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Biderivations and Bihomomorphisms in Banach Algebras

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Abstract. In this paper, we solve the following bi-additive s-functional inequalities

$$\begin{aligned}&\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\|\\&\quad\leq\left\|s\left(2f\left(x+y,z-w\right)+2f\left(x-y,z+w\right)-4f(x,z)+4f(y,w)\right)\right\|\end{aligned}\tag{1}$$

and

$$\begin{aligned}&\left\|2f\left(x+y,z-w\right)+2f\left(x-y,z+w\right)-4f(x,z)+4f(y,w)\right\|\\&\quad\leq\left\|s\left(f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\right)\right\|,\end{aligned}\tag{2}$$

where s is a fixed nonzero complex number with $|s|<1$.

Moreover, we prove the Hyers-Ulam stability of biderivations and bihomomorphismsions in Banach algebras and unital C^*-algebras, associated with the bi-additive s-functional inequalities (1) and (2).

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [10] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\|\leq\|f(x+y)\|\tag{3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [22]. Fechner [8] and Gilányi [11] proved the Hyers-Ulam stability of the functional inequality (3).

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Park [18–20] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [2, 4–7, 16, 25]).

Maksa [14, 15] introduced and investigated biderivations and symmetric biderivations on rings. Öztürk and Sapanci [17], Vukman [24] and Yazarli [26] investigated some properties of symmetric biderivations on rings.

Definition 1.1. [14, 15] *Let A be a ring. A bi-additive mapping $D : A \times A \to A$ is called a symmetric biderivation on A if D satisfies*

$$\begin{aligned} D(xy, z) &= D(x, z)y + xD(y, z), \\ D(x, y) &= D(y, x) \end{aligned}$$

for all $x, y, z \in A$.

In this paper, we introduce biderivations and bihomomorphisms in a Banach algebra.

Definition 1.2. *Let A be a complex Banach algebra. A $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is called a biderivation on A if D satisfies*

$$\begin{aligned} D(xy, z) &= D(x, z)y + xD(y, z), \\ D(x, zw) &= D(x, z)w + zD(x, w) \end{aligned}$$

for all $x, y, z, w \in A$.

It is easy to show that if D is a biderivation, then

$$D(xy, zw) = D(x, z)wy + zD(x, w)y + xD(y, z)w + xzD(y, w)$$

for all $x, y, z, w \in A$.

Definition 1.3. *Let A and B be complex Banach algebras. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a bihomomorphism if H satisfies*

$$\begin{aligned} H(xy, z^2) &= H(x, z)H(y, z), \\ H(x^2, zw) &= H(x, z)H(x, w) \end{aligned}$$

for all $x, y, z, w \in A$.

This paper is organized as follows: In Sections 2 and 3, we solve the bi-additive s-functional inequalities (1) and (2) and prove the Hyers-Ulam stability of the bi-additive s-functional inequalities (1) and (2) in complex Banach spaces. In Section 4, we investigate biderivations on Banach algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2). In Section 5, we investigate bihomomorphisms in Banach algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2).

Throughout this paper, let X be a complex normed space and Y a complex Banach space. Let A and B be complex Banach algebras. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2. Bi-additive s-functional inequality (1)

We solve and investigate the bi-additive s-functional inequality (1) in complex normed spaces.

Lemma 2.1. *If a mapping $f : X^2 \to Y$ satisfies $f(0, z) = f(x, 0) = 0$ and*

$$\begin{aligned} &\|f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) - 4f(x,z)\| \\ &\leq \left\| s\left(2f\left(x+y, z-w\right) + 2f\left(x-y, z+w\right) - 4f(x,z) + 4f(y,w)\right)\right\| \end{aligned} \tag{4}$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is bi-additive.

Proof. Assume that $f : X^2 \to Y$ satisfies (4).

Letting $x = y$ and $w = 0$ in (4), we get $f(2x, z) = 2f(x, z)$ for all $x, z \in X$.

