are ready to state our results.

Theorem 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be a positive operator and $r \geq 1$. Then we have*

$$ber^r(B^*A) \leq \frac{1}{2} ber\left(|A|^{2r} + |B|^{2r}\right).$$

Proof. Using the Schwarz inequality, we get

$$\left|\widetilde{(B^*A)}(\lambda)\right| = \left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \leq \left\|A\widehat{k}_\lambda\right\| \left\|B\widehat{k}_\lambda\right\| \tag{7}$$
$$= \left(\widetilde{(A^*A)}(\lambda)\right)^{1/2} \left(\widetilde{(B^*B)}(\lambda)\right)^{1/2}$$

for all $\lambda \in \Omega$.

Applying the arithmetic-geometric mean inequality and the convexity of the function $f(t) = t^r$, $r \geq 1$, we obtain that

$$\left(\widetilde{(A^*A)}(\lambda)\right)^{1/2} \left(\widetilde{(B^*B)}(\lambda)\right)^{1/2} \leq \frac{1}{2}\left(\widetilde{(A^*A)}(\lambda)\right)^{1/2} + \frac{1}{2}\left(\widetilde{(B^*B)}(\lambda)\right)^{1/2} \leq \left(\frac{\left(\widetilde{(A^*A)}(\lambda)\right)^r + \left(\widetilde{(B^*B)}(\lambda)\right)^r}{2}\right)^{1/r}$$

for all $\lambda \in \Omega$.

Using Lemma 3.2 for $r \geq 1$, we obtain

$$\left(\frac{\left(\widetilde{(A^*A)}(\lambda)\right)^r + \left(\widetilde{(B^*B)}(\lambda)\right)^r}{2}\right)^{1/r} \leq \left(\frac{\left(\widetilde{(A^*A)^r}(\lambda)\right) + \left(\widetilde{(B^*B)^r}(\lambda)\right)}{2}\right)^{1/r} = \left(\frac{\left(\widetilde{(A^*A)^r + (B^*B)^r}(\lambda)\right)}{2}\right)^{1/r} \tag{8}$$

for all $\lambda \in \Omega$. Therefore, we obtain from (7) and (8) that

$$\left|\widetilde{(B^*A)}(\lambda)\right|^r \leq \frac{1}{2}\left(\widetilde{(A^*A)^r + (B^*B)^r}(\lambda)\right) \leq \frac{1}{2}\left(\widetilde{|A|^{2r} + |B|^{2r}}(\lambda)\right)$$

and hence

$$\left|\widetilde{(B^*A)}(\lambda)\right|^r \leq \frac{1}{2} \sup_{\lambda \in \Omega}\left(\widetilde{|A|^{2r} + |B|^{2r}}(\lambda)\right) = \frac{1}{2} ber\left(|A|^{2r} + |B|^{2r}\right)$$

for all $\lambda \in \Omega$. This implies that

$$ber^r(B^*A) \leq \frac{1}{2} ber\left(|A|^{2r} + |B|^{2r}\right).$$

This completes the proof. $\square$

In the following theorem, we consider a different approach to obtain Berezin number inequality.

Theorem 3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\alpha \in (0,1)$ and $r \geq 1$. Then we have*

$$ber^{2r}(B^*A) \leq \left[\alpha\, ber\left(|A|^{\frac{2r}{\alpha}}\right) + (1-\alpha)\, ber\left(|B|^{\frac{2r}{1-\alpha}}\right)\right].$$

Proof. From Schwarz inequality, we obtain

$$\left|\widetilde{(B^*A)}(\lambda)\right|^2 \leq \widetilde{(A^*A)}(\lambda)\, \widetilde{(B^*B)}(\lambda) = \left(\widetilde{(A^*A)^{\frac{1}{\alpha}}}\right)^\alpha(\lambda) \left(\widetilde{(B^*B)^{\frac{1}{1-\alpha}}}\right)^{1-\alpha}(\lambda)$$

for all $\lambda \in \Omega$.

Then we get from Lemma 3.2 for $0 < \alpha < 1$

$$\left(\widetilde{(A^*A)^{\frac{1}{\alpha}}}\right)^\alpha (\lambda) \left(\widetilde{(B^*B)^{\frac{1}{1-\alpha}}}\right)^{1-\alpha} (\lambda) \leq \left[\widetilde{(A^*A)^{\frac{1}{\alpha}}} (\lambda)\right]^\alpha \left[\widetilde{(B^*B)^{\frac{1}{1-\alpha}}} (\lambda)\right]^{1-\alpha}$$

for all $\lambda \in \Omega$.