Letting $w = 0$ in (4), we get $f(x + y, z) + f(x - y, z) = 2f(x, z)$ and so $f(x_1, z) + f(y_1, z) = 2f\left(\frac{x_1+y_1}{2}, z\right) = f(x_1 + y_1, z)$ for all $x_1 := x + y, y_1 := x - y, z \in X$, since $|s| < 1$ and $f(0, z) = 0$ for all $z \in X$. So $f : X^2 \to Y$ is additive in the first variable.

Similarly, one can show that $f : X^2 \to Y$ is additive in the second variable. Hence $f : X^2 \to Y$ is bi-additive. $\square$

We prove the Hyers-Ulam stability of the bi-additive s-functional inequality (4) in complex Banach spaces.

Theorem 2.2. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) - 4f(x,z)\| \\ &\leq \left\| s\left(2f\left(x+y, z-w\right) + 2f\left(x-y, z+w\right) - 4f(x,z) + 4f(y,w)\right)\right\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{5}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that

$$\|f(x,z) - A(x,z)\| \leq \frac{\theta}{(1-|s|)(2^r - 2)}\|x\|^r\|z\|^r \tag{6}$$

for all $x, z \in X$.

Proof. Letting $w = 0$ and $y = x$ in (5), we get

$$2(1-|s|)\|f(2x,z) - 2f(x,z)\| \leq 2\theta\|x\|^r\|z\|^r \tag{7}$$

for all $x, z \in X$. So

$$\left\|f(x,z) - 2f\left(\frac{x}{2}, z\right)\right\| \leq \frac{1}{(1-|s|)2^r}\theta\|x\|^r\|z\|^r$$

for all $x, z \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}, z\right) - 2^m f\left(\frac{x}{2^m}, z\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}, z\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}, z\right)\right\| \\ &\leq \frac{1}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{(1-|s|)2^{rj}}\theta\|x\|^r\|z\|^r \end{aligned} \tag{8}$$

for all nonnegative integers m and l with $m > l$ and all $x, z \in X$. It follows from (8) that the sequence $\{2^k f(\frac{x}{2^k}, z)\}$ is Cauchy for all $x, z \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k}, z)\}$ converges. So one can define the mapping $A : X^2 \to Y$ by

$$A(x,z) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}, z\right)$$

for all $x, z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (8), we get (6).

It follows from (5) that

$$
\begin{aligned}
&\|A(x+y,z+w)+A(x+y,z-w)+A(x-y,z+w)+A(x-y,z-w)-4A(x,z)\| \\
&= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n},z+w\right)+f\left(\frac{x+y}{2^n},z-w\right)+f\left(\frac{x-y}{2^n},z+w\right)\right.\right. \\
&\qquad \left.\left.+f\left(\frac{x-y}{2^n},z-w\right)-4f\left(\frac{x}{2^n},z\right)\right)\right\| \\
&\le \lim_{n\to\infty}\left\|2^n s\left(2f\left(\frac{x+y}{2^n},z-w\right)+2f\left(\frac{x-y}{2^n},z+w\right)-4f\left(\frac{x}{2^n},z\right)+4f\left(\frac{y}{2^n},w\right)\right)\right\| \\
&\qquad + \lim_{n\to\infty}\frac{2^n}{2^{rn}}\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r) \\
&\le \left\|s\left(2A\left(x+y,z-w\right)+A\left(x-y,z+w\right)-4A(x,z)+4A(y,w)\right)\right\|
\end{aligned}
$$

for all $x, y, z, w \in X$. So

$$
\begin{aligned}
&\|A(x+y,z+w)+A(x+y,z-w)+A(x-y,z+w)+A(x-y,z-w)-4A(x,z)\| \\
&\le \left\|s\left(2A\left(x+y,z-w\right)+2A\left(x-y,z+w\right)-4A(x,z)+4A(y,w)\right)\right\|
\end{aligned}
$$

for all $x, y, z, w \in X$. By Lemma 2.1, the mapping $A : X^2 \to Y$ is bi-additive.

Now, let $T : X^2 \to Y$ be another bi-additive mapping satisfying (6). Then we have

$$
\begin{aligned}
\|A(x,z)-T(x,z)\| &= \left\|2^q A\left(\frac{x}{2^q},z\right)-2^q T\left(\frac{x}{2^q},z\right)\right\| \\
&\le \left\|2^q A\left(\frac{x}{2^q},z\right)-2^q f\left(\frac{x}{2^q},z\right)\right\|+\left\|2^q T\left(\frac{x}{2^q},z\right)-2^q f\left(\frac{x}{2^q},z\right)\right\| \\
&\le \frac{2\theta}{(1-|s|)(2^r-2)}\frac{2^q}{2^{qr}}\|x\|^r\|z\|^r,
\end{aligned}
$$

which tends to zero as $q \to \infty$ for all $x, z \in X$. So we can conclude that $A(x,z) = T(x,z)$ for all $x, z \in X$. This proves the uniqueness of A, as desired. □

Theorem 2.3. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying* (5) *and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that*

$$
\|f(x,z)-A(x,z)\| \le \frac{\theta}{(1-|s|)(2-2^r)}\|x\|^r\|z\|^r \tag{9}
$$

for all $x, z \in X$.

Proof. It follows from (7) that

$$
\left\|f(x,z)-\frac{1}{2}f(2x,z)\right\| \le \frac{\theta}{2(1-|s|)}\|x\|^r\|z\|^r
$$

for all $x, z \in X$. Hence

$$
\begin{aligned}
\left\|\frac{1}{2^l}f(2^l x,z)-\frac{1}{2^m}f(2^m x,z)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x,z\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x,z\right)\right\| \\
&\le \sum_{j=l}^{m-1}\frac{2^{rj}}{(1-|s|)2^{j+1}}\theta\|x\|^r\|z\|^r
\end{aligned} \tag{10}
$$

for all nonnegative integers m and l with $m > l$ and all $x, z \in X$. It follows from (10) that the sequence $\{\frac{1}{2^n} f(2^n x, z)\}$ is a Cauchy sequence for all $x, z \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x, z)\}$ converges. So one can define the mapping $A : X^2 \to Y$ by

$$A(x,z) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x, z)$$

for all $x, z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (10), we get (9).

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Bi-additive s-functional inequality (2)

We solve and investigate the bi-additive s-functional inequality (2) in complex normed spaces.

Lemma 3.1. *If a mapping $f : X^2 \to Y$ satisfies $f(0,z) = f(x,0) = 0$ and*

$$\begin{aligned} &\left\|2f(x+y, z-w) + 2f(x-y, z+w) - 4f(x,z) + 4f(y,w)\right\| \\ &\quad \le \left\|s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right)\right\| \end{aligned} \tag{11}$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is bi-additive.

Proof. Assume that $f : X^2 \to Y$ satisfies (11).

Letting $y = x$ and $w = 0$ in (11), we get $2f(2x, z) = 4f(x, z)$ for all $x, z \in X$.

Letting $w = 0$ in (4), we get $f(x + y, z) + f(x - y, z) = 2f(x, z)$ and so $f(x_1, z) + f(y_1, z) = 2f\left(\frac{x_1+y_1}{2}, z\right) = f(x_1 + y_1, z)$ for all $x_1 := x + y, y_1 := x - y, z \in X$, since $|s| \le 1$ and $f(0, z) = 0$ for all $z \in X$. So $f : X^2 \to Y$ is additive in the first variable.

Similarly, one can show that $f : X^2 \to Y$ is additive in the second variable. Hence $f : X^2 \to Y$ is bi-additive. □

We prove the Hyers-Ulam stability of the bi-additive s-functional inequality (11) in complex Banach spaces.

Theorem 3.2. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\left\|2f(x+y, z-w) + 2f(x-y, z+w) - 4f(x,z) + 4f(y,w)\right\| \\ &\le \left\|s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right)\right\| \\ &\quad +\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{12}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that

$$\|f(x,z) - A(x,z)\| \le \frac{\theta}{(1-|s|)(2^r-2)} \|x\|^r \|z\|^r \tag{13}$$

for all $x, z \in X$.