Therefore, we obtain from Lemma 3.1 (*a*) that

$$\left[\widetilde{(A^*A)^{\frac{1}{\alpha}}} (\lambda)\right]^\alpha \left[\widetilde{(B^*B)^{\frac{1}{1-\alpha}}} (\lambda)\right]^{1-\alpha} \leq \alpha \widetilde{(A^*A)^{\frac{1}{\alpha}}} (\lambda) + (1-\alpha) \widetilde{(B^*B)^{\frac{1}{1-\alpha}}} (\lambda)$$

$$\leq \left[\alpha \left(\widetilde{(A^*A)^{\frac{1}{\alpha}}} (\lambda)\right)^r + (1-\alpha) \left(\widetilde{(B^*B)^{\frac{1}{1-\alpha}}} (\lambda)\right)^r\right]^{\frac{1}{r}} \leq \left[\alpha \widetilde{(A^*A)^{\frac{r}{\alpha}}} (\lambda) + (1-\alpha) \widetilde{(B^*B)^{\frac{r}{1-\alpha}}} (\lambda)\right]^{\frac{1}{r}}$$

and then

$$\left|\widetilde{(B^*A)} (\lambda)\right|^{2r} \leq \left[\alpha \widetilde{(A^*A)^{\frac{r}{\alpha}}} (\lambda) + (1-\alpha) \widetilde{(B^*B)^{\frac{r}{1-\alpha}}} (\lambda)\right] \leq \left[\alpha \widetilde{|A|^{\frac{2r}{\alpha}}} (\lambda) + (1-\alpha) \widetilde{|B|^{\frac{2r}{1-\alpha}}} (\lambda)\right]$$

for all $\lambda \in \Omega$.

Taking the supremum over $\lambda \in \Omega$ above inequality, we obtain

$$ber^{2r} (B^*A) \leq \left[\alpha \, ber \left(|A|^{\frac{2r}{\alpha}}\right) + (1-\alpha) \, ber \left(|B|^{\frac{2r}{1-\alpha}}\right)\right].$$

This proves the theorem. □

Now we give an inequality for the sum of two products.

Theorem 3.5. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ be four positive operators and $r, s \geq 1$. Then*

$$ber^2 \left(\frac{(B^*A + D^*C)}{2}\right) \leq ber^{\frac{1}{r}} \left(\frac{|A|^{2r} + |C|^{2r}}{2}\right) ber^{\frac{1}{s}} \left(\frac{|B|^{2s} + |D|^{2s}}{2}\right).$$

Proof. We get from Schwarz inequality that

$$\left|\left\langle (B^*A + D^*C) \widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|^2 = \left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle + \left\langle D^*C\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|^2 \tag{9}$$

$$\leq \left[\left|\left\langle B^*A\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right| + \left|\left\langle D^*C\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|\right]^2$$

$$\leq \left[\left(\widetilde{A^*A} (\lambda)\right)^{1/2} \left(\widetilde{B^*B} (\mu)\right)^{1/2} + \left(\widetilde{C^*C} (\lambda)\right)^{1/2} \left(\widetilde{D^*D} (\mu)\right)^{1/2}\right]^{1/2}$$

for all $\lambda, \mu \in \Omega$.

Using the following elementary inequality for $a, b, c, d \in \mathbb{R}$

$$(ab + cd)^2 \leq \left(a^2 + c^2\right) \left(b^2 + d^2\right),$$

we have

$$\left[\left(\widetilde{A^*A} (\lambda)\right)^{1/2} \left(\widetilde{B^*B} (\mu)\right)^{1/2} + \left(\widetilde{C^*C} (\lambda)\right)^{1/2} \left(\widetilde{D^*D} (\mu)\right)^{1/2}\right]^2 \tag{10}$$

$$\leq \left[\widetilde{A^*A} (\lambda) + \widetilde{C^*C} (\lambda)\right] \left[\widetilde{B^*B} (\mu) + \widetilde{D^*D} (\mu)\right]$$

for all $\lambda, \mu \in \Omega$.