Proof. Letting $y = x$ and $w = 0$ in (12), we get

$$2(1-|s|)\left\|f(2x,z) - 2f(x,z)\right\| \le 2\theta \|x\|^r \|z\|^r \tag{14}$$

for all $x, z \in X$. So

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}, z\right) - 2^m f\left(\frac{x}{2^m}, z\right)\right\| &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}, z\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}, z\right)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{2^j \theta}{(1-|s|)2^{rj+r}} \|x\|^r \|z\|^r \end{aligned} \tag{15}$$

for all nonnegative integers m and l with $m > l$ and all $x, z \in X$. It follows from (15) that the sequence $\{2^k f\left(\frac{x}{2^k}, z\right)\}$ is Cauchy for all $x, z \in X$. Since Y is a Banach space, the sequence $\{2^k f\left(\frac{x}{2^k}, z\right)\}$ converges. So one can define the mapping $A : X^2 \to Y$ by

$$A(x,z) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}, z\right)$$

for all $x, z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (15), we get (13).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying* (12) *and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that*

$$\|f(x,z) - A(x,z)\| \le \frac{\theta}{(1-|s|)(2-2^r)}\|x\|^r\|z\|^r \tag{16}$$

for all $x, z \in X$.

Proof. It follows from (14) that

$$\left\|f(x,z) - \frac{1}{2}f(2x,z)\right\| \le \frac{\theta}{2(1-|s|)}\|x\|^r\|z\|^r$$

for all $x, z \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x,z) - \frac{1}{2^m}f(2^m x,z)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x,z\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x,z\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{2^{rj}}{(1-|s|)2^{j+1}}\theta\|x\|^r\|z\|^r\end{aligned} \tag{17}$$

for all nonnegative integers m and l with $m > l$ and all $x, z \in X$. It follows from (17) that the sequence $\{\frac{1}{2^n}f(2^n x,z)\}$ is a Cauchy sequence for all $x, z \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x,z)\}$ converges. So one can define the mapping $A : X^2 \to Y$ by

$$A(x,z) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x,z)$$

for all $x, z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (17), we get (16).

The rest of the proof is similar to the proof of Theorem 2.2. □

4. Biderivations on Banach algebras

Now, we investigate biderivations on complex Banach algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2).

Lemma 4.1. [3, Lemma 2.1] *Let $f : X^2 \to Y$ be a bi-additive mapping such that $f(\lambda x, \mu z) = \lambda\mu f(x,z)$ for all $x, z \in X$ and $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$. Then f is $\mathbb{C}$-bilinear.*

Theorem 4.2. *Let A be a complex Banach algebra. Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\|f(\lambda(x+y),\mu(z+w)) + f(\lambda(x+y),\mu(z-w)) + f(\lambda(x-y),\mu(z+w)) \\ &\quad + f(\lambda(x-y),\mu(z-w)) - 4\lambda\mu f(x,z)\| \\ &\le \left\|s\left(2f(x+y,z-w) + 2f(x-y,z+w) - 4f(x,z) + 4f(y,w)\right)\right\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)\end{aligned} \tag{18}$$

for all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that

$$\|f(x,z) - D(x,z)\| \le \frac{\theta}{(1-|s|)(2^r - 2)} \|x\|^r \|z\|^r \tag{19}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f(xy,z) - f(x,z)y - xf(y,z)\| \le \theta(\|x\|^r + \|y\|^r)\|z\|^r, \tag{20}$$

$$\|f(x,zw) - f(x,z)w - zf(x,w)\| \le \theta\|x\|^r(\|z\|^r + \|w\|^r) \tag{21}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to A$ is a biderivation.

Proof. Let $\lambda = \mu = 1$ in (18). By Theorem 2.2, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (19) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

Letting $y = w = 0$ in (18), we get $f(\lambda x, \mu z) = \lambda\mu f(x, z)$ for all $x, z \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. By Lemma 4.1, the bi-additive mapping $D : A^2 \to A$ is $\mathbb{C}$-bilinear.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $D(x, z) = f(x, z)$ for all $x, z \in A$.

It follows from (20) that

$$\begin{aligned} \|D(xy,z) - D(x,z)y - xD(y,z)\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}, z\right) - f\left(\frac{x}{2^n}, z\right)\frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}, z\right)\right\| \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{2^{rn}} (\|x\|^r + \|y\|^r)\|z\|^r = 0 \end{aligned}$$

for all $x, y, z \in A$. Thus

$$D(xy,z) = D(x,z)y + xD(y,z)$$

for all $x, y, z \in A$.

Similarly, one can show that

$$D(x,zw) = D(x,z)w + zD(x,w)$$

for all $x, z, w \in A$. Hence the mapping $f : A^2 \to A$ is a biderivation. □

Theorem 4.3. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying* (18) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x,z) - D(x,z)\| \le \frac{\theta}{(1-|s|)(2 - 2^r)} \|x\|^r \|z\|^r \tag{22}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (20), (21) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.*

Proof. The proof is similar to the proof of Theorem 4.2. □

Similarly, we can obtain the following results.

Theorem 4.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned}&\left\|2f(\lambda(x+y), \mu(z-w)) + 2f(\lambda(x-y), \mu(z+w)) - 4\lambda\mu f(x,z) + 4\lambda\mu f(y,w)\right\| \\ &\leq \left\|s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right)\right\| \\ &\quad +\theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)\end{aligned} \tag{23}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that

$$\|f(x,z) - D(x,z)\| \leq \frac{\theta}{(1-|s|)(2^r-2)}\|x\|^r\|z\|^r \tag{24}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (20), (21) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.*

Theorem 4.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying* (23) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x,z) - D(x,z)\| \leq \frac{\theta}{(1-|s|)(2-2^r)}\|x\|^r\|z\|^r \tag{25}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (20), (21) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.*

From now on, assume that A is a unital C^*-algebra with unit e and unitary group $U(A)$.

Theorem 4.6. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying* (18) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying* (19).

If, in addition, the mapping $f : A^2 \to A$ satisfies (21), *$f(2x, z) = 2f(x, z)$ and*

$$\|f(uy,z) - f(u,z)y - uf(y,z)\| \leq \theta(1 + \|y\|^r)\|z\|^r, \tag{26}$$

$$\|f(x,zv) - f(x,z)v - zf(x,v)\| \leq \theta(1 + \|y\|^r)\|z\|^r \tag{27}$$

for all $u, v \in U(A)$ and all $x, y, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.

Proof. By the same reasoning as in the proof of Theorem 4.2, there is a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying (19) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $D(x, z) = f(x, z)$ for all $x, z \in A$.

By the same reasoning as in the proof of Theorem 4.2, $D(uy, z) = D(u, z)y + uD(y, z)$ for all $u, v \in U(A)$ and all $y, z \in A$.

Since D is $\mathbb{C}$-linear in the first variable and each $x \in A$ is a finite linear combination of unitary elements (see [13]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$),

$$\begin{aligned} D(xy,z) &= D(\sum_{j=1}^m \lambda_j u_j y, z) = \sum_{j=1}^m \lambda_j D(u_j y, z) = \sum_{j=1}^m \lambda_j (D(u_j,z)y + u_j D(y,z)) \\ &= (\sum_{j=1}^m \lambda_j) D(u_j, z)y + (\sum_{j=1}^m \lambda_j u_j) D(y,z) = D(x,z)y + xD(y,z)\end{aligned}$$

for all $x, y, z \in A$.

Similarly, one can show that $D(x, zw) = D(x, z)w + zD(x, w)$ for all $x, z, w \in A$. Thus $f : A^2 \to A$ is a biderivation. □

Theorem 4.7. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying* (18) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying* (22).

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.*

Proof. The proof is similar to the proof of Theorem 4.6. □

Similarly, we can obtain the following results.

Theorem 4.8. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying* (23) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying* (24).

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.*

Theorem 4.9. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying* (23) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying* (25).

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a biderivation.*

5. Bihomomorphisms in Banach algebras

Now, we investigate bihomomorphisms in complex Banach algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2).

Theorem 5.1. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and* (18). *Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (19), *where D is replaced by H in* (19).

If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f(xy, z^2) - f(x, z)f(y, z)\| \le \theta(\|x\|^r + \|y\|^r)\|z\|^r, \tag{28}$$

$$\|f(x^2, zw) - f(x, z)f(x, w)\| \le \theta\|x\|^r(\|z\|^r + \|w\|^r) \tag{29}$$

for all $x, y, z, w \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.2, there is a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$, which is defined by

$$H(x, z) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $H(x, z) = f(x, z)$ for all $x, z \in A$.