As in proof of Theorem 3.3, using the arithmetic-geometric mean inequality and the convexity of the function, we get

$$\left[\widetilde{A^*A}\,(\lambda) + \widetilde{C^*C}\,(\lambda)\right]\left[\widetilde{B^*B}\,(\mu) + \widetilde{D^*D}\,(\mu)\right] \tag{11}$$

$$\leq 4\left[\frac{\widetilde{(A^*A)^r + (C^*C)^r}}{2}\,(\lambda)\right]^{1/r}\left[\frac{\widetilde{(B^*B)^s + (D^*D)^s}}{2}\,(\mu)\right]^{1/s}$$

$$\leq 4\left[\frac{\widetilde{|A|^{2r} + |C|^{2r}}}{2}\,(\lambda)\right]^{1/r}\left[\frac{\widetilde{|B|^{2s} + |D|^{2s}}}{2}\,(\mu)\right]^{1/s}$$

for all $\lambda, \mu \in \Omega$ and $r, s \geq 1$.

Thus, we obtain by $(9) - (11)$

$$\left|\left\langle \frac{\widetilde{(B^*A + D^*C)}}{2}\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|^2 \leq \left[\frac{\widetilde{|A|^{2r} + |C|^{2r}}}{2}\,(\lambda)\right]^{1/r}\left[\frac{\widetilde{|B|^{2s} + |D|^{2s}}}{2}\,(\mu)\right]^{1/s}$$

for all $\lambda, \mu \in \Omega$. Now by replacing $\lambda = \mu$ and taking the supremum over $\lambda \in \Omega$ above inequality, we obtain

$$ber^2\left(\frac{(B^*A + D^*C)}{2}\right) \leq ber^{\frac{1}{r}}\left(\frac{|A|^{2r} + |C|^{2r}}{2}\right)ber^{\frac{1}{s}}\left(\frac{|B|^{2s} + |D|^{2s}}{2}\right).$$

This completes the proof. □

Before giving our next result, we set $\|A\|_{ber} := \sup\limits_{\lambda \in \Omega}\left\|A\widehat{k}_\lambda\right\|$ and $m(A) := \inf\limits_{\lambda \in \Omega}\left|\widetilde{A}\,(\lambda)\right|$.

The following theorem proves a new inequality for the Berezin number of the commutator $AB - BA$ for any two operators A and B.

Theorem 3.6. *Let* $A, B \in \mathcal{H} = \mathcal{H}(\Omega)$ *be two operators. Then*

$$ber\,([A, B]) \leq ber\,(A)\,ber\,(B) + \sqrt{\left(\|A^*\|_{ber}^2 - m(A)^2\right)\left(\|B\|_{ber}^2 - m(B)^2\right)}. \tag{12}$$

Proof. Indeed, for every $\lambda \in \Omega$ we have:

$$\begin{aligned}
\widetilde{[A, B]}\,(\lambda) &= \left\langle (AB - BA)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle = \left\langle AB\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - \left\langle BA\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&= \left\langle B\widehat{k}_\lambda, A^*\widehat{k}_\lambda \right\rangle - \left\langle A\widehat{k}_\lambda, B^*\widehat{k}_\lambda \right\rangle \\
&= \left\langle \left(B\widehat{k}_\lambda - \widetilde{B}\,(\lambda)\widehat{k}_\lambda\right) + \widetilde{B}\,(\lambda)\widehat{k}_\lambda, \left(A^*\widehat{k}_\lambda - \widetilde{A^*}\,(\lambda)\widehat{k}_\lambda\right) + \widetilde{A^*}\,(\lambda)\widehat{k}_\lambda \right\rangle \\
&\quad - \left\langle \left(A\widehat{k}_\lambda - \widetilde{A}\,(\lambda)\widehat{k}_\lambda\right) + \widetilde{A}\,(\lambda)\widehat{k}_\lambda, \left(B^*\widehat{k}_\lambda - \widetilde{B^*}\,(\lambda)\widehat{k}_\lambda\right) + \widetilde{B^*}\,(\lambda)\widehat{k}_\lambda \right\rangle \\
&= \left\langle B\widehat{k}_\lambda - \widetilde{B}\,(\lambda)\widehat{k}_\lambda, A^*\widehat{k}_\lambda - \widetilde{A^*}\,(\lambda)\widehat{k}_\lambda \right\rangle + \widetilde{A}\,(\lambda)\left\langle B\widehat{k}_\lambda - \widetilde{B}\,(\lambda)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\quad + \widetilde{B}\,(\lambda)\left\langle \widehat{k}_\lambda, A^*\widehat{k}_\lambda - \widetilde{A^*}\,(\lambda)\widehat{k}_\lambda \right\rangle + \widetilde{A}\,(\lambda)\widetilde{B}\,(\lambda) \\
&= \widetilde{A}\,(\lambda)\widetilde{B}\,(\lambda) + \left\langle B\widehat{k}_\lambda - \widetilde{B}\,(\lambda)\widehat{k}_\lambda, A^*\widehat{k}_\lambda - \widetilde{A^*}\,(\lambda)\widehat{k}_\lambda \right\rangle.
\end{aligned}$$