It follows from (28) that

$$\begin{aligned} \|H(xy, z^2) - H(x, z)H(y, z)\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}, z^2\right) - f\left(\frac{x}{2^n}, z\right) f\left(\frac{y}{2^n}, z\right) \right\| \\ &\le \lim_{n\to\infty} \frac{4^n \theta}{2^{rn}} (\|x\|^r + \|y\|^r)\|z\|^r = 0 \end{aligned}$$

for all $x, y, z \in A$. Thus

$$H(xy, z^2) = H(x, z)H(y, z)$$

for all $x, y, z \in A$.

Similarly, one can show that

$$H(x^2, zw) = H(x, z)H(x, w)$$

for all $x, z, w \in A$. Hence the mapping $f : A^2 \to B$ is a bihomomorphism. □

Theorem 5.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying* (18) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (22)*, where D is replaced by H in* (22).

If, in addition, the mapping $f : A^2 \to B$ satisfies (28), (29) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.*

Proof. The proof is similar to the proof of Theorem 5.1. □

Similarly, we can obtain the following results.

Theorem 5.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and* (23)*. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (24)*, where D is replaced by H in* (24).

If, in addition, the mapping $f : A^2 \to B$ satisfies (28), (29) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.*

Theorem 5.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying* (23) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (25)*, where D is replaced by H in* (25).

If, in addition, the mapping $f : A^2 \to B$ satisfies (28), (29) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.*

From now on, assume that A is a unital C^*-algebra with unit e and unitary group $U(A)$.

Theorem 5.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying* (18) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (19)*, where D is replaced by H in* (19).

If, in addition, the mapping $f : A^2 \to B$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f(uy, z^2) - f(u, z)f(y, z)\| \leq \theta(1 + \|y\|^r)\|z\|^r, \tag{30}$$

$$\|f(x^2, zv) - f(x, z)f(x, v)\| \leq \theta(1 + \|y\|^r)\|z\|^r, \tag{31}$$

for all $u, v \in U(A)$ and all $x, y, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.2, there is a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying (19) defined by

$$H(x, z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $H(x, z) = f(x, z)$ for all $x, z \in A$.

By the same reasoning as in the proof of Theorem 4.2, $H(uy, z^2) = H(u, z)H(y, z)$ for all $u, v \in U(A)$ and all $y, z \in A$.

Since H is $\mathbb{C}$-linear in the first variable and each $x \in A$ is a finite linear combination of unitary elements (see [13]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$),

$$\begin{aligned} H(xy, z^2) &= H(\sum_{j=1}^{m} \lambda_j u_j y, z^2) = \sum_{j=1}^{m} \lambda_j H(u_j y, z^2) = \sum_{j=1}^{m} \lambda_j (H(u_j, z)H(y, z)) \\ &= (\sum_{j=1}^{m} \lambda_j)H(u_j, z)H(y, z) = H(x, z)H(y, z) \end{aligned}$$

for all $x, y, z \in A$.

Similarly, one can show that $H(x, zw) = H(x, z)H(x, w)$ for all $x, z, w \in A$. Thus $f : A^2 \to B$ is a bihomomorphism. □

Theorem 5.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying* (18) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (22), *where D is replaced by H in* (22).

If, in addition, the mapping $f : A^2 \to B$ satisfies (30), (31) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.*

Proof. The proof is similar to the proof of Theorem 5.7. □

Similarly, we can obtain the following results.

Theorem 5.7. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying* (23) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (24), *where D is replaced by H in* (24).

If, in addition, the mapping $f : A^2 \to B$ satisfies (30), (31) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.*

Theorem 5.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to B$ be a mapping satisfying* (23) *and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $H : A^2 \to B$ satisfying* (25), *where D is replaced by H in* (25).

If, in addition, the mapping $f : A^2 \to B$ satisfies (30), (31) *and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to B$ is a bihomomorphism.*

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