Thus

$$\widetilde{[A, B]}\,(\lambda) = \widetilde{A}\,(\lambda)\widetilde{B}\,(\lambda) + \left\langle B\widehat{k}_\lambda - \widetilde{B}\,(\lambda)\widehat{k}_\lambda, A^*\widehat{k}_\lambda - \widetilde{A^*}\,(\lambda)\widehat{k}_\lambda \right\rangle$$

for all $\lambda \in \Omega$. Then

$$\left|\widetilde{[A,B]}(\lambda)\right| \leq \left|\widetilde{A}(\lambda)\right|\left|\widetilde{B}(\lambda)\right| + \left\|\widehat{Bk_\lambda} - \widetilde{B}(\lambda)\widehat{k_\lambda}\right\|\left\|A^*\widehat{k_\lambda} - \widetilde{A^*}(\lambda)\widehat{k_\lambda}\right\|$$

$$= \left|\widetilde{A}(\lambda)\right|\left|\widetilde{B}(\lambda)\right| + \left(\left\|A^*\widehat{k_\lambda}\right\|^2 - \left|\widetilde{A}(\lambda)\right|^2\right)^{1/2}\left(\left\|\widehat{Bk_\lambda}\right\|^2 - \left|\widetilde{B}(\lambda)\right|^2\right)^{1/2}$$

$$\leq ber(A)\,ber(B) + \left[\left(\|A^*\|_{ber}^2 - \inf_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|^2\right)\left(\|B\|_{ber}^2 - \inf_{\lambda\in\Omega}\left|\widetilde{B}(\lambda)\right|^2\right)\right]^{1/2}$$

$$= ber(A)\,ber(B) + \sqrt{\left(\|A^*\|_{ber}^2 - m(A)^2\right)\left(\|B\|_{ber}^2 - m(B)^2\right)}$$

for all $\lambda \in \Omega$, and hence

$$ber([A,B]) \leq ber(A)\,ber(B) + \sqrt{\left(\|A^*\|_{ber}^2 - m(A)^2\right)\left(\|B\|_{ber}^2 - m(B)^2\right)},$$

as desired. $\square$

Corollary 3.7. *If $A \in \mathcal{B}(\mathcal{H}(\Omega))$, then*

$$ber(A)^2 \geq ber([A,A^*]) + m(A)^2 - \|A^*\|_{ber}^2.$$

The following proposition gives in case $B = A^*$ and A is hyponormal better estimate than the estimate (12).

Proposition 3.8. *If $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is a hyponormal operator (i.e., $[A,A^*] \geq 0$), then*

$$ber([A,A^*]) \leq \|A^*\|_{ber}^2 - m(A)^2.$$

Proof. In fact, it follows from hyponormality of A that

$$0 \leq \widetilde{[A,A^*]}(\lambda) = \left\|A^*\widehat{k_\lambda}\right\|^2 - \left\|A\widehat{k_\lambda}\right\|^2 \quad \leq \left\|A^*\widehat{k_\lambda}\right\|^2 - \left|\widetilde{A}(\lambda)\right|^2 \leq \left\|A^*\widehat{k_\lambda}\right\|^2 - \inf_{\mu\in\Omega}\left|\widetilde{A}(\mu)\right|^2 \quad \leq \|A^*\|_{ber}^2 - m(A)^2$$

for all $\lambda \in \Omega$. Hence

$$ber([A,A^*]) \leq \|A^*\|_{ber}^2 - m(A)^2,$$

as desired. $\square$

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